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RESEARCH PAPER

Limitations of pullback attractors for processes

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Pullback convergence has been investigated in numerous papers as an appropriate attraction concept for nonautonomous problems. However, in this note it is illustrated through some simple examples that pullback attractors do not give a complete picture of asymptotic behaviour when the nonautonomous dynamical systems that they generate are formulated as processes. It is then shown how the problem can be resolved by using a skew product formulation of the nonautonomous dynamical systems when the state space of the driving system is compact.

Keywords: Pullback attractor; two-parameter semigroup; skew product flow

AMS Subject Classification: Primary 39A11; Secondary 37B55

Dedicated to Francisco Balibrea on the occasion of his sixtieth birthday

1. Introduction

Consider a nonautonomous difference equation

$$x_{k+1} = f_k(x_k) \tag{1}$$

on a metric space (X, d) with continuous mappings $f_k : X \to X$, $k \in \mathbf{Z}$.

The above difference equation (1) generates a discrete time nonautonomous dynamical system described by a discrete time *process* or *two-parameter semigroup*

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$\varphi : \{(k, \ell, x) \in \mathbf{Z} \times \mathbf{Z} \times X : \ell \leq k\} \to X$, which satisfies

$$\varphi(k, \ell, x) = \begin{cases} f_{k-1} \circ \cdots \circ f_\ell(x) : \ell < k, \\ \qquad x \qquad\quad : \ell = k. \end{cases}$$

Invariant sets and pullback attractors for processes are introduced and discussed in section 2 and the limitations of pullback attractors for processes are indicated through some examples in section 3. In particular, the examples show how the forward asymptotic behaviour is not always captured by the pullback attractors of processes. Finally, in subsection 4.2 it is shown how this behaviour is included in the dynamics of the system when formulated as skew product flows.

For arbitrary nonempty sets A, $B \subset X$ and $x \in X$, define the *distance* of x to A by $\operatorname{dist}(x, A) := \inf \{d(x, y) : y \in A\}$ and the *Hausdorff semi-distance* of A and B by $\operatorname{dist}(A, B) := \sup \{d(x, B) : x \in A\}$.

2. Invariant sets and pullback attractors for processes

An attractor of a discrete time process φ on a metric space (X, d) consists of a family $\mathcal{A} = \{A_k : k \in \mathbf{Z}\}$ of nonempty compact subsets of X rather than a single set as in autonomous dynamical systems (cf. [1]). The reader is referred to [5, 6, 8] for motivation and details.

A family $\mathcal{A} = \{A_k : k \in \mathbf{Z}\}$ of nonempty compact subsets of X is said to be φ-invariant if

$$\varphi(k, \ell, A_\ell) = A_k \quad \text{for all } \ell \leq k,$$

which is equivalent to

$$A_{k+1} = f_k(A_k) \quad \text{for all } k \in \mathbf{Z}.$$

An *entire solution* of a process φ is a sequence $\chi = \{\chi_k\}_{k \in \mathbf{Z}}$ such that

$$\varphi(k, \ell, \chi_\ell) = \chi_k \quad \text{for all } \ell \leq k,$$

which is equivalent to $\chi_{k+1} = f_k(\chi_k)$ for all $k \in \mathbf{Z}$. An entire sequence $\chi = \{\chi_k\}_{k \in \mathbf{Z}}$ is a φ-invariant set consisting of singleton sets $A_k = \{\chi_k\}$, $k \in \mathbf{Z}$.

In general, a φ-invariant set consists of entire solutions. This is essentially due to the fact that processes are onto between the components sets. The backward solutions, however, need not be uniquely determined, since the mappings f_k are usually not assumed to be one-to-one.

Proposition 2.1. *A family $\mathcal{A} = \{A_k : k \in \mathbf{Z}\}$ is φ-invariant if and only if for every pair $\kappa \in \mathbf{Z}$ and $x \in A_\kappa$ there exists an entire solution χ such that $\chi_\kappa = x$ and $\chi_k \in A_k$ for all $k \in \mathbf{Z}$. The entire solution χ is uniquely determined, provided every mapping*

$$f_k : X \to X \quad \textit{is one-to-one for all } k \in \mathbf{Z}. \tag{2}$$

Proof. ($\Rightarrow$) Let $\mathcal{A}$ be φ-invariant and choose $x \in A_\kappa$. For $k \geq \kappa$, define the sequence $\chi_k := \varphi(k, \kappa, x)$. Then the φ-invariance of $\mathcal{A}$ yields $\chi_k \in A_k$. On the other hand, $A_\kappa = \varphi(\kappa, k, A_k)$ for $k \leq \kappa$, so there exists a sequence $x_k \in A_k$ with $x = \varphi(\kappa, k, x_k)$

and $x_k = \varphi(k, k-1, x_{k-1})$ for all $k < \kappa$. Hence define $\chi_k := x_k$ for $k < \kappa$ and χ becomes an entire solution with the desired properties. Under (2) the sequence x_k is given uniquely.

($\Leftarrow$) Suppose for arbitrary $\kappa \in \mathbf{Z}$ and $x \in A_\kappa$ there is an entire solution χ satisfying $\chi_\kappa = x$ and $\chi_k \in A_k$ for all $k \in \mathbf{Z}$. Hence $\varphi(k, \kappa, x) = \varphi(k, \kappa, \chi_\kappa) = \chi_k \in A_k$ holds for all $k \geq \kappa$. From this ones concludes that $f_k(A_k) \subseteq A_{k+1}$. The remaining inclusion $f_k(A_k) \supseteq A_{k+1}$ follows from the fact that $x = \varphi(\kappa, k, \chi_k) \in \varphi(\kappa, k, A_k)$ for $k \leq \kappa$. □

Definition 2.2. A φ-invariant family $\mathcal{A} = \{A_k : k \in \mathbf{Z}\}$ of nonempty uniformly bounded compact subsets of X is called a *pullback attractor* if

$$\lim_{n\to\infty} \operatorname{dist}\left(\varphi(k, k-n, B), A_k\right) = 0 \quad \text{for all } k \in \mathbf{Z} \tag{3}$$

and all bounded subsets B of X.

Pullback attraction is essentially a concept of attraction for the past of the system. In the next example, due to $|\lambda_+| > 1$, different solutions diverge when time is positive. Specifically, the behaviour of the system on $\mathbf{Z}_0^+ := \{0, 1, 2, \ldots\}$ has no influence on pullback attractivity.

Example 2.3 Consider the linear difference equation

$$x_{k+1} = \lambda_k x_k, \qquad\qquad \lambda_k := \begin{cases} \lambda_+, & k \geq 0 \\ \lambda_-, & k < 0 \end{cases} \tag{4}$$

with $\lambda_-, \lambda_+ \in \mathbf{R}$ satisfying $0 < |\lambda_-| < 1 < |\lambda_+|$, which generates the process

$$\varphi(k, \ell, x) = \begin{cases} \lambda_+^{k-\ell} x & \text{for } 0 \leq \ell \leq k, \\ \lambda_+^k \lambda_-^{-\ell} x & \text{for } \ell \leq 0 \leq k, \\ \lambda_-^{k-\ell} x & \text{for } \ell \leq k \leq 0. \end{cases}$$

The family of sets $\mathcal{A}$ with $A_k \equiv \{0\}$ for all $k \in \mathbf{Z}$ is obviously φ-invariant and also pullback attracting, since for any bounded subset $B \subseteq [-R, R]$ for some $R > 0$

$$0 \leq \operatorname{dist}(\varphi(k, k-n, B), A_k) \leq \operatorname{dist}(\varphi(k, k-n, [-R, R]), \{0\})$$

$$\leq R\,|\lambda_-|^n \begin{cases} \left|\frac{\lambda_+}{\lambda_-}\right|^k & \text{for } k \geq 0, \\ 1 & \text{for } k < 0 \end{cases} \xrightarrow[n\to\infty]{} 0 \quad \text{for all } k \in \mathbf{Z}.$$

As pointed out above, the pullback attraction concept reflects the dynamical behaviour in the past and does not necessarily imply forward convergence, which is the appropriate concept for the future dynamics. Example 2.3 shows that

$$\lim_{k\to\infty} \operatorname{dist}(\varphi(k, \ell, x), A_k) = 0 \quad \Longleftrightarrow \quad x = 0 \quad \text{for all } \ell \in \mathbf{Z},$$

i.e., there is no forward convergence towards $\mathcal{A}$.

The assumption that pullback attractors are uniformly bounded in the above definition ensures that they are uniquely determined.

Example 2.4 Discard the uniform boundedness assumption in Definition 2.2 and consider the process φ from the above Example 2.3. Given an arbitrary $\gamma \in \mathbf{R}$, it is easy to see that

$$\chi_k^\gamma := \gamma \begin{cases} \lambda_+^k & \text{for } k \geq 0, \\ \lambda_-^k & \text{for } k < 0 \end{cases}$$

defines an entire solution for equation (4). Hence, the families $\{0 : k \in \mathbf{Z}\}$ and $\chi^\gamma := \{\chi_k^\gamma : k \in \mathbf{Z}\}$ are invariant, as well as their union $\mathcal{A}_\gamma := \{\{0, \chi_k^\gamma\} : k \in \mathbf{Z}\}$. In Example 2.3 it was shown that $\{0 : k \in \mathbf{Z}\}$ attracts uniformly bounded sets and the definition of the Hausdorff semidistance guarantees that also $\mathcal{A}_\gamma$ has this property (cf. (3)). Since all members $A_k^\gamma = \{0, \chi_k^\gamma\}$, $k \in \mathbf{Z}$, of the family $\mathcal{A}_\gamma$ are compact (in fact finite), each $\mathcal{A}_\gamma$ would be a pullback attractor for φ, among which $\mathcal{A}^0$ is the unique uniformly bounded pullback attractor.

Since pullback attractors are uniformly bounded (by definition), they can be characterized by the bounded entire solutions of the process.

PROPOSITION 2.5. *A pullback attractor $\mathcal{A} = \{A_k : k \in \mathbf{Z}\}$ admits the dynamical characterization: for each $\kappa \in \mathbf{Z}$*

$$x \in A_\kappa \;\Leftrightarrow\; \text{there exists a bounded entire solution } \chi \text{ with } \chi_\kappa = x.$$

It is therefore uniquely determined.

Proof. For the implication ($\Rightarrow$) pick $\kappa \in \mathbf{Z}$ and $x \in A_\kappa$ arbitrarily. Then due to the φ-invariance of the pullback attractor $\mathcal{A}$, Proposition 2.1 provides the existence of an entire solution χ with $\chi_\kappa = x$ and $\chi_k \in A_k$ for each $k \in \mathbf{Z}$. Moreover, φ is bounded since the component sets of the pullback attractor are uniformly bounded.

For the converse implication ($\Leftarrow$), if there exists a bounded entire solution χ to equation (1), then the set of points $B_\chi := \{\chi_k : k \in \mathbf{Z}\}$ is bounded in X. Since $\mathcal{A}$ pullback attracts bounded subsets of X, it follows that

$$0 \leq \operatorname{dist}(\chi_k, A_k) \leq \lim_{n\to\infty} \operatorname{dist}(\varphi(k, k-n, B_\chi), A_k) = 0 \quad \text{for all } k \in \mathbf{Z},$$

so $\chi_k \in A_k$. □

3. Limitations

The limitations of pullback attraction are illustrated in this section through some examples, which show that pullback attractors do not capture the complete dynamics of nonautonomous systems defined through processes.

First consider the autonomous scalar difference equation

$$x_{k+1} = \frac{\lambda x_k}{1 + |x_k|} \tag{5}$$

depending on a real parameter $\lambda > 0$. Its zero solution $\bar{x} = 0$ exhibits a pitchfork bifurcation at $\lambda = 1$, and one obtains the following global behaviour (see Figure 1):

- If $\lambda \leq 1$, then $\bar{x} = 0$ is the only fixed point, it is globally asymptotically stable and is thus the global attractor of the autonomous dynamical system generated by the difference equation (5).

- If $\lambda > 1$, then there exist in addition two nontrivial fixed points $x_{\pm} := \pm(\lambda - 1)$. The trivial solution $\bar{x} = 0$ is an unstable steady state solution and the symmetric interval $A = [x_-, x_+]$ is the global attractor.

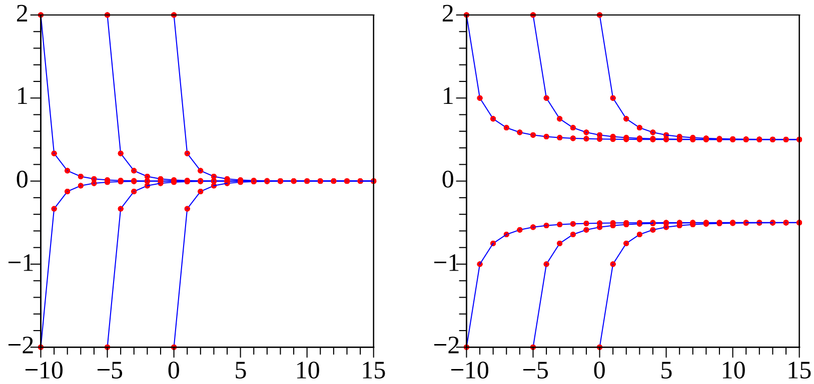

Figure 1. Trajectories of the autonomous difference equation (5) with $\lambda = 0.5$ (left) and $\lambda = 1.5$ (right)

3.1 Piecewise autonomous equation

Consider now the piecewise autonomous equation

$$x_{k+1} = \frac{\lambda_k x_k}{1 + |x_k|}, \qquad \lambda_k := \begin{cases} \lambda, & k \geq 0, \\ \lambda^{-1}, & k < 0 \end{cases} \tag{6}$$

for some $\lambda > 1$, which corresponds to a switch between the two autonomous problems (5) at $k = 0$.

Due to Proposition 2.5 the pullback attractor $\mathcal{A}$ of the resulting nonautonomous system has component sets $A_k \equiv \{0\}$ for all $k \in \mathbf{Z}$ corresponding to the zero entire solution. Note that the trivial fixed point $\bar{x} = 0$ is "asymptotically stable" for $k < 0$ and then "unstable" for $k \geq 0$. Moreover the interval $[x_-, x_+]$ is like a global attractor for the whole equation on $\mathbf{Z}$, but it is not really one since it is not invariant or minimal for $k < 0$.

The nonautonomous difference equation (6) is asymptotic autonomous in both directions, but the pullback attractor does not reflect the full limiting dynamics (see Figure 2 (left)), in particular in the forwards time direction.

If the λ_k do not switch from one constant to another, but increase monotonically, e.g. such as $\lambda_k = 1 + \frac{0.9k}{1+|k|}$, then the dynamics is similar, although the limiting dynamics is not so obvious from the equations. See Fig. 2 (left).

3.2 Fully nonautonomous equation

Now let $\{\lambda_k\}_{k\in\mathbf{Z}}$ be a monotonically increasing sequence with $\lim_{k\to\pm\infty} \lambda_k = \bar{\lambda}^{\pm 1}$ for $\bar{\lambda} > 1$. The nonautonomous problem

$$x_{k+1} = f_k(x_k) := \frac{\lambda_k x_k}{1 + |x_k|}. \tag{7}$$

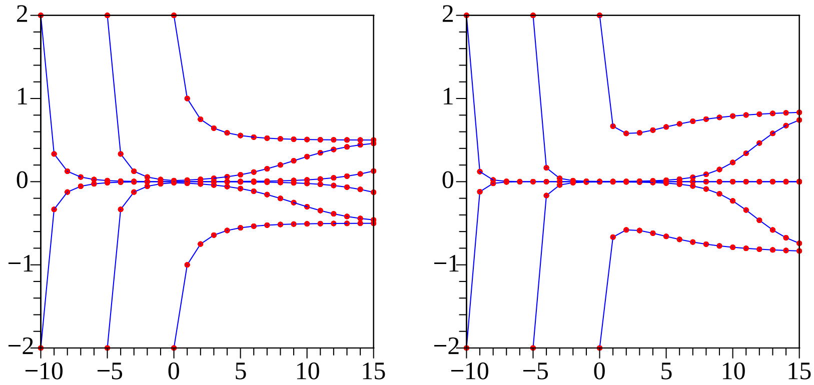

Figure 2. Trajectories of the piecewise autonomous difference equation (6) with $\lambda = 1.5$ (left) and the asymptotically autonomous difference equation (7) with $\lambda_k = 1 + \frac{0.9k}{1+|k|}$ (right)

is asymptotically autonomous in both directions with the limiting systems given in the previous subsection.

Its pullback attractor $\mathcal{A}$ has component sets $A_k \equiv \{0\}$ for all $k \in \mathbf{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution. As above, the trivial solution $\bar{x} = 0$ is "asymptotically stable" for $k < 0$ and then "unstable" for $k \geq 0$. However the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, both of which are not solutions at all. In particular, they are not entire solutions, so cannot belong to an attractor, forward or pullback, since these consist of entire solutions. See Figure 2 (right).

4. Resolution of the problem

The problem arises in part due to the use of the process formulation of the nonautonomous dynamical system. Pullback attraction alone does not characterize fully the bounded limiting behaviour of a nonautonomous system formulated as a process. In fact, something in addition like nonautonomous limit sets [6, 8], limiting equations [2] or asymptotic invariant sets [3] and eventual asymptotic stability [4] or a mixture of these ideas is needed to complete the picture. However, this varies from example to example and is somewhat ad hoc.

4.1 Resolution in terms of processes

The examples above show that a pullback attractor only reflects the limiting behaviour for $k \to -\infty$. A description for $k \to \infty$, however, is possible if one considers future attractors in addition to pullback attractors.

Here, a (global) future or forward attractor of (1) is given by an invariant nonautonomous set $\mathcal{A} \subset \mathbf{Z} \times X$ which attracts all uniformly bounded nonautonomous sets in forward time. More precisely, a forward attractor fulfills

$$\lim_{n\to\infty} \operatorname{dist}\left(\varphi(n,k,B), A_n\right) = 0$$

for all $k \in \mathbf{Z}$ and all bounded subsets B of X.

However, pullback and forward attractors are completely different objects. For instance, an analogue to Corollary 2.5 does not hold for future attractors. This follows from a modified version of Example 2.3.

Example 4.1 Consider the linear difference equation

$$x_{k+1} = a_k x_k, \qquad a_k := \begin{cases} \alpha_+, & k \geq 0, \\ \alpha_-, & k < 0, \end{cases}$$

generating the process

$$\varphi(k, \kappa, \xi) = \begin{cases} \alpha_+^{k-\kappa}\xi & \text{for } 0 \leq \kappa \leq k, \\ \alpha_+^k \alpha_-^{-\kappa}\xi & \text{for } \kappa \leq 0 \leq k, \\ \alpha_-^{k-\kappa}\xi & \text{for } \kappa \leq k \leq 0 \end{cases}$$

with $\alpha_-, \alpha_+ \in \mathbf{R}$ satisfying $0 < |\alpha_+| < 1 < |\alpha_-|$. It is easy to see that every entire solution of this system is uniformly bounded and a forward attractor.

For this reason, forward attractors are intrinsically nonunique, which makes it more difficult to deal with them. However, they are necessary to describe and understand the full dynamics of the system. A detailed discussion of the interplay between pullback and forward attractors can be found in [10].

The problem is also resolved if one considers *uniform attractors*. These are essentially pullback attractors (or forward attractors), where the attraction is uniform in time, i.e., the limit relation in (3) is uniform w.r.t. $k \in \mathbf{Z}$. Uniform attractors are both pullback and forward attractors and are able to describe attractivity for both the past and the future of the system.

4.2 Resolution of the problem through skew product flows

A more elegant way to resolve the problem is to consider the discrete skew product flow formalism of a nonautonomous dynamical system (cf. [11]). This includes an autonomous dynamical system as a driving mechanism, which is responsible for the temporal change in the dynamics of the nonautonomous difference equation. Moreover, it includes the dynamics of the asymptotically autonomous difference equations above and their limiting autonomous systems.

Definition 4.2. A *skew product flow* (θ, ϕ) is defined in terms of a *cocycle mapping* ϕ on a state space X, which is driven by an autonomous dynamical system θ acting on a base space P, where (P, d_P) is a metric space with metric d_P. Specifically, the *dynamical system* θ on P is a group of homeomorphisms $\{\theta_n\}_{n\in\mathbf{Z}}$ under composition on P with the properties that

- $\theta_0(p) = p$ for all $p \in P$,
- $\theta_{k+\ell}(p) = \theta_\ell(\theta_k(p))$ for all k, $\ell \in \mathbf{Z}$ and $p \in P$,
- the mapping $p \mapsto \theta_k(p)$ is continuous for each $k \in \mathbf{Z}$,

and the *cocycle mapping* $\phi : \mathbf{Z}_0^+ \times P \times X \to X$ satisfies

- $\phi(0, p, x) = x$ for all $p \in P$ and $x \in X$,
- $\phi(k + \ell, p, x) = \phi(\ell, \theta_k(p), \phi(k, p, x))$ for all k, $\ell \in \mathbf{Z}_0^+$, $p \in P$ and $x \in X$,
- the mapping $(p, x) \mapsto \phi(k, p, x)$ is continuous for each $k \in \mathbf{Z}$.

The word "skew" in skew product is due to the skew or triangular nature of the dependence of the components on the product space $P \times X$ with the driving system on the "base" space independent of the cocycle system on X, but affecting its evolution.

A family $\mathcal{A} = \{A_p : p \in P\}$ of compact subsets of X is said to be ϕ-invariant if

$$\phi(k, p, A_p) = A_{\theta_k(p)} \quad \text{for all } k \in \mathbf{Z}_0^+, p \in P.$$

An *entire solution* of a skew product flow is a sequence $\chi = \{\chi_p\}_{p\in P}$ such that

$$\phi(k, p, \chi_p) = \chi_{\theta_k(p)} \quad \text{for all } k \in \mathbf{Z}_0^+, p \in P,$$

and is thus an example of a ϕ-invariant family consisting of singleton sets.

Definition 4.3. A ϕ-invariant family $\mathcal{A} = \{A_p : p \in P\}$ of uniformly bounded compact subsets of X is called a pullback attractor of a skew product flow (θ, ϕ) on $P \times X$ if

$$\lim_{n\to\infty} \operatorname{dist}\left(\phi(n, \theta_{-n}(p), B), A_p\right) = 0 \quad \text{for all } p \in P \tag{8}$$

and all bounded subsets B of X.

The counterpart of the crucial Proposition 2.5 holds for skew product flows too, i.e., the pullback attractor consists of all the bounded entire solutions of the system.

A process φ admits a formulation as a skew product flow with $P = \mathbf{Z}$, the time shift $\theta_\ell(k) := k + \ell$ and the cocycle mapping $\phi(\ell, k, x) : = \varphi(k + \ell, k, x)$ for $\ell \geq 0$ and $x \in X$. Here P is a locally compact space.

The real advantage of the somewhat more complicated skew product flow formulation of nonautonomous dynamical systems occurs when P is compact. Then $\cup_{p\in P} A_p$ is precompact and its closure includes the possible limiting future behaviour of the system [6]. In fact,

$$\lim_{n\to\infty} \operatorname{dist}\left(\phi(n, p, x), \overline{\cup_{p\in P} A_p}\right) = 0 \quad \text{for all } p \in P,\ x \in X. \tag{9}$$

The nonautonomous difference equation* (7) *as a skew product flow

The nonautonomous difference equation (7) can be formulated as a skew product flow with the driving system defined in terms of the shift operator θ on the space of bi-infinite sequences

$$\Lambda_L = \{\lambda = \{\lambda_k\}_{k\in\mathbf{Z}} : \lambda_k \in [0, L] \quad \text{for all } k \in \mathbf{Z}\},$$

for some $L > 1$, which is a compact metric space with the metric

$$d_{\Lambda_L}\left(\lambda^{(1)}, \lambda^{(2)}\right) := \sum_{k\in\mathbf{Z}} 2^{-|k|} \left|\lambda_k^{(1)} - \lambda_k^{(2)}\right|.$$

This is coupled with a cocycle state space mapping with values $x_k = \phi(k, \lambda, x_0)$ generated by the difference equation (7) with a given coefficient sequence λ.

For the sequence λ used in (7), the limit of the shifted sequences $\theta_n(\lambda)$ in the above metric as $n \to \infty$ is λ_+^* with all components equal to $\bar{\lambda}$, while the limit as $n \to -\infty$ is λ_-^* with all components equal to $\bar{\lambda}^{-1}$.

The pullback attractor of the corresponding skew product flow (θ, ϕ) consists of compact subsets A_λ of $\mathbf{R}$ for each $\lambda \in \Lambda_L$. It is easy to see that $A_\lambda = \{0\}$ for any λ with components $\lambda_k < 1$ for $k \leq 0$, which includes the constant sequence λ^*_- as well as the switched sequence in (7). On the other hand, $A_{\lambda^*_+} = [-\bar{\lambda}, \bar{\lambda}]$. Here $\cup_{\lambda\in\Lambda_L} A_\lambda$ is precompact, so contains all future limiting dynamics.

The pullback attractor of the skew product flow includes that of the process for a given bi-infinite coefficient sequence, but also includes its forward asymptotic limits and much more. The coefficient sequence set Λ_L includes all possibilities, in fact, far more than may be of interest in particular situation. If one is interested in the dynamics of a process corresponding to a specific $\hat{\lambda} \in \Lambda_L$, then it would suffice to consider the skew product flow with respect to the driving system on the smaller space $\Lambda_{\hat{\lambda}}$ defined as the hull of this sequence, i.e., the set of accumulation points of the set $\{\theta_n \hat{\lambda} : n \in \mathbf{Z}\}$ in the metric space $(\Lambda_L, d_{\Lambda_L})$. In particular, if $\hat{\lambda}$ is the specific sequence in (7), then $\cup_{\lambda\in\Lambda_{\hat{\lambda}}} A_\lambda = A_{\lambda^*_+} = [-\bar{\lambda}, \bar{\lambda}]$ contains all future limiting dynamics, i.e.,

$$\lim_{n\to\infty} \text{dist}\left(\phi(n, \lambda, x), [-\bar{\lambda}, \bar{\lambda}]\right) = 0 \quad \text{for all } x \in X.$$

The example described by nonautonomous difference equation (7) is asymptotically autonomous with $\Lambda_\lambda = \{\lambda^*_\pm\} \cup \{\theta_n\lambda : n \in \mathbf{Z}\}$. The forward limit points $\pm(\bar{\lambda} - 1)$ of the process generated by (7), which were not steady states of the process, are now locally asymptotic steady states of the skew product flow with base space $P = \bar{\Lambda}$ consisting of the single constant sequence $\lambda_k \equiv \bar{\lambda}$, when the skew product system is interpreted as an autonomous semidynamical system on the product space $P \times X$.

More generally, the skew product flow formalism includes the asymptotically periodic case as well as more general forms of asymptotic recurrence.

5. Conclusion

The asymptotic behaviour or nonautonomous dynamical systems is considerably more complicated than that of autonomous dynamical systems [1, 6]. In the literature there are now various concepts of nonautonomous attractors including the pullback attractor, which is convenient since, unlike forward attraction, pullback attraction provides a means for actually constructing the components subsets of the pullback attraction. However, pullback attraction is essentially a concept of attraction for the past of the system and does not necessarily provide information about the future asymptotic behaviour of the system, which can be considerably more complicated, see the monographs [6, 8, 9] and the papers [5, 10] for more information.

The process formulation of a nonautonomous dynamical system is simpler and more intuitive than the skew product flow formulation. However, in itself, it often does not provide complete information about the future asymptotic behaviour of the system, as the above examples show. In contrast, this information is built into the skew product flow formulation when the state space P of the driving system is compact. Essentially, the skew product flow already includes the limiting dynamics and no further ad hoc methods are needed to determine it. The compactness of the space state P of the driving system is a restriction, but nevertheless holds for a very wide class of nonautonomous systems [7].

References

[1] J.K. Hale. *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.
[2] J. Kato, A.A. Martynyuk and A.A. Shestakov, *Stability of motion of nonautonomous systems (method of limiting equations)*. Gordon and Breach Publishers, Amsterdam, 1996.
[3] P.E. Kloeden, Asymptotic invariance and limit sets of general control systems, *J. Differential Equations* **19** (1975), 91–105.
[4] P.E. Kloeden, Eventual stability in general control systems, *J. Differential Equations* **19** (1975), 106–124.
[5] P.E. Kloeden, Pullback attractors in nonautonomous difference equations. *J. Difference Eqns. Applns.* **6**, (2000), 33–52.
[6] P.E. Kloeden and M. Rasmussen. *Nonautonomous Dynamical Systems.* Manuscript. Amer. Math. Soc., 2011.
[7] P.E. Kloeden and H.M. Rodrigues, Dynamics of a class of ODEs more general than almost periodic, *Nonlinear Analysis TMA* **74** (2011), 2695–2719.
[8] C. Pötzsche. *Geometric theory of discrete nonautonomous dynamical systems*, *Lect. Notes Math.* 2002, Springer, Berlin, 2010.
[9] M. Rasmussen. *Attractivity and Bifurcation for Nonautonomous Dynamical Systems*, *Lect. Notes Math.* 1907, Springer, Berlin, 2007.
[10] M. Rasmussen. Morse decompositions of nonautonomous dynamical systems, *Transactions Amer. Math. Soc.* **359** (2007), 5091–5115.
[11] R.J. Sacker and G.R. Sell, *Lifting properties in skew-product flows and applications to differential equations*, Memoirs of the AMS, vol. 190, AMS, Providence, RI, 1977.