Two perturbation results for semi-linear dynamic equations on measure chains

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Abstract In this note we investigate semi-linear parameter dependent dynamic equations on Banach spaces and provide sufficient criteria for them to possess exponentially bounded solutions in forward and backward time. Apart from classical stability theory, these results can be applied in the construction of non-autonomous invariant manifolds.

Keywords Time Scale, Measure Chain, Dynamic Equation, Perturbation

AMS Subject Classification 39A11, 39A12, 37C75

1 Introduction and preliminaries

It is well-known that the exponential stability of linear difference or differential equations is robust under sufficiently small perturbations. With regard to this, the present paper has two main goals:

- We can unify the corresponding results for ordinary differential equations (ODEs) and difference equations (OΔEs) within the calculus on measure chains or time scales (cf. [BP01, Hil88, Hil90]).
- Exponential stability is weakened to a certain exponential boundedness of solutions, namely the so-called *quasiboundedness* (see Definition 1).

However, the two results of this paper (Theorems 2 and 4) are originally and basically designed as tools to construct invariant manifolds using a Lyapunov-Perron technique, and an application on general measure chains can be found in [Pöt03], while ODEs and OΔEs are considered in, e.g., [AW96] and [Aul98, APS02], respectively. Indeed the Theorems 2 and 4 carry a certain technical amount in such situations. Related constructions for dynamic equations are provided in [Hil96, Theorem 3.1] and [Kel99, p. 41, Satz 3.2.7, pp. 42–43,

[1]Research supported by the "Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik" (GRK 283) financed by the Deutsche Forschungsgemeinschaft and the State of Bavaria.

Satz 3.2.8]. Both references consider linear regressive equations, while we allow nonlinear perturbations and try to avoid regressivity as far as possible. The general nonlinear case on homogeneous measure chains, i.e., when the graininess μ^* (cf. [Hil90, Section 2.3]) is constant, is treated in [Hil88, pp. 62–63, Satz 10.1, p. 71, Satz 10.3]. Apart from this, our approach has its roots in [Aul87], where finite-dimensional ODEs have been considered; for non-autonomous OΔEs see [Aul95].

Concerning our notation, $\mathbb{R}$ is the real field. Throughout this paper Banach spaces $\mathcal{X}, \mathcal{Y}$ are all real or complex and their norm is denoted by $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ is the Banach algebra of linear continuous endomorphisms on $\mathcal{X}$ and $I_{\mathcal{X}}$ the identity mapping on $\mathcal{X}$. We also introduce some notions which are specific to the calculus on measure chains. In all the subsequent considerations we deal with a measure chain $(\mathbb{T}, \preceq, \mu)$ unbounded above and below with bounded graininess μ^*, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and with $\tau \in \mathbb{T}$ we write $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}$, $\mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \preceq \tau\}$. $\chi_{\mathbb{T}_\tau^+} : \mathbb{T} \to \{0, 1\}$ is the characteristic function of $\mathbb{T}_\tau^+$. $\mathcal{C}_{rd}(I, \mathcal{L}(\mathcal{X}))$ denotes the rd-continuous and $\mathcal{C}_{rd}\mathcal{R}(I, \mathcal{L}(\mathcal{X}))$ the rd-continuous, regressive mappings from a $\mathbb{T}$-interval I into $\mathcal{L}(\mathcal{X})$. Recall that $\mathcal{C}_{rd}^+\mathcal{R}(I, \mathbb{R}) := \{a \in \mathcal{C}_{rd}\mathcal{R}(I, \mathbb{R}) : 1 + \mu^*(t)a(t) > 0 \text{ for } t \in I\}$ forms the so-called *positively regressive group* with respect to the addition $(a \oplus b)(t) := a(t) + b(t) + \mu^*(t)a(t)b(t)$ for $t \in I$ and $a, b \in \mathcal{C}_{rd}^+\mathcal{R}(I, \mathbb{R})$. An element $a \in \mathcal{C}_{rd}^+\mathcal{R}(I, \mathbb{R})$ is denoted as a *growth rate*, if $\sup_{t\in I} \mu^*(t)a(t) < \infty$ holds. Moreover we define the relations $a \lhd b :\Leftrightarrow 0 < \lfloor b - a \rfloor := \inf_{t\in I}(b(t) - a(t))$ and $e_a(t, s) \in \mathbb{R}$ stands for the real exponential function on $\mathbb{T}$ (cf. [Hil90, Section 7]).

Given $A \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, the *transition operator* $\Phi_A(t, \tau) \in \mathcal{L}(\mathcal{X})$, $\tau \preceq t$, of a linear dynamic equation $x^\Delta = A(t)x$ is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(\tau) = I_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$ and if A is regressive then $\Phi_A(t, \tau)$ is defined for all $\tau, t \in \mathbb{T}$. The partial derivative of $\Phi_A(t, \tau)$ with respect to the first variable is denoted by $\Delta_1\Phi_A(t, \tau)$.

To bring these preliminaries to an end we introduce the so-called *quasiboundedness* which is a handy notion describing exponential growth of functions. For a further motivation see [AW96, Section 3].

Definition 1. *For a growth rate* $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$, $\tau_0 \in \mathbb{T}$, *a Banach space* $\mathcal{X}$, *a* $\mathbb{T}$*-interval* I *and a rd-continuous function* $\lambda : I \to \mathcal{X}$ *we say that* λ *is*

(a) c^+-quasibounded *if* $I = \mathbb{T}_{\tau_0}^+$ *and* $\|\lambda\|_{\tau,c}^+ := \sup_{t\in\mathbb{T}_\tau^+} \|\lambda(t)\| \, e_{\ominus c}(t, \tau) < \infty$ *for* $\tau \in \mathbb{T}_{\tau_0}^+$,

(b) c^--quasibounded *if* $I = \mathbb{T}_{\tau_0}^-$ *and* $\|\lambda\|_{\tau,c}^- := \sup_{t\in\mathbb{T}_\tau^-} \|\lambda(t)\| \, e_{\ominus c}(t, \tau) < \infty$ *for* $\tau \in \mathbb{T}_{\tau_0}^-$,

(c) $c^\pm$-quasibounded *if* $I = \mathbb{T}$ *and* $\|\lambda\|_{\tau,c}^\pm := \sup_{t\in\mathbb{T}} \|\lambda(t)\| \, e_{\ominus c}(t, \tau) < \infty$ *for* $\tau \in \mathbb{T}$.

With $\mathcal{B}_{\tau,c}^+(\mathcal{X}), \mathcal{B}_{\tau,c}^-(\mathcal{X})$ *and* $\mathcal{B}_c^\pm(\mathcal{X})$ *we denote the sets of all* c^+-, c^-- *and* $c^\pm$-*quasibounded functions* $\lambda : I \to \mathcal{X}$, *respectively.*

Obviously the three sets $\mathcal{B}^+_{\tau,c}(\mathcal{X}), \mathcal{B}^-_{\tau,c}(\mathcal{X})$ and $\mathcal{B}^\pm_c(\mathcal{X})$ are non-empty and using [Hil90, Theorem 4.1(iii)] one can show that they define Banach spaces (cf. [Pöt02, p. 76, Lemma 1.4.3]).

2 Perturbation results

After the above preparations we can tackle the problem of the existence and uniqueness of quasibounded solutions in forward and backward time, respectively. We begin with dynamic equations where the linear part is not necessarily regressive. For difference equations see [Aul98, Lemma 3.3, Lemma 3.2] and [AW96, Lemma 3.2, Lemma 3.4] for Carathéodory differential equations.

Theorem 2. *Assume that $K_1 \geq 1$, $L_1, M \geq 0$, $a \in \mathcal{C}^+_{rd}\mathcal{R}(I,\mathbb{R})$ is a growth rate, I denotes some closed $\mathbb{T}$-interval, $\mathcal{X}$ is a Banach space and $\mathcal{P}$ a topological space satisfying the first axiom of countability. Let us consider the parameter dependent dynamic equation*

$$\boxed{x^\Delta = A(t)x + F(t,x,p) + f(t,p),} \tag{1}$$

where $A \in \mathcal{C}_{rd}(I, \mathcal{L}(\mathcal{X}))$ and $F : I \times \mathcal{X} \times \mathcal{P} \to \mathcal{X}$, $f : I \times \mathcal{P} \to \mathcal{X}$ are rd-continuous mappings satisfying

$$\|\Phi_A(t,s)\| \leq K_1 e_a(t,s) \quad \text{for } s,t \in I,\ s \preceq t, \tag{2}$$

$$F(t,0,p) = 0 \quad \text{for } t \in I,\ p \in \mathcal{P}, \tag{3}$$

$$\|F(t,x,p) - F(t,\bar{x},p)\| \leq L_1 \|x - \bar{x}\| \quad \text{for } t \in I,\ x,\bar{x} \in \mathcal{X},\ p \in \mathcal{P}. \tag{4}$$

Then for every growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(I,\mathbb{R})$, $a + K_1L_1 \lhd c$ and $\tau \in I$ we get the following:

(a) Supposed I is unbounded above and $f(\cdot,p) \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$, $p \in \mathcal{P}$, allows the estimate $\|f(\cdot,p)\|^+_{\tau,c} \leq M$ for $p \in \mathcal{P}$, then every solution $\nu(\cdot,p) : I \to \mathcal{X}$, $p \in \mathcal{P}$, of (1) is c^+-quasibounded with

$$\|\nu(\cdot,p)\|^+_{\tau,c} \leq K_1 \|\nu(\tau,p)\| + \frac{K_1 M}{\lfloor c - a - K_1 L_1 \rfloor} \quad \text{for } p \in \mathcal{P}. \tag{5}$$

Moreover, the mapping $\nu : I \times \mathcal{P} \to \mathcal{X}$ is continuous.

(b) Supposed I is unbounded below and $f(\cdot,p) \in \mathcal{B}^-_{\tau,c}(\mathcal{X})$, $p \in \mathcal{P}$, allows the estimate $\|f(\cdot,p)\|^-_{\tau,c} \leq M$ for $p \in \mathcal{P}$, then there exists exactly one c^--quasibounded solution $\nu_*(\cdot,p) : I \to \mathcal{X}$, $p \in \mathcal{P}$, of (1), which furthermore satisfies

$$\|\nu_*(\cdot,p)\|^-_{\tau,c} \leq \frac{K_1 M}{\lfloor c - a - K_1 L_1 \rfloor} \quad \text{for } p \in \mathcal{P}. \tag{6}$$

Moreover, the mapping $\nu_* : I \times \mathcal{P} \to \mathcal{X}$ is continuous.

(c) *Supposed $I = \mathbb{T}$ and $f(\cdot,p) \in \mathcal{B}_c^{\pm}(\mathcal{X})$, $p \in \mathcal{P}$, allows the estimate $\|f(\cdot,p)\|_{\tau,c}^{\pm} \leq M$ for $p \in \mathcal{P}$, then there exists exactly one $c^{\pm}$-quasi-bounded solution $\nu_*(\cdot,p) : \mathbb{T} \to \mathcal{X}$, $p \in \mathcal{P}$, of* (1), *which furthermore satisfies*

$$\|\nu_*(\cdot,p)\|_{\tau,c}^{\pm} \leq \frac{K_1 M}{\lfloor c - a - K_1 L_1 \rfloor} \quad \textit{for } p \in \mathcal{P}.$$

Moreover, the mapping $\nu_* : \mathbb{T} \times \mathcal{P} \to \mathcal{X}$ is continuous.

Remark 3. *A version of Theorem 2 for parameter independent dynamic equations, where the linear part $x^{\Delta} = A(t)x$ is allowed to possess an exponential dichotomy, can be found in [Pöt01, Theorem 3.4] (for assertion (a)) and [Pöt02, p. 111, Satz 2.2.12] (for assertion (c)). Hence we can weaken the assumption* (2) *and the corresponding remark applies to Theorem 4 also.*

Proof. Let $\tau \in \mathbb{T}$ and the growth rate $c \in \mathcal{C}_{rd}^{+}\mathcal{R}(I,\mathbb{R})$, $a + K_1 L_1 \lhd c$, be given arbitrarily. It is easy to see that the subsequently quoted results from [Hil88, Hil90] are valid in forward time ($t \in \mathbb{T}_\tau^+$) without assuming regressivity of the dynamic equation (1).
(a) All solutions of (1) starting at time $\tau \in \mathbb{T}$ exist throughout the $\mathbb{T}$-interval $\mathbb{T}_\tau^+$ according to [Hil90, Theorem 5.7]. If $\nu : \mathbb{T}_\tau^+ \times \mathcal{P} \to \mathcal{X}$ denotes such an arbitrary solution of (1) then the variation of constants formula (cf. [Hil90, Theorem 6.4(ii)]) yields

$$\nu(t,p) = \Phi_A(t,\tau)\nu(\tau,p) + \int_\tau^t \Phi_A(t,\sigma(s)) \left[F(s,\nu(s,p),p) + f(s,p)\right] \Delta s$$

for $t \in \mathbb{T}_\tau^+$, $p \in \mathcal{P}$ and with (2), (3) as well as (4) we obtain the estimate

$$\begin{aligned}
&\|\nu(t,p)\|\, e_{\ominus a}(t,\tau) \\
&\leq K_1 \|\nu(\tau,p)\| + K_1 \int_\tau^t e_a(\tau,\sigma(s))\, \|F(s,\nu(s,p),p) - F(s,0,p)\|\, \Delta s \\
&\quad + K_1 \int_\tau^t e_a(\tau,\sigma(s))\, \|f(s,p)\|\, \Delta s \\
&\leq K_1 \|\nu(\tau,p)\| + K_1 L_1 \int_\tau^t e_a(\tau,\sigma(s))\, \|\nu(s,p)\|\, \Delta s \\
&\quad + K_1 \int_\tau^t e_a(\tau,\sigma(s)) e_c(s,\tau)\, \|f(s,p)\|\, e_{\ominus c}(s,\tau)\, \Delta s \\
&\leq K_1 \|\nu(\tau,p)\| + \int_\tau^t \frac{K_1 L_1}{1 + \mu^*(s) a(s)} \|\nu(s,p)\|\, e_{\ominus a}(s,\tau)\, \Delta s \\
&\quad + K_1 M \int_\tau^t e_a(\tau,\sigma(s)) e_c(s,\tau)\, \Delta s
\end{aligned}$$

for $t \in \mathbb{T}_\tau^+$ and parameters $p \in \mathcal{P}$. Now Gronwall's Lemma (cf. [Hil88, p. 49, Satz 7.2]) leads to

$$
\begin{aligned}
&\|\nu(t,p)\| \, e_{\ominus a}(t,\tau) \\
&\le e_{a_0}(t,\tau)\left[K_1 \|\nu(\tau,p)\| + K_1 M \int_\tau^t e_{a+K_1L_1}(\tau,\sigma(s)) e_c(s,\tau)\,\Delta s\right] \\
&\le e_{a_0}(t,\tau)\left[K_1 \|\nu(\tau,p)\| + \frac{K_1 M}{\lfloor c-a-K_1L_1\rfloor}\left(e_{c\ominus(a+K_1L_1)}(t,\tau)-1\right)\right]
\end{aligned}
$$

for $t \in \mathbb{T}_\tau^+$, $p \in \mathcal{P}$, where we have abbreviated $a_0(t) := \frac{K_1L_1}{1+\mu^*(s)a(s)}$ and used [Pöt01, Lemma 3.1] to evaluate the integral. Finally, multiplying both sides of the above estimate by $e_{a\ominus c}(t,\tau) > 0$ we get

$$
\begin{aligned}
&\|\nu(t,p)\| \, e_{\ominus c}(t,\tau) \\
&\le K_1 e_{a\oplus a_0\ominus c}(t,\tau) \|\nu(\tau,p)\| + \frac{K_1 M}{\lfloor c-a-K_1L_1\rfloor}\left(1-e_{a_0\oplus a\ominus c}(t,\tau)\right) \\
&\le K_1 \|\nu(\tau,p)\| + \frac{K_1 M}{\lfloor c-a-K_1L_1\rfloor} \quad \text{for } t \in \mathbb{T}_\tau^+,\, p \in \mathcal{P}
\end{aligned}
$$

with the aid of [Hil90, Theorem 7.4]. This means that $\nu(\cdot,p) : \mathbb{T}_\tau^+ \to \mathcal{X}$, $p \in \mathcal{P}$, is c^+-quasibounded and satisfies (5). The continuity of $\nu : \mathbb{T}_\tau^+ \times \mathcal{P} \to \mathcal{X}$ follows from [Hil88, p. 51, Satz 7.4].
(b) We subdivide the proof of statement (b) into four steps.
Step 1 – Claim: *The zero solution of the linear homogeneous equation*

$$x^\Delta = A(t)x \tag{7}$$

is the only solution of (7) *in* $\mathcal{B}_{\tau,c}^-(\mathcal{X})$.
Because the system (7) evidently possesses an exponential dichotomy on $\mathbb{T}_\tau^-$ with the invariant projector $P(t) \equiv I_\mathcal{X}$ by (2), we can apply [Pöt01, Corollary 2.11(b)] to prove that any c^--quasibounded solution $\nu : \mathbb{T}_\tau^- \to \mathcal{X}$ of (7) vanishes identically.
Step 2 – Claim: *There exists exactly one c^--quasibounded solution $\nu_*(\cdot,p)$: $\mathbb{T}_\tau^- \to \mathcal{X}$, $p \in \mathcal{P}$, of the linear inhomogeneous equation*

$$x^\Delta = A(t)x + f(t,p), \tag{8}$$

which furthermore satisfies

$$\|\nu_*(\cdot,p)\|_{\tau,c}^- \le \frac{K_1 M}{\lfloor c-a\rfloor} \quad \textit{for } p \in \mathcal{P}. \tag{9}$$

Above all the function $\nu_* : \mathbb{T}_\tau^- \times \mathcal{P} \to \mathcal{X}$, $\nu_*(t,p) := \int_{-\infty}^t \Phi_A(t,\sigma(s)) f(s,p)\,\Delta s$ is well-defined, since the integrand is rd-continuous (in s) and the estimate

$$
\begin{aligned}
&\|\nu_*(t,p)\| \, e_{\ominus c}(t,\tau) \le \int_{-\infty}^t \|\Phi_A(t,\sigma(s))\| \, \|f(s,p)\| \, \Delta s e_{\ominus c}(t,\tau) \\
&\overset{(2)}{\le} K_1 \int_{-\infty}^t e_a(t,\sigma(s)) e_c(s,\tau)\,\Delta s e_{\ominus c}(t,\tau) \, \|f(\cdot,p)\|_{\tau,c}^+ \le \frac{K_1 M}{\lfloor c-a\rfloor}
\end{aligned}
$$

for $t \in \mathbb{T}_\tau^-$ holds true, where the improper integral has been evaluated using [Pöt01, Lemma 3.1]. Additionally the inclusion $\nu_*(\cdot,p) \in \mathcal{B}_{\tau,c}^-(\mathcal{X})$, $p \in \mathcal{P}$, yields by passing over to the least upper bound over $t \in \mathbb{T}_\tau^-$. The derivative of ν_* with respect to $t \in \mathbb{T}_\tau^-$ is given by

$$\nu_*^\Delta(t,p) \equiv f(t,p) + \int_{-\infty}^{t} \Delta_1\Phi_A(t,\sigma(s))f(s,p)\,\Delta s \equiv A(t)\nu_*(t,p) + f(t,p)$$

on $\mathbb{T}_\tau^-$, and the integral has been differentiated using a result dual to [Pöt01, Lemma 4.2]. Therefore $\nu_*(\cdot,p)$, $p \in \mathcal{P}$, is a c^--quasibounded solution of (8) satisfying the estimate (9). Finally the uniqueness statement immediately results from Step 1, because the difference of two c^--quasibounded solutions of (8) is a c^--quasibounded solution of (7) and consequently identically vanishing. In order to prove the continuity of the mapping $\nu_* : \mathbb{T}_\tau^- \times \mathcal{P} \to \mathcal{X}$, let the pair $(t_0,p_0) \in \mathbb{T}_\tau^- \times \mathcal{P}$ be arbitrarily fixed and consider the alternative representation

$$\nu_*(t,p) = \int_{-\infty}^{\tau} \chi_{\mathbb{T}_t^-}(s)\Phi_A(t,\sigma(s))f(s,p)\,\Delta s \quad \text{for } t \in \mathbb{T}_\tau^-,\ p \in \mathcal{P}.$$

As $(t,p) \to (t_0,p_0)$ the integrand converges to $\chi_{\mathbb{T}_{t_0}^-}(s)\Phi_A(t_0,\sigma(s))f(s,p_0)$ for all $s \in \mathbb{T}_\tau^-$ and the inequality

$$\begin{aligned}\left\|\chi_{\mathbb{T}_t^-}(s)\Phi_A(t,\sigma(s))f(s,p)\right\| &\overset{(2)}{\leq} K_1 e_a(t,\sigma(s))e_c(s,\tau)\,\|f(\cdot,p_0)\|_{\tau,c}^- \\ &\leq K_1 M e_a(t,\sigma(s))e_c(s,\tau) \quad \text{for } \sigma(s) \preceq t\end{aligned} \tag{10}$$

is valid. Because of $a \lhd c$ we can use [Pöt01, Lemma 3.1] to evaluate the integral from $-\infty$ to τ over the right hand side of the estimate (10), and we may apply Lebesgue's dominated convergence theorem (cf. [Nei01, p. 161, Satz 313]) to get the convergence $\lim_{(t,p)\to(t_0,p_0)} \nu_*(t,p) = \nu_*(t_0,p_0)$, which proves the desired continuity of $\nu_* : \mathbb{T}_\tau^- \times \mathcal{P} \to \mathcal{X}$.

Step 3 – Claim: *There exists exactly one c^--quasibounded solution $\nu_*(\cdot,p) : \mathbb{T}_\tau^- \to \mathcal{X}$, $p \in \mathcal{P}$, of the semi-linear equation* (1), *which moreover satisfies* (6). In order to set up the framework of Banach's fixed point theorem we define the sets

$$\mathcal{B} := \left\{ \nu : \mathbb{T}_\tau^- \times \mathcal{P} \to \mathcal{X} \;\middle|\; \begin{array}{c} \nu : \mathbb{T}_\tau^- \times \mathcal{P} \to \mathcal{X} \text{ is continuous,} \\ \nu(\cdot,p) \in \mathcal{B}_{\tau,c}^-(\mathcal{X}) \text{ for all } p \in \mathcal{P}, \\ \sup_{p\in\mathcal{P}} \|\nu(\cdot,p)\|_{\tau,c}^- < \infty \end{array} \right\},$$

which are readily seen to be Banach spaces equipped with the norm $\|\nu\|_{\tau,c}^{-,0} := \sup_{p\in\mathcal{P}} \|\nu(\cdot,p)\|_{\tau,c}^-$. As to the construction of an appropriate contraction operator on $\mathcal{B}$ we choose any $\nu \in \mathcal{B}$ and consider the linear inhomogeneous dynamic equation

$$x^\Delta = A(t)x + F(t,\nu(t,p),p) + f(t,p). \tag{11}$$

Since (3) and (4) imply the estimate

$$\begin{aligned} &\|F(t,\nu(t,p),p)+f(t,p)\|\, e_{\ominus c}(t,\tau) \\ &\le L_1 \|\nu(t,p)\|\, e_{\ominus c}(t,\tau)+\|f(t,p)\|\, e_{\ominus c}(t,\tau) \\ &\le L_1 \|\nu(\cdot,p)\|^-_{\tau,c}+\|f(\cdot,p)\|^-_{\tau,c} \le L_1 \|\nu\|^{-,0}_{\tau,c}+M \quad \text{for } t\in\mathbb{T}^-_\tau,\, p\in\mathcal{P}, \end{aligned} \tag{12}$$

we may apply Step 2 of the present proof to equation (11). Hence there exists a continuous mapping $\nu_*:\mathbb{T}^-_\tau\times\mathcal{P}\to\mathcal{X}$ such that the function $\nu_*(\cdot,p)$ is the unique c^--quasibounded solution of (11) to the parameter value $p\in\mathcal{P}$, and for arbitrary $p\in\mathcal{P}$ we get the estimate

$$\|\nu_*(\cdot,p)\|^-_{\tau,c} \overset{(9)}{\le} \frac{K_1}{\lfloor c-a\rfloor}\left(L_1\|\nu\|^{-,0}_{\tau,c}+M\right) \quad \text{for } p\in\mathcal{P},$$

which shows the inclusion $\nu_*\in\mathcal{B}$. Now we are defining the designated contraction mapping $\mathcal{T}:\mathcal{B}\to\mathcal{B}$, $\nu\mapsto\nu_*$. Then (12) implies the estimate

$$\|(\mathcal{T}\nu)(\cdot,p)\|^-_{\tau,c} \le \frac{K_1}{\lfloor c-a\rfloor}\left[L_1\|\nu(\cdot,p)\|^-_{\tau,c}+\|f(\cdot,p)\|^-_{\tau,c}\right] \tag{13}$$

for $p\in\mathcal{P}$, $\nu\in\mathcal{B}$. Now we claim that the mapping $\mathcal{T}:\mathcal{B}\to\mathcal{B}$ actually is a contraction. In order to verify this, let $\nu,\bar\nu\in\mathcal{B}$ be arbitrary. Then the difference $(\mathcal{T}\nu-\mathcal{T}\bar\nu)(\cdot,p)$, $p\in\mathcal{P}$, is a c^--quasibounded solution of the linear inhomogeneous system

$$x^\Delta = A(t)x+F(t,\nu(t,p),p)-F(t,\bar\nu(t,p),p) \tag{14}$$

for every parameter $p\in\mathcal{P}$ and similar to equation (11) also (14) satisfies all the assumptions of Step 2. Consequently, (4) implies

$$\|(\mathcal{T}\nu-\mathcal{T}\bar\nu)(\cdot,p)\|^-_{\tau,c} \le \frac{K_1L_1}{\lfloor c-a\rfloor}\|(\nu-\bar\nu)(\cdot,p)\|^-_{\tau,c} \le \frac{K_1L_1}{\lfloor c-a\rfloor}\|\nu-\bar\nu\|^{0,-}_{\tau,c}$$

for $p\in\mathcal{P}$, and we therefore get $\|\mathcal{T}\nu-\mathcal{T}\bar\nu\|^{0,-}_{\tau,c}\le\frac{K_1L_1}{\lfloor c-a\rfloor}\|\nu-\bar\nu\|^{0,-}_{\tau,c}$. According to the assumptions we have $a+K_1L_1\lhd c$, which is sufficient for $0\le\frac{K_1L_1}{\lfloor c-a\rfloor}<1$. Thus $\mathcal{T}$ is a contraction and Banach's fixed point theorem implies a unique fixed point $\bar\nu_*\in\mathcal{B}$. Applying this fixed point argument to the dynamic equation (1), it is immediately seen that $\bar\nu_*$ is a c^--quasibounded solution of (1). Because of $\bar\nu_*\in\mathcal{B}$ the mapping $\bar\nu_*:\mathbb{T}^-_\tau\times\mathcal{P}\to\mathcal{X}$ is continuous. In order to conclude the proof of Step 3 we only have to verify the estimate (6). Since $\bar\nu_*$ is a fixed point of $\mathcal{T}$ together with (13) yields $\|(\mathcal{T}\nu)(\cdot,p)\|^-_{\tau,c}\le\frac{K_1L_1}{\lfloor c-a\rfloor}\|\nu(\cdot,p)\|^-_{\tau,c}+\frac{K_1L_1}{\lfloor c-a\rfloor}\|f(\cdot,p)\|^-_{\tau,c}$ for $p\in\mathcal{P}$ and from this we get (6).

Step 4: For an arbitrary $\mathbb{T}$-interval I which is unbounded below, Step 3 guarantees the unique existence of the c^--quasibounded solution $\nu_\tau(\cdot,p):\mathbb{T}^-_\tau\to\mathcal{X}$, $p\in\mathcal{P}$, of (1) for every $\tau\in I$ and ν_τ is continuous. Defining $\nu_*:I\times\mathcal{P}\to\mathcal{X}$ as $\nu_*(t,p):=\nu_t(t,p)$ it is easy to see that ν_* is continuous and

has all the properties claimed in Theorem 2(b). Just note that for all $p \in \mathcal{P}$ and $\tau_1, \tau_2, t \in \mathbb{T}$, $\tau_1 \preceq \tau_2 \preceq t$ we have $\nu_{\tau_1}(t,p) = \nu_{\tau_2}(t,p)$.
(c) The proof of statement (c) follows along the lines of the Steps 1 to 3 above. One only has to replace the $\mathbb{T}$-interval $\mathbb{T}_\tau^-$ by $\mathbb{T}$ and the c^--quasiboundedness by $c^\pm$-quasibounded functions. □

Now we study dynamic equations where the unperturbed system is assumed to possess quasibounded solutions in backward time. Hence we have to make the hypothesis of its regressivity. Additionally a smallness condition on the Lipschitz constant for the nonlinear perturbation is involved. The next result is similar to [Aul98, Lemma 3.5, Lemma 3.4] and [AW96, Lemma 3.6, Lemma 3.7].

Theorem 4. *Assume that $K_2 \geq 1$, $L_2, M \geq 0$, $b \in \mathcal{C}_{rd}^+\mathcal{R}(I,\mathbb{R})$ is a growth rate, I denotes some closed $\mathbb{T}$-interval, $\mathcal{Y}$ is a Banach space and $\mathcal{P}$ a topological space satisfying the first axiom of countability. Let us consider the parameter dependent dynamic equation*

$$\boxed{y^\Delta = B(t)y + G(t,y,p) + g(t,p),} \tag{15}$$

where $B \in \mathcal{C}_{rd}\mathcal{R}(I, \mathcal{L}(\mathcal{Y}))$ and $G : I \times \mathcal{Y} \times \mathcal{P} \to \mathcal{Y}$, $g : I \times \mathcal{P} \to \mathcal{Y}$ are rd-continuous mappings satisfying

$$\|\Phi_B(t,s)\| \leq K_2 e_b(t,s) \quad \text{for } s,t \in I,\ t \preceq s, \tag{16}$$
$$G(t,0,p) = 0 \quad \text{for } t \in I,\ p \in \mathcal{P},$$
$$\|G(t,y,p) - G(t,\bar{y},p)\| \leq L_2 \|y - \bar{y}\| \quad \text{for } t \in I,\ y,\bar{y} \in \mathcal{Y},\ p \in \mathcal{P}. \tag{17}$$

Then for every growth rate $d \in \mathcal{C}_{rd}^+\mathcal{R}(I,\mathbb{R})$, $d \lhd b - K_2L_2$ and $\tau \in I$ we get the following:

(a) Supposed I is unbounded below,

$$\mu^*(t)\,[K_2L_2 - b(t)] < 1 \quad \text{for } t \in I \tag{18}$$

and $g(\cdot,p) \in \mathcal{B}_{\tau,d}^-(\mathcal{Y})$, $p \in \mathcal{P}$, allows the estimate $\|g(\cdot,p)\|_{\tau,d}^- \leq M$ for $p \in \mathcal{P}$, then every solution $\upsilon(\cdot,p) : I \to \mathcal{Y}$, $p \in \mathcal{P}$, of (15) is d^--quasibounded with

$$\|\upsilon(\cdot,p)\|_{\tau,d}^- \leq K_2 \|\upsilon(\tau,p)\| + \frac{K_2M}{\lfloor b - d + K_2L_2 \rfloor} \quad \text{for } p \in \mathcal{P}.$$

Moreover, the mapping $\upsilon : I \times \mathcal{P} \to \mathcal{Y}$ is continuous.

(b) Supposed I is unbounded above and $g(\cdot,p) \in \mathcal{B}_{\tau,d}^+(\mathcal{Y})$, $p \in \mathcal{P}$, allows the estimate $\|g(\cdot,p)\|_{\tau,d}^+ \leq M$ for $p \in \mathcal{P}$, then there exists exactly one d^+-quasibounded solution $\upsilon_*(\cdot,p) : I \to \mathcal{Y}$, $p \in \mathcal{P}$, of (15), which furthermore satisfies

$$\|\upsilon_*(\cdot,p)\|_{\tau,d}^+ \leq \frac{K_2M}{\lfloor b - d + K_2L_2 \rfloor} \quad \text{for } p \in \mathcal{P}.$$

Moreover, the mapping $\upsilon_* : I \times \mathcal{P} \to \mathcal{Y}$ is continuous.

(c) Supposed $I = \mathbb{T}$ and $g(\cdot, p) \in \mathcal{B}_d^{\pm}(\mathcal{Y})$, $p \in \mathcal{P}$, allows the estimate $\|g(\cdot, p)\|_{\tau,d}^{\pm} \leq M$ for $p \in \mathcal{P}$, then there exists exactly one $d^{\pm}$-quasibounded solution $\upsilon_*(\cdot, p) : \mathbb{T} \to \mathcal{Y}$, $p \in \mathcal{P}$, of (15), which furthermore satisfies

$$\|\upsilon_*(\cdot, p)\|_{\tau,d}^{\pm} \leq \frac{K_2 M}{\lfloor b - d + K_2 L_2 \rfloor} \quad \textit{for } p \in \mathcal{P}.$$

Moreover, the mapping $\upsilon_* : \mathbb{T} \times \mathcal{P} \to \mathcal{Y}$ is continuous.

Proof. Since the argumentation is dual to Theorem 2 we only give a very rough sketch of the proof. To begin, we prove that the right-hand side of (15) is regressive under the assumption (18). To this end let $t \in I$ be arbitrary. According to the assumptions, the coefficient operator B is regressive and due to the estimates (17) and

$$\left\| \left[I_{\mathcal{Y}} + \mu^*(t) B(t) \right]^{-1} \right\| = \|\Phi_B(t, \sigma(t))\| \stackrel{(16)}{\leq} K_2 e_b(t, \sigma(t)) = \frac{K_2}{1 + \mu^*(t) b(t)}$$

for $t \in I$, we obtain that $I_{\mathcal{Y}} + \mu^*(t)B(t) + \mu^*(t)G(t, \cdot, p) + g(t, p) : \mathcal{Y} \to \mathcal{Y}$ is a bijective mapping (cf. (18) and [Aul98, Corollary 6.2]). Therefore equation (15) is regressive and consequently solutions exist and are unique in backward time ($t \in \mathbb{T}_\tau^-$).
(a) This part of the proof resembles the proof of Theorem 2(a). Hereby growth rates occurring in the Gronwall estimates are positively regressive because of the assumption (18).
(b) One also proceeds in four steps. Above all the zero solution is the unique d^+-quasibounded solution of $y^\Delta = B(t)y$, which follows from [Pöt01, Corollary 2.11(a)], since the above equation trivially possesses an exponential dichotomy on $\mathbb{T}_\tau^+$ with the invariant projector $P(t) \equiv 0$. We define $\upsilon_* : \mathbb{T}_\tau^+ \times \mathcal{P} \to \mathcal{Y}$, $\upsilon_*(t, p) := -\int_t^\infty \Phi_B(t, \sigma(s)) g(s, p) \, \Delta s$ as the unique d^+-quasibounded solution of $y^\Delta = A(t)x + g(t, p)$. The well-definedness and the solution property can be verified using [Pöt01, Lemma 3.2] and [Pöt01, Lemma 4.2], respectively. Finally a fixed point argument similar to Step 3 gives us the general assertion.
(c) One can verify the statement (c) along the lines of (b). Here one has to replace the $\mathbb{T}$-interval $\mathbb{T}_\tau^+$ by $\mathbb{T}$ and the d^+-quasiboundedness by $d^{\pm}$-quasibounded functions. □

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