Delay Equations on Measure Chains: Basics and Linearized Stability

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Abstract We introduce the notion of a dynamic delay equation, which includes differential and difference equations with possibly time-dependent backward delays. After proving a basic global existence and uniqueness theorem for appropriate initial value problems, we derive a criterion for the asymptotic stability of such equations in case of bounded delays.

Keywords Dynamic delay equation, Stability, Time scale, Measure chain

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1 Introduction and Preliminaries

In this paper we briefly introduce dynamic equations on measure chains (or time scales), where time-dependent backward delays are present. Our approach provides a framework sufficiently flexible to include ordinary differential and difference equations without delays ($\dot{x}(t) = F(t, x(t))$ for $t \in \mathbb{R}$ and $\Delta x(t) = F(t, x(t))$ for $t \in \mathbb{Z}$, resp.), equations with constant delays ($\dot{x}(t) = F(t, x(t), x(t-r))$ for $t \in \mathbb{R}$, $r > 0$, and $\Delta x(t) = F(t, x(t), x(t-r))$ for $t, r \in \mathbb{Z}$, $r > 0$, resp.), as well as equations with proportional delays, like, e.g., the *pantograph equation* $\dot{x}(t) = A(t)x(t) + B(t)x(qt)$, $q \in (0, 1)$.

We prove an existence and uniqueness theorem for initial value problems of such equations under global Lipschitz conditions, which basically extends [Hil90, Section 5], who considers equations without delays. Section 3 contains sufficient conditions for the exponential decay of solutions for semi-linear equations and bounded delays. On this occasion, the delay term is interpreted as a perturbation of a linear delay-free dynamic equation, since we avoid the use of a general variation of constants formula for linear delay equations.

From now on, $\mathbb{Z}$ stands for the integers, $\mathbb{R}$ for the reals and $\mathbb{R}_+$ for the nonnegative real numbers. Throughout this paper, Banach spaces $\mathcal{X}$ are all

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real or complex and their norm is denoted by $\|\cdot\|$. The closed ball in $\mathcal{X}$ with center 0 and radius $r > 0$ is given by $\bar{B}_r := \{x \in \mathcal{X} : \|x\| \le r\}$. If I is a topological space, then $\mathcal{C}(I, \mathcal{X})$ are the continuous functions between I and $\mathcal{X}$. Finally, we write $D_{(2,3)}f$ for the partial Fréchet derivative of a mapping $f : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ w.r.t. the variables in $\mathcal{X} \times \mathcal{X}$, provided it exists.

We also sketch the basic terminology from the calculus on measure chains (cf. [Hil90, BP01]). In all the subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [Hil90, Axiom 2]) with growth calibration $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ (see [Hil90, Axiom 3]). The most intuitive and relevant examples of measure chains are *time scales*, where $\mathbb{T}$ is a canonically ordered closed subset of $\mathbb{R}$ and μ is given by $\mu(t, s) = t - s$. Continuing, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and the *graininess* $\mu^* : \mathbb{T} \to \mathbb{R}$ is defined by $\mu^*(t) := \mu(\sigma(t), t)$. If $\mathbb{T}$ has a left-scattered maximum m, we set $\mathbb{T}^\kappa := \mathbb{T} \setminus \{m\}$ and $\mathbb{T}^\kappa := \mathbb{T}$ otherwise. For $\tau, t \in \mathbb{T}$ we abbreviate $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}$, $\mathbb{T}_\tau^- := \{s \in \mathbb{T},\ s \preceq \tau\}$ and $[\tau, t]_{\mathbb{T}} := \{s \in \mathbb{T} : \tau \preceq s \preceq t\}$. Any other notation concerning measure chains is taken from [Hil90].

2 Dynamic Delay Equations

Let $\theta : \mathbb{T}^\kappa \to \mathbb{T}$ be a nondecreasing function satisfying $\theta(t) \preceq t$ for all $t \in \mathbb{T}^\kappa$. Then we denote θ as *delay function* and say an equation of the form

$$\boxed{x^\Delta(t) = F\big(t, x(t), x(\theta(t))\big)} \qquad (1)_F$$

is a *dynamic delay equation* with *right-hand side* $F : \mathbb{T}^\kappa \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. With given $\tau \in \mathbb{T}$, we abbreviate $\mathcal{C}_\tau(\theta) := \mathcal{C}([\theta(\tau), \tau]_{\mathbb{T}}, \mathcal{X})$. For $\phi_\tau \in \mathcal{C}_\tau(\theta)$, a continuous function $\nu : I \to \mathcal{X}$ is said to solve the *initial value problem* (IVP)

$$x^\Delta(t) = F\big(t, x(t), x(\theta(t))\big), \qquad (\tau, \phi_\tau), \qquad (2)$$

if I is a $\mathbb{T}$-interval with $[\theta(\tau), \tau]_{\mathbb{T}} \subseteq I$, $\nu(t) = \phi_\tau(t)$ for all $t \in [\theta(\tau), \tau]_{\mathbb{T}}$ and $\nu^\Delta(t) = F\big(t, \nu(t), \nu(\theta(t))\big)$ for $t \in I$, $\tau \preceq t$ holds, where $\nu^\Delta(\tau) \in \mathcal{X}$ is understood as right-sided derivative of ν in case of a right-dense $\tau \in \mathbb{T}$. Any solution satisfying the IVP (2) will be denoted by $\varphi(\cdot; \tau, \phi_\tau)$.

A tool solely important for the proof of Theorem 2.4 is given by means of the mapping $F^{\tau]} : \mathbb{T}_\tau^- \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, which is defined for a fixed $\tau \in \mathbb{T}^\kappa$ by

$$F^{\tau]}(t, x, y) := \begin{cases} F(t, x, y) & \text{for } t \prec \tau,\ (x, y) \in \mathcal{X} \times \mathcal{X} \\ \lim\limits_{\substack{(s,\xi,\eta) \to (\tau,x,y) \\ s \prec \tau}} F(s, \xi, \eta) & \text{for } t = \tau,\ (x, y) \in \mathcal{X} \times \mathcal{X} \end{cases}.$$

Lemma 2.1. *Suppose $\theta : \mathbb{T}^\kappa \to \mathbb{T}$ is a continuous delay function, let I be a $\mathbb{T}$-interval, $\tau, r \in I$ with $\tau \preceq r$ and define $I_\tau := [\theta(\tau), \tau]_{\mathbb{T}} \cup I$. Then a function $\nu : I_\tau \to \mathcal{X}$ is a (unique) solution of the IVP* (2), *if and only if*

(i) $\nu_1 := \nu|_{\mathbb{T}_r^- \cap I_\tau}$ *is a (unique) solution of the IVP* $(1)_{F^{r]}}$, (τ, ϕ_τ),

(ii) $\nu|_{\mathbb{T}_r^+ \cap I_\tau}$ *is a (unique) solution of the IVP* $(1)_F$, $(r, \nu_1|_{[\theta(r),r]_{\mathbb{T}}})$.

Proof. The proof is similar to [Hil90, Theorem 5.3] and omitted here. $\square$

Lemma 2.2. *Suppose* $\theta : \mathbb{T}^\kappa \to \mathbb{T}$ *is a continuous delay function and define* $I := [a,b]_{\mathbb{T}}$ *for* $a, b \in \mathbb{T}$, $a \prec b$. *Moreover, let* $\ell : I \to \mathbb{R}_+$ *be rd-continuous,*

$$\int_a^b \ell(s)\,\Delta s < 1 \tag{3}$$

and assume the rd-continuous mapping $F : \mathbb{T}^\kappa \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *satisfies*

$$\|F(t,x,y) - F(t,\bar{x},\bar{y})\| \le \ell(t) \left\| \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\| \quad \text{for all } x, y, \bar{x}, \bar{y} \in \mathcal{X} \tag{4}$$

and $t \in I$. *Then, for any* $\tau \in I$ *and any* $\phi_\tau \in \mathcal{C}_\tau(\theta)$, *the IVP* (2) *possesses exactly one solution* $\nu : [\theta(\tau), \tau]_{\mathbb{T}} \cup I \to \mathcal{X}$.

Proof. Let $\tau \in I$ and $\phi_\tau \in \mathcal{C}_\tau(\theta)$. We define the $\mathbb{T}$-interval $I_\tau := I \cup [\theta(\tau), \tau]_{\mathbb{T}}$ and $\mathcal{C}(I_\tau, \mathcal{X})$ is complete w.r.t. the norm $\|\nu\|_{\mathcal{C}(I_\tau,\mathcal{X})} := \max_{t \in I_\tau} \|\nu(t)\|$. Now consider the operator $\mathcal{T}_\tau : \mathcal{C}(I_\tau, \mathcal{X}) \to \mathcal{C}(I_\tau, \mathcal{X})$,

$$\mathcal{T}_\tau(\nu)(t) := \begin{cases} \phi_\tau(t) & \text{for } \theta(\tau) \preceq t \prec \tau \\ \phi_\tau(\tau) + \int_\tau^t F\big(s, \nu(s), \nu(\theta(s))\big)\,\Delta s & \text{for } \tau \preceq t \end{cases}, \tag{5}$$

which is well-defined due to [Hil90, Theorem 4.4]. Then $\nu \in \mathcal{C}(I_\tau, \mathcal{X})$ is a fixed point of $\mathcal{T}_\tau$, if and only if ν solves the IVP (2).

Because of [Hil90, Theorem 4.3(iii)], and for $\nu, \bar{\nu} \in \mathcal{C}(I_\tau, \mathcal{X})$, one obtains

$$\begin{aligned} \|\mathcal{T}_\tau(\nu)(t) - \mathcal{T}_\tau(\bar{\nu})(t)\| &\overset{(5)}{\le} \int_\tau^t \left\|F\big(s, \nu(s), \nu(\theta(s))\big) - F\big(s, \bar{\nu}(s), \bar{\nu}(\theta(s))\big)\right\| \Delta s \\ &\overset{(4)}{\le} \int_\tau^t \ell(s) \left\| \begin{pmatrix} \nu(s) - \bar{\nu}(s) \\ \nu(\theta(s)) - \bar{\nu}(\theta(s)) \end{pmatrix} \right\| \Delta s \\ &\le \int_a^b \ell(s)\,\Delta s\, \|\nu - \bar{\nu}\|_{\mathcal{C}(I_\tau,\mathcal{X})} \quad \text{for all } t \in I_\tau,\ \tau \preceq t \end{aligned}$$

and by passing over to the least upper bound for $t \in I_\tau$, we get

$$\|\mathcal{T}_\tau(\nu) - \mathcal{T}_\tau(\bar{\nu})\|_{\mathcal{C}(I_\tau,\mathcal{X})} \le \int_a^b \ell(s)\,\Delta s\, \|\nu - \bar{\nu}\|_{\mathcal{C}(I_\tau,\mathcal{X})}.$$

Using (3), we know that $\mathcal{T}_\tau$ is a contraction on $\mathcal{C}(I_\tau, \mathcal{X})$ and the contraction mapping principle yields that $\mathcal{T}_\tau$ possesses exactly one fixed point ν. $\square$

To show, e.g., the continuous dependence of solutions on the initial functions, we need a generalized version of Gronwall's inequality.

Lemma 2.3. *Let $\tau \in \mathbb{T}$, suppose $\theta : \mathbb{T}^\kappa \to \mathbb{T}$ is a continuous delay function, $C \geq 0$ and $b_1, b_2 : \mathbb{T}_\tau^+ \to \mathbb{R}_+$, $y : \mathbb{T}_{\theta(\tau)}^+ \to \mathbb{R}_+$ are rd-continuous. Then*

$$y(t) \leq C + \int_\tau^t b_1(s)y(s)\,\Delta s + \int_\tau^t b_2(s)y(\theta(s))\,\Delta s \quad \text{for all } t \in \mathbb{T}_\tau^+ \tag{6}$$

implies $y(t) \leq C e_{b_1+b_2}(t,\tau)$ for all $t \in \mathbb{T}_\tau^+$ with $\tau \preceq \theta(t)$.

Proof. The function $z : \mathbb{T}_\tau^+ \to \mathbb{R}$, $z(t) := \int_\tau^t b_1(s)y(s)\Delta s + \int_\tau^t b_2(s)y(\theta(s))\Delta s$ satisfies $z(\tau) = 0$ and is nondecreasing. Furthermore, we have

$$\begin{aligned} z^\Delta(t) &\leq b_1(t)y(t) + b_2(t)y(\theta(t)) \\ &\overset{(6)}{\leq} C(b_1(t)+b_2(t)) + b_1(t)z(t) + b_2(t)z(\theta(t)) \\ &\leq C(b_1(t)+b_2(t)) + (b_1(t)+b_2(t))z(t) \quad \text{for all } t \in \mathbb{T}_\tau^+,\ \tau \preceq \theta(t) \end{aligned}$$

and [BP01, p. 255, Theorem 6.1] yields

$$z(t) \leq C \int_\tau^t e_{b_1+b_2}(t,\sigma(s))(b_1(s)+b_2(s))\,\Delta s = C\left[e_{b_1+b_2}(t,\tau) - 1\right]$$

for all $t \in \mathbb{T}_\tau^+$, $\tau \preceq \theta(t)$. Hence the claim follows because of $y(t) \leq C + z(t)$. $\square$

Theorem 2.4 (global existence and uniqueness)**.** *Suppose $\theta : \mathbb{T}^\kappa \to \mathbb{T}$ is a continuous delay function, $L_1, L_2 : \mathbb{T}^\kappa \to \mathbb{R}_+$ are rd-continuous, and that the rd-continuous mapping $F : \mathbb{T}^\kappa \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfies the condition:*

For each $t \in \mathbb{T}^\kappa$ there exists a compact $\mathbb{T}$-neighborhood U_t of t such that

$$\begin{aligned} \left\|F^{t]}(s,x,y) - F^{t]}(s,\bar{x},y)\right\| &\leq L_1(t)\,\|x-\bar{x}\|, \\ \left\|F^{t]}(s,x,y) - F^{t]}(s,x,\bar{y})\right\| &\leq L_2(t)\,\|y-\bar{y}\| \end{aligned} \tag{7}$$

for all $s \in U_t^\kappa$, $x, \bar{x}, y, \bar{y} \in \mathcal{X}$ hold.

Then, for any $\tau \in \mathbb{T}^\kappa$ and $\phi_\tau \in \mathcal{C}_\tau(\theta)$, the IVP (2) *admits exactly one solution $\varphi(\cdot;\tau,\phi_\tau) : \mathbb{T}_{\theta(\tau)}^+ \to \mathcal{X}$. Moreover, for $\phi_\tau, \bar{\phi}_\tau \in \mathcal{C}_\tau(\theta)$ and $t \in \mathbb{T}_\tau^+$ we have*

$$\begin{aligned} &\left\|\varphi(t;\tau,\phi_\tau) - \varphi(t;\tau,\bar{\phi}_\tau)\right\| \\ &\leq \begin{cases} e_{L_1+L_2}(t,\tau)\left\|\phi_\tau(\tau) - \bar{\phi}_\tau(\tau)\right\| & \text{for } \tau \preceq \theta(t) \\ e_{L_1}(t,\tau)\left(1 + \int_\tau^t L_2(s)\,\Delta s\right)\sup_{s\in[\theta(\tau),\tau]_\mathbb{T}}\left\|\phi_\tau(s) - \bar{\phi}_\tau(s)\right\| & \text{for } \theta(t) \preceq \tau \end{cases} \end{aligned} \tag{8}$$

Proof. Let $\tau \in \mathbb{T}^\kappa$ and $\phi_\tau \in \mathcal{C}_\tau(\theta)$ be given arbitrarily.

(I) To show the existence and uniqueness of solutions, we apply the induction principle (cf. [Hil90, Theorem 1.4(c)] for $r \in (\mathbb{T}_\tau^+)^\kappa$ to the statement:

$$\mathcal{A}(r) : \begin{cases} \text{The IVP} \\ \qquad x^\Delta(t) = F^{r]}\big(t, x(t), x(\theta(t))\big), \qquad (\tau,\phi_\tau) \\ \text{possesses exactly one solution } \nu_r : [\theta(\tau), r]_\mathbb{T} \to \mathcal{X}. \end{cases} \tag{9}$$

(i): Obviously there exists a unique continuous mapping $\nu_\tau : [\theta(\tau), \tau]_{\mathbb{T}} \to \mathcal{X}$ satisfying $\nu_\tau(t) = \phi_\tau(t)$ for $t \in [\theta(\tau), \tau]_{\mathbb{T}}$ and $\nu_\tau^\Delta(t) = F^{\tau]}\big(t, \nu_\tau(t), \nu_\tau(\theta(t))\big)$ for all $t \in \{\tau\}^\kappa = \emptyset$.
(ii): Let r be a right-scattered point. Using the induction hypothesis $\mathcal{A}(r)$, the IVP in (9) possesses exactly one solution $\nu_r : [\theta(\tau), r]_{\mathbb{T}} \to \mathcal{X}$. We define its continuous extension $\nu_{\sigma(r)} : [\theta(\tau), \sigma(r)]_{\mathbb{T}} \to \mathcal{X}$ as

$$\nu_{\sigma(r)}(t) := \begin{cases} \nu_r(t) & \text{for } t \in [\theta(\tau), r]_{\mathbb{T}} \\ \nu_r(r) + \mu^*(r) F\big(r, \nu_r(r), \nu_r(\theta(r))\big) & \text{for } t = \sigma(r) \end{cases},$$

which, by Lemma 2.1, is the unique solution of the above IVP, since the restriction on $[\theta(\tau), r]_{\mathbb{T}}$ is the unique solution of (9) and the restriction on $[\theta(r), \sigma(r)]_{\mathbb{T}}$ is the unique solution of $(1)_F$, $(r, \nu_r|_{[\theta(r), r]_{\mathbb{T}}})$ on $[\theta(r), \sigma(r)]_{\mathbb{T}}$.
(iii): Let r be right-dense. Due to the induction hypothesis $\mathcal{A}(r)$ we have a unique solution ν_r of (9). Let $[a_r, b_r]_{\mathbb{T}} \subseteq U_r$ be a compact $\mathbb{T}$-neighborhood of r, such that the function $\ell : \mathbb{T}^\kappa \to \mathbb{R}_+$, $\ell(t) := \max\{L_1(r), L_2(r)\}$ for all $t \in [a_r, b_r]_{\mathbb{T}}$ from Lemma 2.2 satisfies $\int_{a_r}^{b_r} \ell(s)\,\Delta s = \ell(r)\mu(b_r, a_r) < 1$. Now Lemma 2.2 guarantees that the IVP $(1)_{F^{s]}}$, $(r, \nu_r|_{[\theta(r), r]_{\mathbb{T}}})$ has exactly one solution $\nu : [\theta(r), s]_{\mathbb{T}} \to \mathcal{X}$ for any $s \in [a_r, b_r]_{\mathbb{T}}$. Because of Lemma 2.1, the function $\nu_s : [\theta(\tau), s]_{\mathbb{T}} \to \mathcal{X}$, defined by

$$\nu_s(t) := \begin{cases} \nu_r(t) & \text{for } t \in [\theta(\tau), r]_{\mathbb{T}} \\ \nu(t) & \text{for } t \in [r, s]_{\mathbb{T}} \end{cases},$$

is the unique solution of (9) for $r = s$. Hence, the statement $\mathcal{A}(s)$ holds for all $s \in [a_r, b_r]_{\mathbb{T}} \cap \mathbb{T}_r^+$.
(iv): Let r be left-dense and we choose a $\mathbb{T}$-interval $[a_r, b_r]_{\mathbb{T}}$ as in (iii). Then there exists a $s \in [a_r, b_r]_{\mathbb{T}}$, $s \prec r$. Using the induction hypothesis $\mathcal{A}(s)$, as well as Lemma 2.2, one shows existence and uniqueness of the solution $\nu_r : [\theta(\tau), r]_{\mathbb{T}} \to \mathcal{X}$ of (9) exactly as in step (iii). Since on every interval $[\theta(\tau), r]_{\mathbb{T}}$, $\tau \preceq r$, there exists exactly one solution ν_r, there is one on $\mathbb{T}^+_{\theta(\tau)}$.

(II) It remains to prove the estimate (8). Thereto, let $\phi_\tau, \bar{\phi}_\tau \in \mathcal{C}_\tau(\theta)$. The solution $\varphi(\cdot; \tau, \phi_\tau)$ of $(1)_F$ satisfies the integral equation

$$\varphi(t; \tau, \phi_\tau) = \phi_\tau(\tau) + \int_\tau^t F(s, \varphi(s; \tau, \phi_\tau), \varphi(\theta(s); \tau, \phi_\tau))\,\Delta s \quad \text{for all } t \in \mathbb{T}_\tau^+,$$

yielding the estimate

$$\begin{aligned} \big\|\varphi(t; \tau, \phi_\tau) - \varphi(t; \tau, \bar{\phi}_\tau)\big\| &\overset{(7)}{\leq} \big\|\phi_\tau(\tau) - \bar{\phi}_\tau(\tau)\big\| \\ &\quad + \int_\tau^t L_1(s)\,\big\|\varphi(s; \tau, \phi_\tau) - \varphi(s; \tau, \bar{\phi}_\tau)\big\|\,\Delta s \\ &\quad + \int_\tau^t L_2(s)\,\big\|\varphi(\theta(s); \tau, \phi_\tau) - \varphi(\theta(s); \tau, \bar{\phi}_\tau)\big\|\,\Delta s \end{aligned}$$

for all $t \in \mathbb{T}_\tau^+$, and with Lemma 2.3 we obtain

$$\left\|\varphi(t;\tau,\phi_\tau) - \varphi(t;\tau,\bar{\phi}_\tau)\right\| \le e_{L_1+L_2}(t,\tau)\left\|\phi_\tau(\tau) - \bar{\phi}_\tau(\tau)\right\|$$

for all $t \in \mathbb{T}_\tau^+$, $\tau \preceq \theta(t)$. On the other hand, in case of $\theta(t) \preceq \tau$, one has

$$\begin{aligned}
&\left\|\varphi(t;\tau,\phi_\tau) - \varphi(t;\tau,\bar{\phi}_\tau)\right\| \\
&\le \left\|\phi_\tau(\tau) - \bar{\phi}_\tau(\tau)\right\| + \int_\tau^t L_2(s)\left\|\phi_\tau\big(\theta(s)\big) - \bar{\phi}_\tau(\theta(s))\right\|\,\Delta s \\
&\quad + \int_\tau^t L_1(s)\left\|\varphi(s;\tau,\phi_\tau) - \varphi(s;\tau,\bar{\phi}_\tau)\right\|\,\Delta s \\
&\le \left\|\phi_\tau(\tau) - \bar{\phi}_\tau(\tau)\right\| + \int_\tau^t L_2(s)\,\Delta s \sup_{s\in[\theta(\tau),\tau]_\mathbb{T}}\left\|\phi_\tau(s) - \bar{\phi}_\tau(s)\right\| \\
&\quad + \int_\tau^t L_1(s)\left\|\varphi(s;\tau,\phi_\tau) - \varphi(s;\tau,\bar{\phi}_\tau)\right\|\,\Delta s
\end{aligned}$$

and Gronwall's Lemma (cf. [BP01, p. 256, Theorem 6.4]) implies the second inequality in (8). This concludes the present proof. □

3 Linearized Asymptotic Stability

Throughout this section, let $\mathbb{T}$ be unbounded above. Moreover, $\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$ is the set of rd-continuous functions $a : \mathbb{T} \to \mathbb{R}$ with $1 + \mu^*(t)a(t) > 0$ for $t \in \mathbb{T}$.

Lemma 3.1. *Let $\tau \in \mathbb{T}$, $K \ge 1$, $a \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, suppose $\theta : \mathbb{T} \to \mathbb{T}$ is a continuous delay function, $A : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ and $f : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are rd-continuous. Consider the dynamic delay equation*

$$\boxed{x^\Delta(t) = A(t)x(t) + f\big(t, x(t), x(\theta(t))\big)} \qquad (10)_f$$

under the following assumptions:

(i) The transition operator of $x^\Delta(t) = A(t)x(t)$ satisfies

$$\|\Phi_A(t,s)\| \le Ke_a(t,s) \quad \text{for all } \tau \preceq s \preceq t, \qquad (11)$$

(ii) $f(t,0,0) \equiv 0$ on $\mathbb{T}$, and there exist reals $L_1, L_2 \ge 0$ such that we have

$$\begin{aligned}
\|f(t,x,y) - f(t,\bar{x},y)\| &\le L_1\|x - \bar{x}\|, \\
\|f(t,x,y) - f(t,x,\bar{y})\| &\le L_2\|y - \bar{y}\|
\end{aligned} \qquad (12)$$

for all $t \in \mathbb{T}$, $x, \bar{x}, y, \bar{y} \in \mathcal{X}$.

Then the solution $\varphi(\cdot;\tau,\phi_\tau)$ of $(10)_f$ satisfies

$$\|\varphi(t;\tau,\phi_\tau)\| \le Ke_{\bar{a}}(t,\tau)\|\phi_\tau(\tau)\| \quad \text{for all } t \in \mathbb{T}_\tau^+,\ \tau \preceq \theta(t), \qquad (13)$$

initial functions $\phi_\tau \in \mathcal{C}_\tau(\theta)$, and $\bar{a}(t) := a(t) + K\left(L_1 + L_2e_a(\theta(t),t)\right)$.

Proof. Let $\tau \in \mathbb{T}$. Due to our present assumptions, one can apply Theorem 2.4 to the dynamical delay equation $(10)_f$ and consequently all solutions $\varphi(\cdot;\tau,\phi_\tau)$ with $\phi_\tau \in \mathcal{C}_\tau(\theta)$ exist on $\mathbb{T}^+_{\theta(\tau)}$. Furthermore, the variation of constants formula (cf. [Pöt02, p. 56, Satz 1.3.11]) implies the identity

$$\varphi(t;\tau,\phi_\tau) = \Phi_A(t,\tau)\phi_\tau(\tau) + \int_\tau^t \Phi_A(t,\sigma(s))f(s,\varphi(s;\tau,\phi_\tau),\varphi(\theta(s);\tau,\phi_\tau))\,\Delta s$$

for all $t \in \mathbb{T}^+_\tau$, and from $f(t,0,0) \equiv 0$ we obtain

$$\begin{aligned}
\|\varphi(t;\tau,\phi_\tau)\| &\overset{(11)}{\leq} Ke_a(t,\tau)\,\|\phi_\tau(\tau)\| \\
&\quad + K\int_\tau^t e_a(t,\sigma(s))\,\|f(s,\varphi(s;\tau,\phi_\tau),\varphi(\theta(s);\tau,\phi_\tau))\|\,\Delta s \\
&\overset{(12)}{\leq} Ke_a(t,\tau)\,\|\phi_\tau(\tau)\| + KL_1\int_\tau^t e_a(t,\sigma(s))\,\|\varphi(s;\tau,\phi_\tau)\|\,\Delta s \\
&\quad + KL_2\int_\tau^t e_a(t,\sigma(s))\,\|\varphi(\theta(s);\tau,\phi_\tau)\|\,\Delta s \quad \text{for all } t \in \mathbb{T}^+_\tau,
\end{aligned}$$

which, in turn, yields (cf. [Hil90, Theorem 6.2])

$$\begin{aligned}
\|\varphi(t;\tau,\phi_\tau)\|\,e_a(\tau,t) &\leq K\,\|\phi_\tau(\tau)\| + \int_\tau^t \frac{K_1L}{1+\mu^*(s)a(s)}e_a(\tau,s)\,\|\varphi(s;\tau,\phi_\tau)\|\,\Delta s \\
&\quad + KL_2\int_\tau^t e_a(\theta(s),\sigma(s))e_a(\tau,\theta(s))\,\|\varphi(\theta(s);\tau,\phi_\tau)\|\,\Delta s
\end{aligned}$$

for all $t \in \mathbb{T}^+_\tau$. Then Lemma 2.3 gives us the desired estimate (13). □

Theorem 3.2. *Let $\tau \in \mathbb{T}$, suppose $\theta : \mathbb{T} \to \mathbb{T}$ is a continuous delay function, $A : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is rd-continuous, $F : \mathbb{T}\times\mathcal{X}\times\mathcal{X} \to \mathcal{X}$ is rd-continuous and continuously differentiable w.r.t. the variables in $\mathcal{X}\times\mathcal{X}$. Consider the dynamic delay equation $(10)_f$ under the following assumptions:*

- *(i) The transition operator of $x^\Delta(t) = A(t)x(t)$ satisfies the estimate (11) with $\sup_{s\in\mathbb{T}^+_\tau} a(s) < 0$ and $\sup_{s\in\mathbb{T}^+_\tau} e_a(\theta(s),s) < \infty$,*
- *(ii) $f(t,0,0) \equiv 0$ on $\mathbb{T}$, and we have*

$$\lim_{(x,y)\to(0,0)} D_{(2,3)}f(t,x,y) = 0 \quad \textit{uniformly in } t \in \mathbb{T}. \tag{14}$$

Then there exists a $\rho > 0$ such that all solutions $\varphi(\cdot,\tau,\phi_\tau)$ of $(10)_f$ with initial functions $\phi_\tau \in \mathcal{C}_\tau(\theta)$, $\sup_{t\in[\theta(\tau),\tau]_\mathbb{T}} \|\phi_\tau(t)\| \leq \rho$ exist uniquely on $\mathbb{T}^+_{\theta(\tau)}$ and decay to 0 exponentially.

Proof. Let $\tau \in \mathbb{T}$. Due to hypothesis (i) there exists a $L > 0$ such that

$$KL\Big(1 + \sup_{s\in\mathbb{T}_\tau^+} e_a(\theta(s), s)\Big) < \inf_{s\in\mathbb{T}_\tau^+}(-a(s)) \tag{15}$$

holds, and the limit relation (14) guarantees that there is a $\rho_1 > 0$ with $\left\|D_{(2,3)}f(t,x,y)\right\| \leq \frac{1}{2}L$ for all $t \in \mathbb{T}$, $x, y \in \bar{B}_{\rho_1}$. Now the mean value inequality implies $\|f(t,x,y) - f(t,\bar{x},\bar{y})\| \leq \frac{1}{2}L \left\|\binom{x-\bar{x}}{y-\bar{y}}\right\|$ for $t \in \mathbb{T}$, $x, \bar{x}, y, \bar{y} \in \bar{B}_{\rho_1}$. Using the radial retraction $R_\rho : \mathcal{X} \to \bar{B}_\rho$, defined by $R_\rho(x) := x$ for $\|x\| \leq \rho$ and $R_\rho(x) := \frac{\rho}{\|x\|}x$ for $\|x\| \geq \rho$, it is well-known that the modified mapping $\tilde{f} : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $\tilde{f}(t,x,y) := f(t, R_{\rho_1}(x), R_{\rho_1}(y))$ coincides with f on the set $\mathbb{T} \times \bar{B}_{\rho_1} \times \bar{B}_{\rho_1}$ and satisfies $\left\|\tilde{f}(t,x,y) - \tilde{f}(t,\bar{x},\bar{y})\right\| \leq L \left\|\binom{x-\bar{x}}{y-\bar{y}}\right\|$ for all $t \in \mathbb{T}$, $x, \bar{x}, y, \bar{y} \in \mathcal{X}$. Therefore, from Theorem 2.4 we get that all solutions $\tilde{\varphi}(\cdot;\tau,\phi_\tau)$, $\phi_\tau \in \mathcal{C}_\tau(\theta)$, of $(10)_{\tilde{f}}$ exist and are unique on $\mathbb{T}^+_{\theta(\tau)}$. Furthermore, from Lemma 3.1 we have the inequality

$$\|\tilde{\varphi}(t;\tau,\phi_\tau)\| \overset{(13)}{\leq} Ke_{\bar{a}}(t,\tau)\,\|\phi_\tau(\tau)\| \quad \text{for all } t \in \mathbb{T}_\tau^+,\ \tau \preceq \theta(t) \tag{16}$$

with $\bar{a}(t) := a(t) + KL(1 + e_a(\theta(t),t))$ and (15) yields $\sup_{s\in\mathbb{T}_\tau^+} \bar{a}(s) < 0$. This implies $\|\tilde{\varphi}(t;\tau,\phi_\tau)\| \leq K\|\phi_\tau(\tau)\| \leq \rho_1$ for all $t \in \mathbb{T}_\tau^+$, $\tau \preceq \theta(t)$, $\phi_\tau \in \bar{B}_{\frac{K}{\rho_1}}$, and from Theorem 2.4 we additionally get

$$\|\tilde{\varphi}(t;\tau,\phi_\tau)\| \overset{(8)}{\leq} e_L(t,\tau)\left(1 + \int_\tau^t L(s)\,\Delta s\right) \sup_{s\in[\theta(\tau),\tau]_{\mathbb{T}}} \|\phi_\tau(s)\|$$

for all $t \in \mathbb{T}_\tau^+$, $\theta(t) \preceq \tau$, which yields the existence of a $\rho_2 > 0$ such that $\|\tilde{\varphi}(t;\tau,\phi_\tau)\| \leq \rho_1$ for all $t \in \mathbb{T}_\tau^+$, $\phi_\tau \in \bar{B}_{\rho_2}$. If we choose $\rho := \min\left\{\frac{\rho_1}{K}, \rho_2\right\}$, then any solution $\tilde{\varphi}(\cdot;\tau,\phi_\tau)$ of $(10)_{\tilde{f}}$ with $\phi_\tau \in \bar{B}_\rho$ is also a solution of $(10)_f$ and together with (16) our assertion follows. □

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