EXTENDED HIERARCHIES OF INVARIANT FIBER BUNDLES FOR DYNAMIC EQUATIONS ON MEASURE CHAINS

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ABSTRACT. If a linear autonomous ordinary differential of difference equation possesses a coefficient operator, which is (pseudo-) hyperbolic or allows a more specific splitting of its spectrum into appropriate spectral sets, then this gives rise to a so-called *hierarchy* of invariant linear subspaces of $\mathcal{X}$ related to the ranges to the corresponding spectral projections. Together with the intersections of these invariant subspaces, we get an *extended hierarchy*. Here, each member of the hierarchy can be characterized dynamically as set of initial points for orbits with a certain asymptotic growth rate in forward or backward time.

In this paper we show that such a scenario persists under perturbations w.r.t. two points of view: In the first instance, the invariant linear spaces become an "extended hierarchy" of invariant manifolds, if the linear part is perturbed by a globally Lipschitzian (or smooth) mapping on $\mathcal{X}$. This will be done in the nonautonomous context of dynamic equations on measure chains or time scales, where the time-varying invariant manifolds are called *invariant fiber bundles*. Secondly, we derive perturbation results well-suited for up-coming applications in analytical discretization theory.

1. MOTIVATION AND NOTATION

Consider a linear autonomous difference equation (recursion)

$$x_{n+1} = Ax_n, \tag{1.1}$$

in a (complex) Banach space $\mathcal{X}$, where the linear bounded operator $A : \mathcal{X} \to \mathcal{X}$ is *pseudo-hyperbolic*. That means, its spectrum $\Sigma(A) \subset \mathbb{C}$ can be separated into two nonempty subsets Σ_1, Σ_2 with $|\lambda| < \alpha$ for all $\lambda \in \Sigma_1$, $\beta < |\lambda|$ for all $\lambda \in \Sigma_2$ and reals $0 < \alpha < \beta$. Let Q, P be the spectral projections corresponding with Σ_1 and Σ_2, respectively. Then we have a direct decomposition $\mathcal{X} = \mathcal{R}(Q) \oplus \mathcal{R}(P)$ of the state space $\mathcal{X}$ into two (forward) invariant subspaces allowing the dynamical characterization

$$\begin{aligned} \mathcal{R}(Q) &= \left\{ \xi \in \mathcal{X} : \sup_{n \geq 0} \|A^n \xi\| \gamma^{-n} < \infty \right\}, \\ \mathcal{R}(P) &= \left\{ \xi \in \mathcal{X} : \begin{array}{l} \text{there exists a sequence } (x_n)_{n \leq 0} \text{ with } x_0 = \xi, \\ x_{n+1} = Ax_n \text{ for } n < 0 \text{ and } \sup_{n \leq 0} \|A^n \xi\| \gamma^{-n} < \infty \end{array} \right\} \end{aligned} \tag{1.2}$$

for any $\gamma \in (\alpha, \beta)$. It is well-known that these invariant subspaces persist under certain globally Lipschitzian perturbations. To be more precise, if $f : \mathcal{X} \to \mathcal{X}$ is a (possibly nonlinear) mapping with a "sufficiently small" global Lipschitz constant, then also the semilinear difference equation

$$x_{n+1} = Ax_n + f(x_n) \tag{1.3}$$

has two invariant manifolds S and R, which are graphs of Lipschitzian mappings over the linear subspaces $\mathcal{R}(Q)$ and $\mathcal{R}(P)$ of (1.1), respectively. This result is well-known as the *generalized Hadamard-Perron* (or *generalized stable manifold*) *theorem* and can be traced back to, e.g., in [Har64, pp. 234–236, Lemma 5.1 and Exercise 5.2] for C^1-mappings on $\mathbb{R}^n$, in [HPS77, pp. 53–54, Theorem 5.1] for C^m-diffeomorphisms, $m \geq 1$, or [Aul98] for general nonautonomous difference equations in Banach spaces.

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If more detailed information is known about the spectrum of the operator A, the generalized Hadamard-Perron theorem can be used to obtain a deeper insight into the dynamics of (1.3). In fact, if we assume the decomposition $\Sigma(A) = \Sigma_1 \cup \ldots \cup \Sigma_N$ for $N \geq 2$, and the existence of reals $\beta_0 < 0$,

$$0 < \alpha_1 < \beta_1 < \ldots < \alpha_{N-1} < \beta_{N-1} < \alpha_N := \infty,$$

such that the moduli of the spectral points in Σ_i are contained in the open interval (β_{i-1}, α_i) for $1 \leq i \leq N$, then one can show the existence of invariant manifolds $C_{i,1}, C_{N,i}$, $1 \leq i < N$, of (1.3). Denoting the spectral projectors corresponding with Σ_i by P_i, the manifold $C_{i,1}$ is a Lipschitzian graph over $\mathcal{R}(P_1) \oplus \ldots \oplus \mathcal{R}(P_i)$, while $C_{N,i}$ is a Lipschitzian graph over $\mathcal{R}(P_{i+1}) \oplus \ldots \oplus \mathcal{R}(P_N)$ for $1 \leq i < N$, and a dynamical characterization of $C_{i,1}, C_{N,i}$ similar to (1.2) leads to the hierarchical inclusions

$$\{0\} \subset C_{1,1} \subset \ldots \subset C_{N-1,1} \subset \mathcal{X}, \qquad \{0\} \subset C_{N,N} \subset \ldots \subset C_{N,2} \subset \mathcal{X}.$$

In addition, also the intersections $C_{i,j} := C_{i,1} \cap C_{N,j}$, $1 < j \leq i < N$, are invariant manifolds of (1.3) representable as graphs over $\mathcal{R}(P_j) \oplus \ldots \oplus \mathcal{R}(P_i)$, and one speaks of an *extended hierarchy* of invariant manifolds. Note that the above scenario is classical in case $N = 3$ and $\beta_1 < 0 < \alpha_2$. Then $C_{1,1}$ is the stable, $C_{2,1}$ the center-stable, $C_{3,3}$ the unstable and $C_{3,2}$ the center-unstable manifold of (1.3), while the center manifold is given by the intersection $C_{2,2} = C_{2,1} \cap C_{3,2}$ (cf. [Kel67] for ODEs).

Analogous results to the discrete case considered above, hold for ODEs or more general evolutionary differential equations under appropriate spectral assumptions on the linear part, i.e., the generator of the corresponding semigroup. Essentially, one has to replace modulii of the spectral points by their real parts. Indeed, the idea to consider (extended) hierarchies of invariant manifolds stands in a certain tradition. In the general setting described above, it can be traced back at least to [Aul87], who considers finite-dimensional autonomous ODEs, while [Aul95] works with nonautonomous difference equations in general Banach spaces, and the case of infinite-dimensional Carathéodory differential equations can be found in [AW96]. These quoted papers also contain examples showing that not all invariant linear subspaces of (1.1) survive under perturbations described above in the sense that they are Lipschitzian graphs. Finally, extended hierarchies of Carathéodory differential equations are constructed in [Sie99].

The recent years saw an increasing interest to study dynamical behavior on time axes different from the reals (as for ODEs) or the integers (as for difference equations). The corresponding tool for such an endeavor is the so-called calculus on measure chains or time scales (cf. [Hil90, BP01]). In general, a time scale is an arbitrary closed subset $\mathbb{T}$ of the real numbers. Thus, instead of studying discrete ($\mathbb{T} = \mathbb{Z}$) and continuous ($\mathbb{T} = \mathbb{R}$) dynamical systems separately, one investigates them on time scales and obtains the classical situations for free as special cases. Beyond that, it turned out that time scales of particular interest include unions of disjoint closed intervals (for population modeling under the influence of hibernation periods) or meshes of discrete points (in analytical discretization theory). Specifically for discretization issues, the time scale calculus is a well-suited device, since two key topics are met:

- *Persistence* under temporal discretization, since continuous and discrete phenomena can be derived simultaneously.
- *Convergence* for small time step-sizes, provided one establishes appropriate perturbation results for time scale, i.e., dynamic equations.

Motivated by these perspectives, the main contributions of this paper essentially consist of two results (see Theorem 3.3 and 4.2).

- We provide a flexible and general version of the Hadamard-Perron theorem, where the invariant manifolds generalize to so-called *invariant fiber bundles*. It applies to nonautonomous, noninvertible dynamic equations on nearly arbitrary time scales. Their linear part is allowed to be pseudo-hyperbolic in terms of an exponential dichotomy with not necessarily constant growth rates. For discretization problems, a dichotomy notion with variable growth rates has the technical advantage to avoid lower bounds for the step-sizes (see [KP05, Lemma 3.3] for details). The nonlinearities can depend on parameters from general metric spaces. Based on this result, we will be able to derive concepts like invariant foliations or asymptotic phases, leading to the reduction principle (see [Pöt06]) and a generalized Hartman-Grobman theorem (see [Pöt07a]). In doing so, we generalize [Aul87, Aul95, AW96, Kel99, Pöt03, PS04] in two ways: Firstly, the equations need to be only (globally) Lipschitzian in their state space variable, which is a minimal requirement to develop a topological linearization theory. Secondly, the equations under consideration are not assumed to possess a decoupled linear part and the usual Lyapunov transformation technique (cf. [AP02]) becomes redundant.

- If we replace the above dichotomy assumption on the linear part by a more general exponential splitting of the extended state space, we are able to prove the existence of extended hierarchies of invariant fiber bundles for dynamic equations on time scales beyond the reals or the integers. Under certain spectral gap conditions, and relying on our earlier work (see [Pöt03, PS04]), we moreover obtain differentiability results for this hierarchy.

These two results provide a theoretical framework to study the *persistence* of hierarchies of invariant manifolds under numerical discretization. To address *convergence* questions, though, it is essential to allow a specific parameter-dependence (cf. equation (2.2)). In applications (see [KP05]) this parameter serves as homotopy parameter between the flow of a nonautonomous ODE and its discretization using a variable step-size one-step method of, e.g., Runge-Kutta-type.

Concerning our general notation, $\mathbb{N}$ are the positive integers, $\mathbb{R}$ denotes the real and $\mathbb{C}$ the complex field. Throughout this paper, Banach spaces $\mathcal{X}$ are over the real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) field, and their norm is denoted by $\|\cdot\|_{\mathcal{X}}$, or simply by $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ is the Banach space of linear bounded endomorphisms and $I_{\mathcal{X}}$ the identity on $\mathcal{X}$, $\mathcal{R}(T) := T\mathcal{X}$ the range and $\mathcal{N}(T) := T^{-1}(\{0\})$ the kernel of $T \in \mathcal{L}(\mathcal{X})$. The direct sum of two linear subspaces $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{X}$ is denoted by $\mathcal{X}_1 \oplus \mathcal{X}_2$ and on the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$ we always use the norm

$$\|(x_1, x_2)\|_{\mathcal{X}_1 \times \mathcal{X}_2} := \max \left\{ \|x_1\|_{\mathcal{X}_1}, \|x_2\|_{\mathcal{X}_2} \right\}.$$

If a mapping f between metric spaces satisfies a Lipschitz condition, then its smallest Lipschitz constant is denoted by $\operatorname{Lip} f$. In case f depends on more than one variable, and if it is Fréchet-differentiable in the ith variable, we write $D_i f$ for the corresponding partial derivative.

2. Measure Chains, Time Scales and Dynamic Equations

In order to keep this article self-contained, we introduce some basic terminology from the calculus on measure chains (cf. [Hil90, BP01]). In all subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [Hil90, Axiom 2]) with growth calibration $\mu : \mathbb{T}^2 \to \mathbb{R}$ (see [Hil90, Axiom 3]), such that $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, is unbounded above. The most intuitive and relevant examples of measure chains are *time scales*, where $\mathbb{T}$ is a canonically ordered closed subset of the reals and μ is given by $\mu(t, s) = t - s$. Continuing, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf \{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and the *graininess* $\mu^* : \mathbb{T} \to \mathbb{R}$, $\mu^*(t) := \mu(\sigma(t), t)$ is assumed to be bounded from now on. For $\tau \in \mathbb{T}$ we abbreviate $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}$ and $\mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \preceq \tau\}$.

$\mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ denotes the set of rd-continuous functions from $\mathbb{T}$ to $\mathcal{X}$ (cf. [Hil90, Section 4.1]). *Growth rates* are functions $a \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ such that $-1 < \inf_{t \in \mathbb{T}} \mu^*(t) a(t), \sup_{t \in \mathbb{T}} \mu^*(t) a(t) < \infty$. Moreover, for $a, b \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ we introduce the relations $\lfloor b - a \rfloor := \inf_{t \in \mathbb{T}} (b(t) - a(t))$,

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \lfloor b - a \rfloor, \qquad\qquad a \unlhd b \quad :\Leftrightarrow \quad 0 \leq \lfloor b - a \rfloor.$$

On the set $\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{a \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}) : a \text{ is a growth rate and } 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ we define the product $(m \odot a)(t) := \lim_{h \searrow \mu^*(t)} \frac{(1+ha(t))^m - 1}{h}$ with $m \in \mathbb{N}$. Besides, for $a \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ the (real) *exponential function* on $\mathbb{T}$ is denoted by $e_a(t, s) \in \mathbb{R}$, $s, t \in \mathbb{T}$ (cf. [Hil90, Theorem 7.3]).

A function $\phi : \mathbb{T} \to \mathcal{X}$ is said to be *T-periodic* for some $T > 0$, if there exists a mapping $\sigma_T : \mathbb{T} \to \mathbb{T}$ satisfying the identities $\mu(\sigma_T(t), t) \equiv T$ and $\phi(\sigma_T(t)) \equiv \phi(t)$ on $\mathbb{T}$. We say ϕ is *differentiable* (in a point $t_0 \in \mathbb{T}$), if there exists a unique *derivative* $\phi^\Delta(t_0) \in \mathcal{X}$, such that for any $\varepsilon > 0$ the estimate

$$\left\|\phi(\sigma(t_0)) - \phi(t) - \mu(\sigma(t_0), t)\phi^\Delta(t_0)\right\| \leq \varepsilon \left|\mu(\sigma(t_0), t)\right| \quad \text{for } t \in U$$

holds in a $\mathbb{T}$-neighborhood U of t_0 (see [Hil90, Section 2.4]). The Lebesgue integral of ϕ is denoted as $\int_\tau^t \phi(s)\, \Delta s$ for $\tau, t \in \mathbb{T}$, provided it exists (cf. [Nei01]). The following example is intended to give readers unfamiliar with time scale a flavor of the above rather abstract objects.

Example 2.1. Above all, a variety of examples for time scales is discussed in [BP01]. Of particular interest, though, are the time scales $\mathbb{T} = \mathbb{R}$ to describe ODEs, as well as discrete meshes

$$\mathbb{T} = \mathbb{D} := \left\{ t_k \in \mathbb{R} : \lim_{k\to\pm\infty} t_k = \pm\infty \text{ and } t_k < t_{k+1} \text{ for all } k \in \mathbb{Z} \right\}$$

to capture numerical schemes for temporal discretizations with varying step-sizes $t_{k+1} - t_k$ (or simply difference equations). On such time scales, the above objects are summarized in the following table:

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{D}$
$\sigma(t)$	$\sigma(t) = t$	$\sigma(t_k) = t_{k+1}$
$\mu^*(t)$	$\mu^*(t) \equiv 0$	$\mu^*(t_k) = t_{k+1} - t_k$
$\mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$	$\mathcal{C}(\mathbb{R}, \mathcal{X})$	$\{\phi : \mathbb{D} \to \mathcal{X}\}$
$\mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$	$\mathcal{C}(\mathbb{R}, \mathbb{R})$	$\{a : \mathbb{D} \to \mathcal{X} \mid 1 + (t_{k+1} - t_k)a(t_k) > 0\}$
$(m \odot a)(t)$	$(m \odot a)(t) = ma(t)$	$(m \odot a)(t_k) = \frac{[1+(t_{k+1}-t_k)a(t_k)]^m - 1}{t_{k+1}-t_k}$
$e_a(t, \tau)$	$e_a(t, \tau) = \exp\left(\int_\tau^t a(a)ds\right)$	$e_a(t_k, t_n) = \prod_{l=n}^{k-1} [1 + (t_{l+1} - t_l)a(t_l)]$
$\phi^\Delta(t)$	$\phi^\Delta(t) = \dot{\phi}(t)$	$\phi^\Delta(t_k) = \frac{\phi(t_{k+1})-\phi(t_k)}{t_{k+1}-t_k}$
$\int_\tau^t \phi(s)\,\Delta s$	$\int_\tau^t \phi(s)\,\Delta s = \int_\tau^t \phi(s)\,ds$	$\int_{t_n}^{t_k} \phi(s)\,\Delta s = \sum_{l=k}^{n-1} (t_{l+1} - t_l)\phi(t_l)$

Note that on more complicated time scales these formulas are more involved than the above differential resp. difference case (think of, e.g., a Cantor set $\mathbb{T}$).

Given a mapping $A \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, a *linear dynamic equation* is of the form

$$x^\Delta = A(t)x; \tag{2.1}$$

here the *transition operator* $\Phi_A(t, s) \in \mathcal{L}(\mathcal{X})$, $s \preceq t$, is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(s) = I_\mathcal{X}$ in $\mathcal{L}(\mathcal{X})$.

Let $(\mathcal{Q}, d)$ be a metric space and keep the parameters $\theta \in \mathbb{F}$, $q \in \mathcal{Q}$ fixed. We consider semilinear perturbations of the dynamic equation (2.1) given by

$$x^\Delta = A(t)x + F_1(t, x; q) + \theta F_2(t, x; q) \tag{2.2}$$

with mappings $F_i : \mathbb{T} \times \mathcal{X} \times \mathcal{Q} \to \mathcal{X}$, such that $F_i(\cdot; q)$, $q \in \mathcal{Q}$, is rd-continuous (see [Hil90, Section 5.1]) and $F_i(t, \cdot)$, $t \in \mathbb{T}$, is continuous for $i = 1, 2$. Further assumptions on F_1, F_2 can be found in Hypothesis 3.1 and 4.1. A *solution* of (2.2) is a function ν satisfying the identity $\nu^\Delta(t) \equiv A(t)\nu(t) + F_1(t, \nu(t); q) + \theta F_2(t, \nu(t); q)$ on a $\mathbb{T}$-interval. φ denotes the *general solution* of (2.2), i.e., $\varphi(\cdot; \tau, \xi; \theta, q)$ solves (2.2) on $\mathbb{T}^+_\tau$ and satisfies the initial condition $\varphi(\tau; \tau, \xi; \theta, q) = \xi$ for $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$. It fulfills the *cocycle property*

$$\varphi(t; s, \varphi(s; \tau, \xi; \theta, q); \theta, q) = \varphi(t; \tau, \xi; \theta, q) \quad \text{for } \tau, s, t \in \mathbb{T},\ \tau \preceq s \preceq t,\ \xi \in \mathcal{X}. \tag{2.3}$$

The dynamic equation (2.2) is said to be *T-periodic* for some $T > 0$, if the mappings A and $F_i(\cdot, x, q)$ possess this property for all $x \in \mathcal{X}$, $q \in \mathcal{Q}$.

As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to nonautonomous equations. In order to be more precise, we call a subset $S(\theta, q)$ of the extended state space $\mathbb{T} \times \mathcal{X}$ an *invariant fiber bundle* of (2.2), if it is forward invariant, i.e., if for any pair $(\tau, \xi) \in S(\theta, q)$ one has $(t, \varphi(t; \tau, \xi; \theta, q)) \in S(\theta, q)$ for all $t \in \mathbb{T}^+_\tau$.

Definition 2.1. For $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\tau \in \mathbb{T}$ we say that $\phi \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ is

(a) c^+-*quasibounded*, if $\|\phi\|^+_{\tau,c} := \sup_{t\in\mathbb{T}^+_\tau} \|\phi(t)\|\, e_c(\tau, t) < \infty$,
(b) c^--*quasibounded*, if $\|\phi\|^-_{\tau,c} := \sup_{t\in\mathbb{T}^-_\tau} \|\phi(t)\|\, e_c(\tau, t) < \infty$,
(c) $c^\pm$-*quasibounded*, if $\|\phi\|^\pm_{\tau,c} := \sup_{t\in\mathbb{T}} \|\phi(t)\|\, e_c(\tau, t) < \infty$.

$\mathcal{X}^+_{\tau,c}$ and $\mathcal{X}^-_{\tau,c}$ denote the sets of c^+- and c^--quasibounded functions on $\mathbb{T}^+_\tau$ and $\mathbb{T}^-_\tau$, respectively.

Obviously $\mathcal{X}_{\tau,c}^{+}$ and $\mathcal{X}_{\tau,c}^{-}$ are nonempty and by [Hil90, Theorem 4.1(iii)], it is immediate that for any $c \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$, $\tau \in \mathbb{T}$, the sets $\mathcal{X}_{\tau,c}^{+}$ and $\mathcal{X}_{\tau,c}^{-}$ are Banach spaces with the norms $\|\cdot\|_{\tau,c}^{+}$ and $\|\cdot\|_{\tau,c}^{-}$, respectively. Finally, $\mathcal{X}_{\tau,c}^{+}$ define a scale of Banach spaces, i.e., for $c, d \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$ we have

$$c \trianglelefteq d \quad \Rightarrow \quad \mathcal{X}_{\tau,c}^{+} \subseteq \mathcal{X}_{\tau,d}^{+} \tag{2.4}$$

and in case $c \lhd d$ the strong inclusion $\mathcal{X}_{\tau,c}^{+} \subset \mathcal{X}_{\tau,d}^{+}$ holds.

3. Existence of Invariant Fiber Bundles

We begin this section by stating some frequently used assumptions for our prototype system (2.2) of dynamic equations. As a preparation, a projection-valued mapping $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is called a *projector*, we speak of an *invariant projector* of (2.1), if

$$P(t)\Phi_A(t,s) = \Phi_A(t,s)P(s) \quad \text{for } s,t \in \mathbb{T},\ s \preceq t \tag{3.1}$$

holds, and finally an invariant projector P is denoted as *regular*, if

$$I_{\mathcal{X}} + \mu^*(t)A(t)|_{\mathcal{R}(P(t))} : \mathcal{R}(P(t)) \to \mathcal{R}\big(P(\sigma(t))\big) \text{ is bijective for all } t \in \mathbb{T}.$$

Then the restriction $\bar{\Phi}_A(t,s) := \Phi_A(t,s)|_{\mathcal{R}(P(s))} : \mathcal{R}(P(s)) \to \mathcal{R}(P(t))$, $s \preceq t$, is a well-defined isomorphism, and we write $\bar{\Phi}_A(s,t)$ for its inverse (cf. [Pöt02, p. 85, Lemma 2.1.8]).

From now on we assume:

Hypothesis 3.1. (i) *There exists a regular invariant projector* $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ *of* (2.1) *such that the dichotomy estimates*

$$\|\Phi_A(t,s)Q(s)\| \le K_1 e_a(t,s), \qquad \big\|\bar{\Phi}_A(s,t)P(t)\big\| \le K_2 e_b(s,t) \quad \textit{for } t \preceq s \tag{3.2}$$

are satisfied, with the complementary projector $Q(t) := I_{\mathcal{X}} - P(t)$, *reals* $K_1, K_2 \ge 1$ *and growth rates* $a, b \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a \lhd b$.

(ii) *We abbreviate* $H_\theta := F_1 + \theta F_2$, *and for* $i = 1,2$ *the identities*

$$F_i(t,0;q) \equiv 0 \quad \textit{on } \mathbb{T} \times \mathcal{Q} \tag{3.3}$$

hold and the mappings F_i *satisfy the following global Lipschitz estimates*

$$L_i := \sup_{(t,q)\in\mathbb{T}\times\mathcal{Q}} \operatorname{Lip} F_i(t,\cdot;q) < \infty. \tag{3.4}$$

Moreover, for some $\delta_{\max} > 0$ *we require*

$$L_1 < \frac{\delta_{\max}}{2(K_1+K_2)}, \tag{3.5}$$

choose a fixed $\delta \in (2(K_1+K_2)L_1, \delta_{\max})$ *and abbreviate* $\Theta := \{\theta \in \mathbb{F} : L_2|\theta| \le L_1\}$,

$$\Gamma := \big\{c \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R}) : a + \delta \lhd c \lhd b - \delta\big\}.$$

(iii) *Assume the partial derivatives* $D_2^n F_i(t,\cdot)$, $t \in \mathbb{T}$, *exist, are continuous on* $\mathcal{X} \times \mathcal{Q}$ *up to order* $m \in \mathbb{N}$, *and suppose they are globally bounded, i.e. for* $2 \le n \le m$ *we have*

$$|F_i|_n := \sup_{(t,x,q)\in\mathbb{T}\times\mathcal{X}\times\mathcal{Q}} \|D_2^n F_i(t,x;q)\| < \infty \quad \textit{for } i = 1,2.$$

Remark 3.1. (1) It is easy to see that the existence of suitable values for δ follows from (3.5). Since we have $0 < \delta < \delta_{\max}$ there exist functions $c \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$ such that $a + \delta \lhd c \lhd b - \delta$ and in addition $a+\delta, b-\delta$ are positively regressive.

(2) As a consequence of (3.4), the partial derivatives D_2F_i are globally bounded on $\mathbb{T} \times \mathcal{X} \times \mathcal{Q}$ by the Lipschitz constants L_i for $i = 1,2$.

(3) Under Hypothesis 3.1(i)–(ii) the solutions $\varphi(\cdot;\tau,\xi;\theta,q)$ exist and are unique on $\mathbb{T}_\tau^{+}$ for arbitrary $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ and $\theta \in \mathbb{F}$, $q \in \mathcal{Q}$ (cf. [Pöt02, p. 38, Satz 1.2.17(a)]).

(4) By means of a cut-off-technique using radial retractions (cf. [AW96]), we can replace the strong assumption on the existence of $L_i < \infty$, $i = 1, 2$, and (3.5) by

$$\lim_{x,\bar{x}\to 0} \frac{F_i(t, x; q) - F_i(t, \bar{x}; q)}{\|x - \bar{x}\|} = 0 \quad \text{uniformly in } t \in \mathbb{T},\ q \in \mathcal{Q}.$$

Similarly, using C^m-bump functions (cf., for instance, [KM97]), one substitutes the existence of $|F_i|_n$, $i = 1, 2$, $n \in \{2, \dots, m\}$ and (3.5) by

$$\lim_{x\to 0} D_2 F_i(t, x; q) = 0 \quad \text{uniformly in } t \in \mathbb{T},\ q \in \mathcal{Q}.$$

Then, however, the obtained results hold only locally in a neighborhood of the origin.

At this point we transplant most of our technical preparations into an abstract lemma. It particularly allows to characterize the quasibounded solutions of (2.2) as fixed points of a suitable operator.

Lemma 3.2. *Assume Hypothesis 3.1(i)–(ii), choose $\tau \in \mathbb{T}$ fixed and set $\delta_{\max} := \frac{\lfloor b-a \rfloor}{2}$. Then for growth rates $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd c \lhd b$, the operator $\mathcal{T}_\tau : \mathcal{X}^+_{\tau,c} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}^+_{\tau,c}$,*

$$\begin{aligned} \mathcal{T}_\tau(\nu; x_0, \theta, q) :=& \Phi_A(\cdot, \tau) Q(\tau) x_0 + \int_\tau^\cdot \Phi_A(\cdot, \sigma(s)) Q(\sigma(s)) H_\theta(s, \nu(s); q)\, \Delta s \\ & - \int_\cdot^\infty \bar{\Phi}_A(\cdot, \sigma(s)) P(\sigma(s)) H_\theta(s, \nu(s); q)\, \Delta s \end{aligned} \tag{3.6}$$

is well-defined and has, for fixed $x_0 \in \mathcal{X}$, $\theta \in \Theta$, $q \in \mathcal{Q}$, the following properties:

(a) *$\nu : \mathbb{T}^+_\tau \to \mathcal{X}$ is a c^+-quasibounded solution of equation (2.2) with $Q(\tau)\nu(\tau) = Q(\tau)x_0$, if and only if $\nu \in \mathcal{X}^+_{\tau,c}$ solves the fixed point problem*

$$\nu = \mathcal{T}_\tau(\nu; x_0, \theta, q). \tag{3.7}$$

Moreover, in case $a + \delta \unlhd c \unlhd b - \delta$, we have:

(b) *$\mathcal{T}_\tau(\cdot; x_0, \theta, q)$ is a uniform contraction with Lipschitz constant*

$$\operatorname{Lip} \mathcal{T}_\tau(\cdot; x_0, \theta, q) \le L(\theta) < 1, \tag{3.8}$$

where $L(\theta) := \frac{K_1+K_2}{\delta}(L_1 + |\theta|\, L_2)$,

(c) *the unique fixed point $\nu^*_\tau(x_0, \theta, q) \in \mathcal{X}^+_{\tau,c}$ of $\mathcal{T}_\tau(\cdot; x_0, \theta, q)$ does not depend on the growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, it satisfies $\nu^*_\tau(0, \theta, q) = 0$, $\nu^*_\tau(x_0, \theta, q) = \nu^*_\tau(Q(\tau)x_0, \theta, q)$ and we have*

$$\operatorname{Lip} P(\tau)\nu^*_\tau(\cdot, \theta, q)(\tau) \le \frac{K_1 K_2 (L_1 + |\theta|\, L_2)}{\delta - (K_1 + K_2)(L_1 + |\theta|\, L_2)}, \tag{3.9}$$

$$\operatorname{Lip} \nu^*_\tau(x_0, \cdot, q) \le \frac{\delta K_1 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|x_0\|, \tag{3.10}$$

(d) *for $c \in \Gamma$ the mapping $\nu^*_\tau : \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}^+_{\tau,c}$ is continuous.*

Proof. Let $\tau \in \mathbb{T}$ be arbitrarily fixed, and choose a growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a + \delta \unlhd c \unlhd b - \delta$. We show the well-definedness of the operator $\mathcal{T}_\tau$. Thereto, let $x_0 \in \mathcal{X}$, $\theta \in \Theta$ and $q \in \mathcal{Q}$ arbitrary. For $\nu, \bar{\nu} \in \mathcal{X}^+_{\tau,c}$ we obtain

$$\begin{aligned} & \|\mathcal{T}_\tau(\nu; x_0, \theta, q)(t) - \mathcal{T}_\tau(\bar{\nu}; x_0, \theta, q)(t)\|\, e_c(\tau, t) \\ \overset{(3.6)}{\le}& \left\| \int_\tau^t \Phi_A(t, \sigma(s)) Q(\sigma(s)) \left[H_\theta(s, \nu(s); q) - H_\theta(s, \bar{\nu}(s); q)\right] \Delta s \right\| e_c(\tau, t) \\ & + \left\| \int_t^\infty \bar{\Phi}_A(t, \sigma(s)) P(\sigma(s)) \left[H_\theta(s, \nu(s); q) - H_\theta(s, \bar{\nu}(s); q)\right] \Delta s \right\| e_c(\tau, t) \\ \overset{(3.2)}{\le}& K_1 \int_\tau^t e_a(t, \sigma(s)) \|H_\theta(s, \nu(s); q) - H_\theta(s, \bar{\nu}(s); q)\|\, \Delta s e_c(\tau, t) \\ & + K_2 \int_t^\infty e_b(t, \sigma(s)) \|H_\theta(s, \nu(s); q) - H_\theta(s, \bar{\nu}(s); q)\|\, \Delta s e_c(\tau, t) \end{aligned} \tag{3.11}$$

$$\overset{(3.4)}{\leq} \left(K_1 \int_\tau^t e_a(t,\sigma(s)) e_c(s,t)\,\Delta s + K_2 \int_t^\infty e_b(t,\sigma(s)) e_c(s,t)\,\Delta s \right) \cdot (L_1 + |\theta|\, L_2)\, \|\nu - \bar{\nu}\|_{\tau,c}^+$$

$$\leq \left(\frac{K_1}{\lfloor c-a \rfloor} + \frac{K_2}{\lfloor b-c \rfloor} \right) (L_1 + |\theta|\, L_2)\, \|\nu - \bar{\nu}\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+,$$

where we have evaluated the integrals using [Pöt02, p. 65, Lemma 1.3.29]. To verify that $\mathcal{T}_\tau$ is well-defined, we observe

$$\|\mathcal{T}_\tau(\nu; x_0, \theta, q)(t)\|\, e_c(\tau, t)$$

$$\leq \|\mathcal{T}_\tau(0; x_0, \theta, q)(t)\|\, e_c(\tau,t) + \|\mathcal{T}_\tau(\nu; x_0, \theta, q)(t) - \mathcal{T}_\tau(0; x_0, \theta, q)(t)\|\, e_c(\tau,t)$$

$$\overset{(3.3)}{\leq} \|\Phi_A(t,\tau) Q(\tau) x_0\|\, e_c(\tau,t) + \|\mathcal{T}_\tau(\nu; x_0, \theta, q) - \mathcal{T}_\tau(0; x_0, \theta, q)\|_{\tau,c}^+$$

$$\overset{(3.2)}{\leq} K_1 \|x_0\| + \left(\frac{K_1}{\lfloor c-a \rfloor} + \frac{K_2}{\lfloor b-c \rfloor} \right) (L_1 + |\theta|\, L_2)\, \|\nu\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+$$

and taking the supremum over $t \in \mathbb{T}_\tau^+$ implies $\mathcal{T}_\tau(\nu; x_0, \theta, q) \in \mathcal{X}_{\tau,c}^+$.

(a) Let $x_0 \in \mathcal{X}$, $\theta \in \Theta$ and $q \in \mathcal{Q}$ be arbitrary.

$(\Rightarrow)$ If $\nu \in \mathcal{X}_{\tau,c}^+$ is a solution of (2.2) with $Q(\tau)\nu(\tau) = Q(\tau)x_0$, then ν also solves the linear-inhomogeneous dynamic equation

$$x^\Delta = A(t)x + H_\theta(t, \nu(t); q) \tag{3.12}$$

on $\mathbb{T}_\tau^+$, where the inhomogeneous part satisfies

$$\|H_\theta(t,\nu(t);q)\|\, e_c(\tau,t) \overset{(3.3)}{=} \|H_\theta(t,\nu(t);q) - H_\theta(t,0;q)\|\, e_c(\tau,t)$$

$$\overset{(3.3)}{\leq} (L_1 + |\theta|\, L_2)\, \|\nu\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+$$

and is therefore in $\mathcal{X}_{\tau,c}^+$. Then [Pöt02, p. 103, Satz 2.2.4(a)] implies that ν is uniquely determined and given by the right hand side of (3.6). So ν satisfies (3.7).

$(\Leftarrow)$ If $\nu \in \mathcal{X}_{\tau,c}^+$ solves the fixed point problem (3.7), then a direct computation in (3.6) yields that ν solves the dynamic equation (3.12) and consequently also (2.2) (cf. [Pöt02, p. 105]). Moreover, from (3.7), (3.6) and [Pöt02, p. 86, Korollar 2.1.9(a)] we have $Q(\tau)\nu(\tau) = Q(\tau)x_0$.

From now on, let $a + \delta \trianglelefteq c \trianglelefteq b - \delta$.

(b) Passing over to the least upper bound for $t \in \mathbb{T}_\tau^+$ in (3.11) yields the estimate

$$\|\mathcal{T}_\tau(\nu; x_0, \theta, q) - \mathcal{T}_\tau(\bar{\nu}; x_0, \theta, q)\|_{\tau,c}^+ \leq L(\theta)\, \|\nu - \bar{\nu}\| \quad \text{for } \nu, \bar{\nu} \in \mathcal{X}_{\tau,c}^+$$

and our choice of δ in Hypothesis 3.1(ii) guarantees $L(\theta) < 1$ for $\theta \in \Theta$. Therefore, the contraction mapping principle implies that there exists a unique fixed point $\nu_\tau^*(x_0, \theta, q) \in \mathcal{X}_{\tau,c}^+$ of $\mathcal{T}_\tau(\cdot; x_0, \theta, q)$, which moreover satisfies

$$\|\nu_\tau^*(x_0, \theta, q)\|_{\tau,c}^+ \leq \frac{K_1}{1 - L(\theta)} \|x_0\| \quad \text{for } \nu \in \mathcal{X}_{\tau,c}^+. \tag{3.13}$$

(c) The fixed point $\nu_\tau^*(x_0, \theta, q) \in \mathcal{X}_{\tau,c}^+$ is independent of the growth rate $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a+\delta \trianglelefteq c \trianglelefteq b - \delta$, because with (2.4) we have the inclusion $\mathcal{X}_{\tau,a+\delta}^+ \subseteq \mathcal{X}_{\tau,c}^+$, and thus, every operator $\mathcal{T}_\tau(\cdot; x_0, \theta, q) : \mathcal{X}_{\tau,c}^+ \to \mathcal{X}_{\tau,c}^+$ has the same fixed point as its restriction $\mathcal{T}_\tau(\cdot; x_0, \theta, q)|_{\mathcal{X}_{\tau,a+\delta}^+}$. Using the assumption (3.3) and the uniqueness of solutions (cf. [Pöt02, p. 38, Satz 1.2.17(a)]), we see $\varphi(t; \tau, 0; \theta, q) \equiv 0$ on $\mathbb{T}_\tau^+$ and since trivially $\varphi(\cdot; \tau, 0; \theta, q) \in \mathcal{X}_{\tau,c}^+$ holds, the assertion (a) with $x_0 = 0$ implies that $\varphi(\cdot; \tau, 0; \theta, q)$ solves the fixed point equation (3.7). This fixed point, in turn, is unique and so we get $\nu_\tau^*(0; \theta, q) = \varphi(\cdot; \tau, 0; \theta, q) = 0$. Directly from (3.6) we obtain the identity $\nu_\tau^*(Q(\tau)x_0, \theta, q) = \mathcal{T}_\tau(\nu_\tau^*(Q(\tau)x_0, \theta, q); Q(\tau)x_0, \theta, q) = \mathcal{T}_\tau(\nu_\tau^*(Q(\tau)x_0, \theta, q); x_0, \theta, q)$ and therefore, $\nu_\tau^*(Q(\tau)x_0, \theta, q)$ is the unique fixed point of $\mathcal{T}_\tau(\cdot; x_0, \theta, q)$, i.e., we have

$$\nu_\tau^*(x_0, \theta, q) = \nu_\tau^*(Q(\tau)x_0, \theta, q) \quad \text{for } x \in \mathcal{X},\ \theta \in \Theta,\ q \in \mathcal{Q}. \tag{3.14}$$

To prove the Lipschitz estimates (3.9), (3.10), we suppress the dependence on the fixed parameter $q \in \mathcal{Q}$. To this end, consider $x_0, \bar{x}_0 \in \mathcal{X}$, fixed $\theta \in \Theta$ and the corresponding fixed points $\nu_\tau^*(x_0,\theta), \nu_\tau^*(\bar{x}_0,\theta) \in \mathcal{X}_{\tau,c}^+$ of $\mathcal{T}_\tau(\cdot;x_0,\theta)$ and $\mathcal{T}_\tau(\cdot;\bar{x}_0,\theta)$, respectively. We have

$$\begin{aligned}\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(\bar{x}_0,\theta)\|_{\tau,c}^+ &\overset{(3.7)}{\leq} \|\mathcal{T}_\tau(\nu_\tau^*(x_0,\theta);x_0,\theta)-\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);x_0,\theta)\|_{\tau,c}^+ \\ &\quad + \|\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);x_0,\theta)-\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);\bar{x}_0,\theta)\|_{\tau,c}^+ \\ &\overset{(3.8)}{\leq} L(\theta)\,\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(\bar{x}_0,\theta)\|_{\tau,c}^+ \\ &\quad + \|\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);x_0,\theta)-\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);\bar{x}_0,\theta)\|_{\tau,c}^+\,,\end{aligned}$$

and thus,

$$\begin{aligned}\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(\bar{x}_0,\theta)\|_{\tau,c}^+ &\leq \frac{1}{1-L(\theta)}\,\|\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);x_0,\theta)-\mathcal{T}_\tau(\nu_\tau^*(\bar{x}_0,\theta);\bar{x}_0,\theta)\|_{\tau,c}^+ \\ &\overset{(3.6)}{=} \frac{1}{1-L(\theta)}\sup_{t\in\mathbb{T}_\tau^+}\|\Phi_A(t,\tau)Q(\tau)\,(x_0-\bar{x}_0)\|\,e_c(\tau,t) \overset{(3.2)}{\leq} \frac{K_1}{1-L(\theta)}\,\|x_0-\bar{x}_0\|\,.\end{aligned} \tag{3.15}$$

Moreover, directly from (3.6) and (3.7) we get the identity

$$P(\cdot)\nu_\tau^*(x_0,\theta) \overset{(3.1)}{=} -\int_\cdot^\infty \bar{\Phi}_A(\cdot,\sigma(s))Q(\sigma(s))H_\theta(s,\nu_\tau^*(x_0,\theta)(s))\,\Delta s$$

and similarly to the proof of (b) this yields

$$\|P(\cdot)\,[\nu_\tau^*(x_0,\theta)-\nu_\tau^*(\bar{x}_0,\theta)]\|_{\tau,c}^+ \leq \frac{K_2}{\lfloor b-c\rfloor}\,(L_1+|\theta|\,L_2)\,\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(\bar{x}_0,\theta)\|_{\tau,c}^+\,,$$

which, together with (3.15), implies (3.9). Analogously, for fixed $x_0 \in \mathcal{X}$ and $\theta, \bar{\theta} \in \Theta$ we obtain

$$\begin{aligned}\left\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(x_0,\bar{\theta})\right\|_{\tau,c}^+ &\overset{(3.8)}{\leq} L(\theta)\left\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(x_0,\bar{\theta})\right\|_{\tau,c}^+ \\ &\quad + \left\|\mathcal{T}_\tau(\nu_\tau^*(x_0,\bar{\theta});x_0,\theta)-\mathcal{T}_\tau(\nu_\tau^*(x_0,\bar{\theta});x_0,\bar{\theta})\right\|_{\tau,c}^+\end{aligned}$$

and consequently, by known arguments, using (3.3)–(3.4)

$$\begin{aligned}&\left\|\nu_\tau^*(x_0,\theta)-\nu_\tau^*(x_0,\bar{\theta})\right\|_{\tau,c}^+ \\ &\overset{(3.6)}{\leq} \frac{1}{1-L(\theta)}\left(\sup_{\tau\preceq t}\left\|\int_\tau^t \Phi_A(t,\sigma(s))Q(\sigma(s))F_2(s,\nu_\tau^*(x_0,\bar{\theta})(s))\,\Delta s\right\|e_c(\tau,t)\right. \\ &\qquad \left.+\sup_{\tau\preceq t}\left\|\int_t^\infty \bar{\Phi}_A(t,\sigma(s))P(\sigma(s))F_2(s,\nu_\tau^*(x_0,\bar{\theta})(s))\,\Delta s\right\|e_c(\tau,t)\right)\left|\theta-\bar{\theta}\right| \\ &\overset{(3.2)}{\leq} \frac{L_2}{1-L(\theta)}\left(\frac{K_1}{\lfloor c-a\rfloor}+\frac{K_2}{\lfloor b-c\rfloor}\right)\left\|\nu_\tau^*(x_0,\bar{\theta})\right\|_{\tau,c}^+\left|\theta-\bar{\theta}\right| \\ &\overset{(3.13)}{\leq} \frac{K_1(K_1+K_2)L_2}{\delta(1-L(\theta))(1-L(\bar{\theta}))}\,\|x_0\|\left|\theta-\bar{\theta}\right|.\end{aligned}$$

If we keep in mind the inequality $L(\theta) \leq 2\frac{K_1+K_2}{\delta}L_1$ for $\theta \in \Theta$, then (3.10) follows. Therefore we have established the assertion (c).

(d) In order to show the continuity of $\nu_\tau^* : \mathcal{X}\times\Theta\times\mathcal{Q} \to \mathcal{X}_{\tau,c}^+$, with $c \in \Gamma$, by [AW96, Lemma B.4] it suffices to prove the following limit relation

$$\lim_{(\theta,q)\to(\theta_0,q_0)} \nu_\tau^*(x_0,\theta,q) = \nu_\tau^*(x_0,\theta_0,q_0) \quad \text{for } \theta_0 \in \Theta,\ q \in \mathcal{Q}, \tag{3.16}$$

since the mappings $\nu_\tau^*(\cdot,\theta,q) : \mathcal{X} \to \mathcal{X}_{\tau,c}^+$ are globally Lipschitzian, where (3.15) implies the uniform estimate $\operatorname{Lip}\nu_\tau^*(\cdot,\theta,q) \leq \frac{\delta K_1}{\delta-2(K_1+K_2)L_1}$ for all $\theta \in \Theta$, $q \in \mathcal{Q}$. We suppress the dependence on $x_0 \in \mathcal{X}$, choose $\theta_0 \in \Theta$, $q_0 \in \mathcal{Q}$ fixed and obtain from (3.2) for $\theta \in \Theta$, $q \in \mathcal{Q}$ the estimate

$$\|\nu_\tau^*(\theta,q)(t)-\nu_\tau^*(\theta_0,q_0)(t)\|$$

$$\stackrel{(3.6)}{\leq} K_1 \int_\tau^t e_a(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta,q)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s$$
$$+ K_2 \int_t^\infty e_b(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta,q)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s$$

for $t \in \mathbb{T}_\tau^+$. Subtraction and addition of $H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q)$ in the corresponding norms leads to $\|\nu_\tau^*(\theta,q)(t) - \nu_\tau^*(\theta_0,q_0)(t)\| \leq I_1 + I_2 + I_3 + I_4$ for all $t \in \mathbb{T}_\tau^+$, where (cf. (3.2), (3.4))

$$I_1 := K_1 \left(L_1 + |\theta| L_2\right) \int_\tau^t e_a(t,\sigma(s)) \left\| \nu_\tau^*(\theta,q)(s) - \nu_\tau^*(\theta_0,q_0)(s) \right\| \Delta s,$$
$$I_2 := K_1 \int_\tau^t e_a(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s,$$
$$I_3 := K_2 \left(L_1 + |\theta| L_2\right) \int_t^\infty e_b(t,\sigma(s)) \left\| \nu_\tau^*(\theta,q)(s) - \nu_\tau^*(\theta_0,q_0)(s) \right\| \Delta s,$$
$$I_4 := K_2 \int_t^\infty e_b(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s.$$

From this, and with the aid of [Pöt02, p. 65, Lemma 1.3.29], we obtain the estimate

$$\begin{aligned} &\left\| \nu_\tau^*(\theta,q)(t) - \nu_\tau^*(\theta_0,q_0)(t) \right\| e_c(\tau,t) \\ &\leq I_1 e_c(\tau,t) + I_3 e_c(\tau,t) + L(\theta) \left\| \nu_\tau^*(\theta,q) - \nu_\tau^*(\theta_0,q_0) \right\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+. \end{aligned}$$

Hence, by passing over to the least upper bound for $t \in \mathbb{T}_\tau^+$, and using (3.7), we get the inequality $\|\nu_\tau^*(\theta,q) - \nu_\tau^*(\theta_0,q_0)\|_{\tau,c}^+ \leq \frac{\delta \max\{K_1,K_2\}}{\delta - 2(K_1+K_2)L_1} \sup_{\tau \preceq t} U(t,\theta,q)$ with

$$U(t,\theta,q) := e_c(\tau,t) \int_\tau^t e_a(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s$$
$$+ e_c(\tau,t) \int_t^\infty e_b(t,\sigma(s)) \left\| H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q) - H_{\theta_0}(s,\nu_\tau^*(\theta_0,q_0)(s);q_0) \right\| \Delta s.$$

Therefore, it is sufficient to prove

$$\lim_{(\theta,q)\to(\theta_0,q_0)} \sup_{\tau \preceq t} U(t,\theta,q) = 0 \tag{3.17}$$

in order to show the limit relation (3.16). We proceed indirectly and assume (3.17) does not hold. Then there is an $\varepsilon > 0$ and sequences $(\theta_i)_{i\in\mathbb{N}}$, $(q_i)_{i\in\mathbb{N}}$ in Θ and $\mathcal{Q}$, resp., with $\lim_{i\to\infty}(\theta_i,q_i) = (\theta_0,q_0)$ and $\sup_{\tau\preceq t} U(t,\theta_i,q_i) > \varepsilon$ for $i \in \mathbb{N}$. This implies the existence of a sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ such that

$$U(t_i,\theta_i,q_i) > \varepsilon \quad \text{for } i \in \mathbb{N}. \tag{3.18}$$

From now on we consider $a + \delta \lhd c$, choose a fixed growth rate $d \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $a + \delta \lhd d \lhd c$ and remark that the inequality $d \lhd c$ will play an important role below. Because of Hypothesis 3.1(ii) and $\nu_\tau^*(\theta,q) \in \mathcal{X}_{\tau,d}^+$ we get (cf. (3.3))

$$\left\| H_\theta(s,\nu_\tau^*(\theta_0,q_0)(s);q) \right\| \stackrel{(3.4)}{\leq} \left(L_1 + |\theta| L_2\right) \left\| \nu_\tau^*(\theta_0,q_0) \right\|_{\tau,d}^+ e_d(s,\tau) \quad \text{for } s \in \mathbb{T}_\tau^+,$$

and the triangle inequality, as well as $|\theta| L_2 \leq L_1$, leads to

$$\begin{aligned} U(t,\theta,q) &\leq 4L_1 \left\| \nu_\tau^*(\theta_0,q_0) \right\|_{\tau,d}^+ e_c(\tau,t) \int_\tau^t e_a(t,\sigma(s)) e_d(s,\tau)\, ds \\ &\quad + 4L_1 \left\| \nu_\tau^*(\theta_0,q_0) \right\|_{\tau,d}^+ e_c(\tau,t) \int_t^\infty e_b(t,\sigma(s)) e_d(s,\tau)\, ds \\ &\leq 8\tfrac{L_1}{\delta} \left\| \nu_\tau^*(\theta_0,q_0) \right\|_{\tau,d}^+ e_{d\ominus c}(t,\tau) \quad \text{for } t \in \mathbb{T}_\tau^+, \end{aligned}$$

where we have evaluated the integrals using [Pöt02, p. 65, Lemma 1.3.29]. Because of $d \lhd c$ and [Pöt02, p. 63, Lemma 1.3.26], passing over to the limit $t \to \infty$ yields $\lim_{t\to\infty} U(t,\theta,q) = 0$ uniformly in $\theta \in \Theta$, $q \in \mathcal{Q}$, and taking into account (3.18), the sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ has to be bounded above, i.e. there exists a time $T \in \mathbb{T}_\tau^+$, $\tau \prec T$, with $t_i \preceq T$ for all $i \in \mathbb{N}$. Hence, by [Hil90, Theorem 7.4(i)],

$$\begin{aligned} U(t_i, \theta_i, q_i) \leq &\int_\tau^T e_c(\tau, \sigma(s)) \left\| H_{\theta_i}(s, \nu_\tau^*(\theta_0, q_0)(s); q_i) - H_{\theta_0}(s, \nu_\tau^*(\theta_0, q_0)(s); q_0) \right\| \Delta s \\ &+ \int_\tau^\infty e_c(\tau, \sigma(s)) e_{b\ominus c}(T, \sigma(s)) \\ &\cdot \left\| H_{\theta_i}(s, \nu_\tau^*(\theta_0, q_0)(s); q_i) - H_{\theta_0}(s, \nu_\tau^*(\theta_0, q_0)(s); q_0) \right\| \Delta s \quad \text{for } i \in \mathbb{N}, \end{aligned}$$

where the first integral tends to 0 for $i \to \infty$ by the continuity of F_1, F_2. Likewise, continuity of F_1, F_2 implies $\lim_{i\to\infty} H_{\theta_i}(s, \nu_\tau^*(\theta_0, q_0)(s); q_i) = H_{\theta_0}(s, \nu_\tau^*(\theta_0, q_0)(s); q_0)$ and with Lebesgue's dominated convergence theorem for the integral on $\mathbb{T}$ (cf. [Nei01, p. 161, Nr. 313]), we get the convergence of the improper integral to 0 for $i \to \infty$. Thus, we derived the relation $\lim_{i\to\infty} U(t_i, \theta_i, q_i) = 0$, which obviously contradicts (3.18). This means (3.16) is verified and the proof of Lemma 3.2 is finished. □

Having collected the preparations in Lemma 3.2, we may now head for our general version of the Hadamard-Perron theorem. Related results for dynamic equations on measure chains (or time scales) can be found in [Kel99, Pöt03, PS04]. Nevertheless, these references assume a decoupled linear part.

Before proceeding, in order to address differentiability issues we introduce a notation needed in the next theorem; for growth rates $a, b \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $m \in \mathbb{N}$ we get from [PS04, Lemma 4.1] that

$$m \odot a \lhd b \quad \Rightarrow \quad \rho_s^m[a,b] := \inf_{t\in\mathbb{T}} \lim_{h\searrow\mu^*(t)} \frac{1+ha(t)}{h}\left(\sqrt[m]{\frac{1+ha(t)+1+hb(t)}{1+ha(t)+(1+ha(t))^m}} - 1\right) > 0,$$

$$a \lhd m \odot b \quad \Rightarrow \quad \rho_r^m[a,b] := \inf_{t\in\mathbb{T}} \lim_{h\searrow\mu^*(t)} \frac{1+hb(t)}{h}\left(1 - \sqrt[m]{\frac{1+ha(t)+1+hb(t)}{1+hb(t)+(1+hb(t))^m}}\right) > 0.$$

Theorem 3.3 (invariant fiber bundles). *Assume Hypothesis 3.1(i)–(ii) is fulfilled with* $\delta_{\max} = \frac{1}{2}\lfloor b-a \rfloor$. *Then the following statements are true:*

(a) *For all* $\theta \in \Theta$, $q \in \mathcal{Q}$ *the set*

$$S(\theta, q) := \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot; \tau, x_0; \theta, q) \in \mathcal{X}_{\tau,c}^+ \text{ for all } c \in \Gamma\right\}$$

is an invariant fiber bundle of (2.2) *possessing the representation*

$$S(\theta, q) = \{(\tau, \xi + s(\tau, \xi; \theta, q)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \xi \in \mathcal{R}(Q(\tau))\} \tag{3.19}$$

with a uniquely determined continuous mapping $s : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfying*

$$s(\tau, x_0; \theta, q) = s(\tau, Q(\tau)x_0; \theta, q) \in \mathcal{R}(P(\tau)) \quad \text{for } \tau \in \mathbb{T},\ x_0 \in \mathcal{X} \tag{3.20}$$

and the invariance equation

$$P(t)\varphi(t; \tau, x_0; \theta, q) = s(t, Q(t)\varphi(t; \tau, x_0; \theta, q); \theta, q) \quad \text{for } (\tau, x_0) \in S(\theta, q),\ \tau \preceq t. \tag{3.21}$$

Furthermore, it holds:

(a_1) $s(\tau, 0; \theta, q) \equiv 0$ *on* $\mathbb{T} \times \Theta \times \mathcal{Q}$,

(a_2) $s : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfies the Lipschitz estimates*

$$\begin{aligned} \operatorname{Lip} s(\tau, \cdot; \theta, q) &\leq \frac{K_1 K_2 (L_1 + |\theta| L_2)}{\delta - (K_1 + K_2)(L_1 + |\theta| L_2)}, \\ \operatorname{Lip} s(\tau, x_0; \cdot, q) &\leq \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|x_0\| \end{aligned} \tag{3.22}$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$,

(a_3) *if additionally Hypothesis 3.1(iii) and the gap condition*

$$m_s \odot a \lhd b$$

holds for some $m_s \in \{1, \dots, m\}$, *and if we set* $\delta_{\max} := \min\left\{\frac{\lfloor b-a\rfloor}{2}, \rho_s^{m_s}[a,b]\right\}$, *then the partial derivatives* $D^n_{(2,3)}s$ *exist, are continuous up to order* m_s, *and there exist reals* $M_s^n, N_s^n > 0$, *such that*

$$\|D_2^n s(\tau, x_0; \theta, q)\| \le M_s^n \quad \text{for } 1 \le n \le m_s,$$
$$\|D_3 D_2^n s(\tau, x_0; \theta, q)\| \le N_s^n \|x_0\| \quad \text{for } 0 \le n < m_s$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$,

(a_4) *if the dynamic equation* (2.2) *and* Q *are* T*-periodic for some* $T > 0$, *then* $s(\cdot, x_0; \theta, q)$ *is* T*-periodic for all* $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$.

(b) *In case* $\mathbb{T}$ *is unbounded below, then for all* $\theta \in \Theta$, $q \in \mathcal{Q}$ *the set*

$$R(\theta, q) := \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l}\text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of (2.2)}\\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}^-_{\tau,c} \text{ for all } c \in \Gamma\end{array}\right\}$$

is an invariant fiber bundle of (2.2) *possessing the representation*

$$R(\theta, q) = \{(\tau, \eta + r(\tau, \eta; \theta, q)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{R}(P(\tau))\}$$

with a uniquely determined mapping $r : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfying*

$$r(\tau, x_0; \theta, q) = r(\tau, P(\tau)x_0; \theta, q) \in \mathcal{R}(Q(\tau)) \quad \text{for } \tau \in \mathbb{T},\ x_0 \in \mathcal{X}$$

and the invariance equation

$$Q(t)\varphi(t; \tau, x_0; \theta, q) = r(t, P(t)\varphi(t; \tau, x_0; \theta, q); \theta, q) \quad \text{for } (\tau, x_0) \in R(\theta, q),\ \tau \preceq t.$$

Furthermore, it holds:

(b_1) $r(\tau, 0; \theta, q) \equiv 0$ *on* $\mathbb{T} \times \Theta \times \mathcal{Q}$,

(b_2) $r : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfies the Lipschitz estimates*

$$\operatorname{Lip} r(\tau, \cdot; \theta, q) \le \frac{K_1 K_2 (L_1 + |\theta| L_2)}{\delta - (K_1 + K_2)(L_1 + |\theta| L_2)}, \tag{3.23}$$
$$\operatorname{Lip} r(\tau, x_0; \cdot, q) \le \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2)L_1]^2} \|x_0\|$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$,

(b_3) *if additionally Hypothesis 3.1(iii) and the gap condition*

$$a \lhd m_r \odot b$$

holds for some $m_r \in \{1, \dots, m\}$, *and if we set* $\delta_{\max} := \min\left\{\frac{\lfloor b-a\rfloor}{2}, \rho_r^{m_r}[a,b]\right\}$, *then the partial derivatives* $D^n_{(2,3)}r$ *exist, are continuous up to order* m_r, *and there exist reals* $M_r^n, N_r^n > 0$, *such that*

$$\|D_2^n r(\tau, x_0; \theta, q)\| \le M_r^n \quad \text{for } 1 \le n \le m_r,$$
$$\|D_3 D_2^n r(\tau, x_0; \theta, q)\| \le N_r^n \|x_0\| \quad \text{for } 0 \le n < m_r$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$,

(b_4) *if the dynamic equation* (2.2) *and* Q *are* T*-periodic for some* $T > 0$, *then* $r(\cdot, x_0; \theta, q)$ *is* T*-periodic for all* $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$.

(c) *In case* $\mathbb{T}$ *is also unbounded below, and if*

$$L_1 < \frac{\delta}{2(K_1 + K_2 + \max\{K_1, K_2\})}, \tag{3.24}$$

only the zero solution of equation (2.2) *is contained both in* $S(\theta, q)$ *and* $R(\theta, q)$, *i.e.* $S(\theta, q) \cap R(\theta, q) = \mathbb{T} \times \{0\}$ *for all* $\theta \in \Theta$, $q \in \mathcal{Q}$, *and hence the zero solution is the only* $c^{\pm}$*-quasibounded solution of* (2.2) *for any growth rate* $c \in \Gamma$.

Proof. Let $\tau \in \mathbb{T}$ be arbitrary, but fixed, and let us choose a $c \in \Gamma$.

(a) The parameters $\theta \in \Theta$, $q \in \mathcal{Q}$ are arbitrary. We want to show that $S(\theta, q)$ is an invariant fiber bundle of (2.2). By definition, the solution $\varphi(\cdot; \tau, \xi_0; \theta, q)$ is c^+-quasibounded for arbitrary pairs of initial values $(\tau, \xi_0) \in S(\theta, q)$. The cocycle property (2.3) now implies for any time $t_0 \in \mathbb{T}_\tau^+$ that $\varphi(t; t_0, \varphi(t_0; \tau, \xi_0; \theta, q); \theta, q) \equiv \varphi(t; \tau, \xi_0; \theta, q)$ on $\mathbb{T}_{t_0}^+$. Hence also $\varphi(\cdot; t_0, \varphi(t_0; \tau, \xi_0; \theta, q); \theta, q)$ is a c^+-quasibounded function and this yields the inclusion $(t_0, \varphi(t_0; \tau, \xi_0; \theta, q)) \in S(\theta, q)$ for $t_0 \in \mathbb{T}_\tau^+$.

For $x_0 \in \mathcal{X}$, by Lemma 3.2(a), the unique fixed point $\nu_\tau^*(x_0, \theta, q) \in \mathcal{X}_{\tau,c}^+$ of $\mathcal{T}_\tau(\cdot; x_0, \theta, q)$ is a solution of the dynamic equation (2.2) satisfying $Q(\tau)\nu_\tau^*(x_0, \theta, q)(\tau) = Q(\tau)x_0$. Now we define

$$s(\tau, x_0; \theta, q) := P(\tau)\nu_\tau^*(x_0, \theta, q)(\tau) \tag{3.25}$$

and evidently have $s(\tau, x_0; \theta, q) \in \mathcal{R}(P(\tau))$. In addition, Lemma 3.2(c) implies $s(\tau, x_0; \theta, q) = s(\tau, Q(\tau)x_0; \theta, q)$ (cf. (3.14)). We postpone the continuity proof for s to the end of part (a) and verify the representation (3.19) and the invariance equation (3.21) now.

$(\subseteq)$ Let $(\tau, x_0) \in S(\theta, q)$, i.e., $\varphi(\cdot; \tau, x_0; \theta, q)$ is c^+-quasibounded. Then $\varphi(\cdot; \tau, x_0; \theta, q)$ trivially satisfies $Q(\tau)\varphi(\tau; \tau, x_0; \theta, q) = Q(\tau)x_0$ and is consequently the unique fixed point of (3.6), i.e., we have $\varphi(\cdot; \tau, x_0; \theta, q) = \nu_\tau^*(x_0, \theta, q)$ (see Lemma 3.2(a)). This implies

$$\begin{aligned} x_0 = \nu_\tau^*(x_0, \theta, q)(\tau) &= Q(\tau)\nu_\tau^*(x_0, \theta, q)(\tau) + P(\tau)\nu_\tau^*(x_0, \theta, q)(\tau) \\ &= Q(\tau)x_0 + P(\tau)\nu_\tau^*(Q(\tau)x_0, \theta, q)(\tau), \end{aligned}$$

since $\nu_\tau^*(x_0, \theta, q) = \nu_\tau^*(Q(\tau)x_0, \theta, q)$ holds due to $\mathcal{T}_\tau(\cdot; x_0, \theta, q) = \mathcal{T}_\tau(\cdot; Q(\tau)x_0, \theta, q)$ (cf. (3.6)). So, setting $\xi := Q(\tau)x_0 \in \mathcal{R}(Q(\tau))$, we have $x_0 = \xi + P(\tau)\nu_\tau^*(\xi, \theta, q) = \xi + s(\tau, \xi; \theta, q)$ by (3.25) and (3.19) is verified.

$(\supseteq)$ On the other hand, let $x_0 \in \mathcal{X}$ be of the form $x_0 = \xi + s(\tau, \xi; \theta, q)$, $\xi \in \mathcal{R}(Q(\tau))$. Then

$$\begin{aligned} x_0 &\overset{(3.25)}{=} \xi + P(\tau)\nu_\tau^*(\xi, \theta, q)(\tau) \\ &= Q(\tau)\nu_\tau^*(\xi, \theta, q)(\tau) + P(\tau)\nu_\tau^*(\xi, \theta, q)(\tau) = \nu_\tau^*(\xi, \theta, q)(\tau) \end{aligned}$$

and therefore, due to the uniqueness of solutions (cf. [Pöt02, p. 38, Satz 1.2.17(a)]), one has $\varphi(\cdot; \tau, x_0; \theta, q) = \varphi(\cdot; \tau, \nu_\tau^*(\xi, \theta, q)(\tau); \theta, q) = \nu_\tau^*(\xi, \theta, q) \in \mathcal{X}_{\tau,c}^+$.

With $(\tau, \xi_0) \in S(\theta, q)$ the invariance of $S(\theta, q)$ implies $\varphi(t; \tau, \xi_0; \theta, q) = Q(t)\varphi(t; \tau, \xi_0; \theta, q) + s(t, Q(t)\varphi(t; \tau, \xi_0; \theta, q); \theta, q)$ for $\tau \preceq t$ and multiplication with $P(t)$ yields (3.21).

(a_1) From Lemma 3.2(c) we get $s(\tau, 0; \theta, q) = P(\tau)\nu_\tau^*(0; \theta, q)(\tau) = 0$ (cf. (3.25)).

(a_2) To prove the claimed Lipschitz estimates, we suppress the dependence on $q \in \mathcal{Q}$. To this end, consider $x_0, \bar{x}_0 \in \mathcal{X}$, fixed $\theta \in \Theta$ and corresponding fixed points $\nu_\tau^*(x_0, \theta), \nu_\tau^*(\bar{x}_0, \theta) \in \mathcal{X}_{\tau,c}^+$ of $\mathcal{T}_\tau(\cdot; x_0, \theta)$ and $\mathcal{T}_\tau(\cdot; \bar{x}_0, \theta)$, respectively. One gets from Lemma 3.2(c)

$$\begin{aligned} \|s(\tau, x_0; \theta) - s(\tau, \bar{x}_0; \theta)\| &\overset{(3.25)}{=} \|P(\tau)\,[\nu_\tau^*(x_0, \theta)(\tau) - \nu_\tau^*(\bar{x}_0, \theta)(\tau)]\| \\ &\overset{(3.9)}{\le} \frac{K_1 K_2\,(L_1 + |\theta|\,L_2)}{\delta - (K_1 + K_2)\,(L_1 + |\theta|\,L_2)}\,\|x_0 - \bar{x}_0\|\,. \end{aligned}$$

Similarly, consider $x_0 \in \mathcal{X}$ fixed, $\theta, \bar{\theta} \in \Theta$ and let $\nu_\tau^*(x_0, \theta), \nu_\tau^*(x_0, \bar{\theta}) \in \mathcal{X}_{\tau,c}^+$ denote the corresponding fixed points of $\mathcal{T}_\tau(\cdot; x_0, \theta)$ and $\mathcal{T}_\tau(\cdot; x_0, \bar{\theta})$, respectively. Then we obtain

$$\begin{aligned} &\left\|s(\tau, x_0; \theta) - s(\tau, x_0; \bar{\theta})\right\| \overset{(3.25)}{=} \left\|P(\tau)\left[\nu_\tau^*(x_0, \theta)(\tau) - \nu_\tau^*(x_0, \bar{\theta})(\tau)\right]\right\| \\ &\overset{(3.2)}{\le} K_2\left\|\nu_\tau^*(x_0, \theta)(\tau) - \nu_\tau^*(x_0, \bar{\theta})(\tau)\right\| \le K_2\left\|\nu_\tau^*(x_0, \theta) - \nu_\tau^*(x_0, \bar{\theta})\right\|_{\tau,c}^+ \\ &\overset{(3.10)}{\le} \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2)L_1]^2}\,\|x_0\|\,\left|\theta - \bar{\theta}\right| \end{aligned}$$

and both Lipschitz estimates are established.

(a_3) Due to its technical complexity, we omit the differentiability proof for the mapping s. It is based on a "formal differentiation" of the fixed point identity (3.7) w.r.t. the variable $(x_0, \theta) \in \mathcal{X} \times \Theta$. Concerning the details, we refer to [PS04, Kel99], where the latter reference is particularly devoted to the dependence on the parameter $\theta \in \Theta$.

It remains to show the continuity statement for s. Thereto, let $\tau_0 \in \mathbb{T}$, $\xi_0 \in \mathcal{X}$, $\theta_0 \in \Theta$ and $q_0 \in \mathcal{Q}$. Then for arbitrary $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ and $q \in \mathcal{Q}$ we obtain the estimate

$$\begin{aligned}\|s(\tau, x_0; \theta, q) - s(\tau, \xi_0; \theta_0, q_0)\| &\overset{(3.22)}{\leq} \frac{2K_1K_2L_1}{\delta - 2(K_1+K_2)L_1}\|x_0 - \xi_0\| \\ &\quad + \|s(\tau, \xi_0; \theta, q) - s(\tau_0, \xi_0; \theta, q)\| + \|s(\tau_0, \xi_0; \theta, q) - s(\tau_0, \xi_0; \theta_0, q_0)\|\end{aligned}$$

and by Lemma 3.2(d) the third sum on the right hand side satisfies

$$\begin{aligned}&\|s(\tau_0, \xi_0; \theta, q) - s(\tau_0, \xi_0; \theta_0, q_0)\| \overset{(3.25)}{\leq} \|P(\tau_0)\| \, \|\nu_\tau^*(\xi_0, \theta, q)(\tau_0) - \nu_\tau^*(\xi_0, \theta_0, q_0)(\tau_0)\| \\ &\overset{(3.2)}{\leq} K_2 \, \|\nu_\tau^*(\xi_0, \theta, q) - \nu_\tau^*(\xi_0, \theta_0, q_0)\|^+_{\tau_0, c} \xrightarrow[(\theta,q)\to(\theta_0,q_0)]{} 0.\end{aligned}$$

Consequently, to verify the continuity of s in $(\tau_0, \xi_0, \theta_0, q_0)$, it remains to prove the limit relation

$$\lim_{\tau\to\tau_0} s(\tau, \xi_0; \theta, q) = s(\tau_0, \xi_0; \theta, q) \quad \text{uniformly in } \theta \in \Theta,\ q \in \mathcal{Q}. \tag{3.26}$$

We abbreviate $\phi(\tau; \theta, q) := \varphi(\tau; \tau_0, Q(\tau_0)\xi_0 + s(\tau_0, \xi_0; \theta, q); \theta, q)$ and remark that the solution $\phi(\cdot; \theta, q)$ of (2.2) exists in a $\mathbb{T}$-neighborhood (uniform in $\theta \in \Theta$, $q \in \mathcal{Q}$) of τ_0, due to [PS04, Satz 2.3(a)]. Moreover, as a preparation we have the estimate (cf. (a_1))

$$\begin{aligned}\|Q(\tau_0)\xi_0 + s(\tau_0, \xi_0; \theta, q)\| &\overset{(3.2)}{\leq} K_1\|\xi_0\| + \|s(\tau_0, \xi_0; \theta, q) - s(\tau_0, 0; \theta, q)\| \\ &\overset{(3.22)}{\leq} \left(K_1 + \frac{2K_1K_2L_1}{\delta - 2(K_1+K_2)L_1}\right)\|\xi_0\|\end{aligned}$$

and we therefore can apply [PS04, Satz 2.3(b)] to equation (2.2) to obtain

$$\lim_{\tau\to\tau_0} \phi(\tau; \theta, q) = \phi(\tau_0; \theta, q) \quad \text{uniformly in } \theta \in \Theta,\ q \in \mathcal{Q}. \tag{3.27}$$

By definition of ϕ we get $P(\tau_0)\phi(\tau_0; \theta, q) = s(\tau_0, \xi_0; \theta, q)$, $Q(\tau_0)\phi(\tau_0; \theta, q) = Q(\tau_0)\xi_0$ from (3.25) and (3.21) implies $P(\tau)\phi(\tau; \theta, q) = s(\tau, Q(\tau)\phi(\tau; \theta, q); \theta, q)$. Hence, we arrive at

$$\begin{aligned}\|s(\tau, \xi_0; \theta, q) - s(\tau_0, \xi_0; \theta, q)\| &\overset{(3.20)}{\leq} \|s(\tau, Q(\tau)\xi_0; \theta, q) - s(\tau, Q(\tau)\phi(\tau; \theta, q); \theta, q)\| \\ &\quad + \|s(\tau, Q(\tau)\phi(\tau; \theta, q); \theta, q) - s(\tau_0, \xi_0; \theta, q)\| \\ &\overset{(3.22)}{\leq} \frac{2K_1^2K_2L_1}{\delta - 2(K_1+K_2)L_1}\|\xi_0 - \phi(\tau; \theta, q)\| \\ &\quad + \|P(\tau)\phi(\tau; \theta, q) - P(\tau_0)\phi(\tau_0; \theta, q)\|,\end{aligned}$$

and so (3.27) readily implies the desired limit relation (3.26), because the invariant projectors $P, Q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ are continuous (cf. [Pöt02, p. 88, Satz 2.1.10]).

(a_4) Choose a growth rate $c \in \Gamma$ and an arbitrary point $\xi_0 \in \mathcal{R}(Q(\tau))$. Then the solution $\nu := \varphi(\cdot; \tau, \xi_0 + s(\tau, \xi_0; \theta, q); \theta, q)$ of (2.2) is c^+-quasibounded. Because of the T-periodicity of (2.2) and [Pöt02, p. 31, Korollar 1.2.4(b)], we know that also $\tilde{\nu} := \nu \circ \sigma_{-T}$ is a c^+-quasibounded solution, where $\sigma_T : \mathbb{T} \to \mathbb{T}$ is a function satisfying the identity $\mu(\sigma_T(t), t) \equiv T$ on $\mathbb{T}$. Hence, we have $(\sigma_T(\tau), \tilde{\nu}(\tau)) \in S(\theta, q)$ and consequently

$$\begin{aligned}s(\sigma_T(\tau), \xi_0; \theta, q) &\overset{(3.20)}{=} s(\sigma_T(\tau), Q(\tau)\nu(\sigma_{-T} \circ \sigma_T(\tau)); \theta, q) = s(\sigma_T(\tau), Q(\sigma_T(\tau))\tilde{\nu}(\sigma_T(\tau)); \theta, q) \\ &\overset{(3.21)}{=} P(\sigma_T(\tau))\tilde{\nu}(\sigma_T(\tau)) \overset{(3.20)}{=} s(\tau, \xi_0; \theta, q),\end{aligned}$$

i.e., we established the T-periodicity of $s(\cdot, \xi_0; \theta, q)$ in case $\xi_0 \in \mathcal{R}(Q(\tau))$. Now the T-periodicity of $s(\cdot, x_0; \theta, q)$ for general $x_0 \in \mathcal{X}$ follows from (3.20).

(b) Since the present part (b) of Theorem 3.3 can be proved along the same lines as part (a), we present only a sketch of the proof. Analogously to (a), for $x_0 \in \mathcal{X}$ and parameters $\theta \in \Theta$, $q \in \mathcal{Q}$, the c^--quasibounded solutions ν the dynamic equation (2.2) with $P(\tau)\nu(\tau) = P(\tau)x_0$ may be characterized as fixed points of the operator $\bar{\mathcal{T}}_\tau : \mathcal{X}^-_{\tau,c} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}^-_{\tau,c}$,

$$\bar{\mathcal{T}}_\tau(\nu; x_0, \theta, q) := \bar{\Phi}_A(\cdot, \tau)P(\tau)x_0 + \int_\tau^\cdot \bar{\Phi}_A(\cdot, \sigma(s))P(\tau)H_\theta(s, \nu(s); q)\,\Delta s + \int_{-\infty}^\cdot \Phi_A(\cdot, \sigma(s))Q(\tau)H_\theta(s, \nu(s); q)\,\Delta s.$$

Here a counterpart to the above Lemma 3.2 holds true in the Banach space $\mathcal{X}^-_{\tau,c}$. It follows from the assumption (3.5) that $\bar{\mathcal{T}}_\tau(\cdot; x_0, \theta, q)$ is a uniform contraction on $\mathcal{X}^-_{\tau,c}$ and if $\nu^*_\tau(x_0, \theta, q) \in \mathcal{X}^-_{\tau,c}$ denotes its unique fixed point we define the mapping $r : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ by $r(\tau, x_0; \theta, q) := Q(\tau)\nu^*_\tau(x_0, \theta, q)(\tau)$. The claimed properties of r can be proved using dual arguments as (a).

(c) Keep $\theta \in \Theta$ and $q \in \mathcal{Q}$ arbitrarily fixed. Let $\nu : \mathbb{T} \to \mathcal{X}$ be any $c^\pm$-quasibounded solution of (2.2). By means of Hypothesis 3.1(ii), the mapping $H_\theta(\cdot, \nu(\cdot); q)$ is $c^\pm$-quasibounded and as a consequence of [Pöt02, p. 106, Satz 2.2.7] the unique $c^\pm$-quasibounded solution of $x^\Delta = A(t)x + H_\theta(t, \nu(t); q)$, which again by [Pöt02, p. 106, Satz 2.2.7] additionally satisfies

$$\|\nu\|^\pm_{\tau,c} \overset{(3.4)}{\le} \frac{K_1 + K_2 + \max\{K_1, K_2\}}{\delta}\,(L_1 + |\theta|\,L_2)\,\|\nu\|^\pm_{\tau,c}\,.$$

Using (3.24), we therefore obtain $\nu = 0$ and the proof of Theorem 3.3 is complete. □

4. Hierarchies of Invariant Fiber Bundles

In the preceding section we provided conditions under which the semilinear dynamic equation (2.2) possesses two nontrivial invariant fiber bundles intersecting along the trivial solution. Now we are going to extend this result to obtain conditions guaranteeing more than just two invariant fiber bundles. Actually, we present conditions implying the existence of a so-called extended hierarchy of invariant fiber bundles canonically ordered via set-theoretical inclusion.

Hereto, we turn our attention to dynamic equations of the form (2.2), where more information is known about their linear part (2.1). Precisely, for an integer $N \ge 2$ we say that projectors $P_1, \dots, P_N$ are *complementary*, in case the identities

$$(4.1) \qquad P_1(t) + \ldots + P_N(t) \equiv I_{\mathcal{X}}, \qquad P_i(t)P_j(t) \equiv 0 \quad \text{for } i \ne j$$

hold on $\mathbb{T}$. Then the system (2.1) is said to have an *exponential N-splitting*, if there exist complementary invariant projectors $P_1, \dots, P_N$, reals $K_1^+, \dots, K_{N-1}^+, K_1^-, \dots, K_{N-1}^- \ge 1$, and growth rates $a_1, \dots, a_{N-1}, b_1, \dots, b_{N-1} \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ such that the following holds:

- $P_2, \dots, P_N$ are regular,
- for $1 \le i < N$ we have the estimates

$$(4.2) \qquad \|\Phi_A(t,s)P_i(s)\| \le K_i^+ e_{a_i}(t,s), \qquad \left\|\bar{\Phi}_A(s,t)P_{i+1}(t)\right\| \le K_i^- e_{b_i}(s,t) \quad \text{for } s \preceq t,$$

- $a_i \lhd b_i$ for $1 \le i < N$ and $b_i \unlhd a_{i+1}$ for $1 \le i < N-1$.

In case $N = 2, 3$ we speak of an *exponential dichotomy* or *trichotomy*, respectively.

Hypothesis 4.1. (i) *The linear dynamic equation* (2.1) *has an exponential N-splitting, i.e., the estimates* (4.2) *hold.*

(ii) *For $i = 1, 2$ one has the identities $F_i(t, 0; q) \equiv 0$ on $\mathbb{T} \times \mathcal{Q}$ and the mappings F_i satisfy the following global Lipschitz estimates*

$$L_i := \sup_{(t,q)\in\mathbb{T}\times\mathcal{Q}} \operatorname{Lip} F_i(t, \cdot; q) < \infty.$$

Moreover, we define $K_1(j) := \sum_{k=1}^j K_k^+$, $K_2(j) := \sum_{k=j}^{N-1} K_k^-$ for $1 \le j < N$, for some real $\delta_{\max} > 0$ we require

$$(4.3) \qquad L_1 < \frac{\delta_{\max}}{2K_{\max}}, \qquad K_{\max} := \max_{i=1}^{N-1}\left(K_1(i) + K_2(i) + K_1(i)K_2(i)\right),$$

choose a fixed $\delta \in (2K_{\max}L_1, \delta_{\max})$ *and abbreviate* $\Theta := \{\theta \in \mathbb{F} : L_2 |\theta| \le L_1\}$,

$$\Gamma_j := \left\{c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}) : a_j + \delta \lhd c \lhd b_j - \delta\right\} \quad \text{for } 1 \le j < N.$$

(iii) *Assume the partial derivatives* $D_2^n F_i(t, \cdot)$, $t \in \mathbb{T}$, *exist, are continuous on* $\mathcal{X} \times \mathcal{Q}$ *up to order* $m \in \mathbb{N}$, *and suppose they are globally bounded, i.e. for* $2 \le n \le m$ *we have*

$$\sup_{(t,x,q)\in\mathbb{T}\times\mathcal{X}\times\mathcal{Q}} \|D_2^n F_i(t, x; q)\| < \infty \quad \text{for } i = 1, 2.$$

Having all preparatory results at hand, for $1 \le i \le j \le N$, we define the mappings $P_i^j : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, $P_i^j(t) := P_i(t) + \ldots + P_j(t)$ and using (4.1), we see that P_i^j is an invariant projector of (2.1); moreover, P_i^j is regular for $i \ge 2$. Defining the complementary subspaces

$$\mathcal{X}_i^j(\tau) := \mathcal{R}(P_i^j(\tau)) = \bigoplus_{k=i}^{j} \mathcal{R}(P_k(\tau)), \qquad \bar{\mathcal{X}}_i^j(\tau) := \mathcal{N}(P_i^j(\tau)) = \bigcap_{k=i}^{j} \mathcal{N}(P_k(\tau))$$

for any $\tau \in \mathbb{T}$, we now head for the second main theorem in this paper. It guarantees the existence of $\frac{(N+2)(N-1)}{2}$ nontrivial invariant fiber bundles of (2.2).

Theorem 4.2 (hierarchies of invariant fiber bundles)**.** *Assume Hypothesis 4.1(i)–(ii) holds with* $\delta_{\max} = \frac{1}{2}\min_{i=1}^{N-1} \lfloor b_i - a_i \rfloor$, *let* $1 \le j \le i \le N$, $(j, i) \ne (1, N)$, *where in case* $j > 1$ *we additionally suppose that* $\mathbb{T}$ *is also unbounded below. Then for all* $\theta \in \Theta$, $q \in \mathcal{Q}$ *the sets*

$$C_{i,j}(\theta, q) := \begin{cases} \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot; \tau, x_0; \theta, q) \in \mathcal{X}^+_{\tau,c} \text{ for all } c \in \Gamma_i\right\} & \text{for } j = 1 \\ \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of (2.2)} \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}^-_{\tau,c} \text{ for all } c \in \Gamma_{j-1} \end{array}\right\} & \text{for } i = N \\ C_{i,1}(\theta, q) \cap C_{N,j}(\theta, q) & \text{else} \end{cases}$$

are invariant fiber bundles of (2.2) *admitting the following so-called* extended hierarchy

$$\begin{array}{ccccccccc} C_{1,1}(\theta,q) & \subset & C_{2,1}(\theta,q) & \subset & \ldots & \subset & C_{N-1,1}(\theta,q) & \subset & \mathbb{T}\times\mathcal{X} \\ & & \cup & & & & \cup & & \cup \\ & & C_{2,2}(\theta,q) & \subset & \ldots & \subset & C_{N-1,2}(\theta,q) & \subset & C_{N,2}(\theta,q) \\ & & & & & & \cup & & \cup \\ & & & & & \ddots & \vdots & & \vdots \\ & & & & & & \cup & & \cup \\ & & & & & & C_{N-1,N-1}(\theta,q) & \subset & C_{N,N-1}(\theta,q) \\ & & & & & & & & \cup \\ & & & & & & & & C_{N,N}(\theta,q). \end{array} \tag{4.4}$$

Each $C_{i,j}(\theta, q)$ *possesses the representation*

$$C_{i,j}(\theta, q) = \left\{(\tau, \eta + c_{i,j}(\tau, \eta; \theta, q)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{X}_j^i(\tau)\right\} \tag{4.5}$$

with a uniquely determined continuous mapping $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfying*

$$c_{i,j}(\tau, x_0; \theta, q) = c_{i,j}(\tau, P_j^i(\tau)x_0; \theta, q) \in \bar{\mathcal{X}}_j^i(\tau) \quad \text{for } \tau \in \mathbb{T},\ x_0 \in \mathcal{X} \tag{4.6}$$

and the invariance equation

$$\left[I_{\mathcal{X}} - P_j^i(t)\right] \varphi(t; \tau, x_0; \theta, q) = c_{i,j}(t, P_j^i(t)\varphi(t; \tau, x_0; \theta, q); \theta, q) \quad \text{for } (\tau, x_0) \in C_{i,j}(\theta, q),\ \tau \preceq t.$$

Furthermore, it holds:

(a) $c_{i,j}(\tau, 0; \theta, q) \equiv 0$ *on* $\mathbb{T} \times \Theta \times \mathcal{Q}$,

(b) $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ *satisfies the Lipschitz estimates*

$$\operatorname{Lip} c_{i,j}(\tau, \cdot; \theta, q) \le \begin{cases} \frac{K_1(i)K_2(i)(L_1+|\theta|L_2)}{\delta-(K_1(i)+K_2(i))(L_1+|\theta|L_2)} & \text{for } j = 1 \\ \frac{K_1(j-1)K_2(j-1)(L_1+|\theta|L_2)}{\delta-(K_1(j-1)+K_2(j-1))(L_1+|\theta|L_2)} & \text{for } i = N \\ \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)(L_1+|\theta|L_2)}{\delta-(K_1(k)+K_2(k)+K_1(k)K_2(k))(L_1+|\theta|L_2)} & \text{else} \end{cases}$$

$$\operatorname{Lip} c_{i,j}(\tau, x_0; \cdot, q) \leq \begin{cases} \frac{\delta K_1(i)K_2(i)(K_1(i)+K_2(i))L_2}{[\delta-2(K_1(i)+K_2(i))L_1]^2} \|x_0\| & \textit{for } j=1 \\ \frac{\delta K_1(j-1)(K_1(j-1)+K_2(j-1))L_2}{[\delta-2(K_1(j-1)+K_2(j-1))L_1]^2} \|x_0\| & \textit{for } i=N \\ \frac{2L_{i,j} \max_{k\in\{i,j-1\}} \frac{\delta K_1(k)K_2(k)(K_1(k)+K_2(k))L_2}{[\delta-2(K_1(k)+K_2(k))L_1]^2}}{1-\max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1}} \|x_0\| & \textit{else} \end{cases} \tag{4.7}$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$, *with*

$$L_{i,j} := 1 + \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta - 2(K_1(k)+K_2(k)+K_1(k)K_2(k))L_1},$$

(c) *if additionally Hypothesis 4.1(iii) and the gap conditions*

$$\begin{cases} m_{i,j} \odot a_i \lhd b_i & \textit{for } j=1 \\ a_{j-1} \lhd m_{i,j} \odot b_{j-1} & \textit{for } i=N \\ m_{i,j} \odot a_i \lhd b_i,\ a_{j-1} \lhd m_{i,j} \odot b_{j-1} & \textit{else} \end{cases} \tag{4.8}$$

hold for some $m_{i,j} \in \{1, \dots, m\}$, *and if we set*

$$\delta_{\max} := \begin{cases} \min\left\{\frac{\lfloor b_i - a_i \rfloor}{2}, \rho_s^{m_{i,j}}[a_i, b_i]\right\} & \textit{for } j=1 \\ \min\left\{\frac{\lfloor b_{j-1} - a_{j-1} \rfloor}{2}, \rho_r^{m_{i,j}}[a_{j-1}, b_{j-1}]\right\} & \textit{for } i=N \\ \min\left\{\frac{\lfloor b_i - a_i \rfloor}{2}, \frac{\lfloor b_{j-1} - a_{j-1} \rfloor}{2}, \rho_s^{m_{i,j}}[a_i, b_i], \rho_r^{m_{i,j}}[a_{j-1}, b_{j-1}]\right\} & \textit{else} \end{cases},$$

then the partial derivatives $D^n_{(2,3)} c_{i,j}$ *exist, are continuous up to order* $m_{i,j}$, *and there exist reals* $M^n_{i,j}, N^n_{i,j} > 0$, *such that*

$$\begin{aligned} \|D_2^n c_{i,j}(\tau, x_0; \theta, q)\| &\leq M^n_{i,j} \quad \textit{for } 1 \leq n \leq m_{i,j}, \\ \|D_3 D_2^n c_{i,j}(\tau, x_0; \theta, q)\| &\leq N^n_{i,j} \|x_0\| \quad \textit{for } 0 \leq n < m_{i,j} \end{aligned} \tag{4.9}$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$,

(d) *if the dynamic equation* (2.2) *and* $P_2, \dots, P_N$ *are* T*-periodic for some* $T > 0$, *then* $c_{i,j}(\cdot, x_0; \theta, q)$ *is* T*-periodic for all* $x_0 \in \mathcal{X}$, $\theta \in \Theta$ *and* $q \in \mathcal{Q}$.

Remark 4.1. (1) We denote the first row in the array (4.4) as *stable hierarchy*, the right column as *unstable hierarchy* and the remaining inclusions as *center hierarchy* of (2.2).

(2) For the special case of a dynamic equation (2.1) possessing an exponential trichotomy with $b_1 \unlhd 0 \unlhd a_2$, we obtain the *classical five invariant fiber bundles*, namely:

- *Stable fiber bundle* $C_{1,1}$: Because of $c_1 \lhd b_1$ and the dynamical characterization in Theorem 4.2(a) all solutions of (2.2) on $C_{1,1}$ converge to 0 exponentially for $t \to \infty$.
- *Center-stable fiber bundle* $C_{2,1}$: All solutions of (2.2) which are not growing too fast as $t \to \infty$ (in the sense that they are c_2^+-quasibounded with $c_2 \unlhd b_2 - \delta$) are contained in $C_{2,1}$, like e.g., solutions bounded in forward time.
- *Center-unstable fiber bundle* $C_{3,2}$: All solutions of (2.2) which exist and are not growing too fast as $t \to -\infty$ (in the sense of c_1^--quasiboundedness with $a_1 + \delta \unlhd c_1$) lie on $C_{3,2}$, like e.g., solutions bounded in backward time.
- *Unstable fiber bundle* $C_{3,3}$: All solutions on the unstable fiber bundle exist in backward time and converge exponentially to 0 as $t \to -\infty$.
- *Center fiber bundle* $C_{2,2}$: The center fiber bundle consists of those solutions which are contained both in the center-stable and the center-unstable fiber bundle. Particularly, all bounded solutions lie on this fiber bundle.

Here we have suppressed the dependence on the parameters $\theta \in \Theta$ and $q \in \mathcal{Q}$.

Proof (of Theorem 4.2): Keep $\tau \in \mathbb{T}$, $q \in \mathcal{Q}$ fixed. We subdivide the proof into four steps.

(I) First of all, we show the extended hierarchy (4.4). Let $\theta \in \Theta$ be fixed, and $1 \leq j \leq i \leq N$ with $(j, i) \neq (1, N)$. Then the facts $c_i \lhd c_{i+1}$ for all $c_i \in \Gamma_i$, $c_{i+1} \in \Gamma_{i+1}$ and $\mathcal{X}^+_{\tau,c_i} \subset \mathcal{X}^+_{\tau,c_{i+1}}$ (cf. (2.4)) imply the hierarchical inclusions

$$C_{i,1}(\theta, q) \subset C_{i+1,1}(\theta, q) \quad \text{for } 1 \leq i < N. \tag{4.10}$$

The definition of $C_{i,j}(\theta,q)$ yields the identity $C_{i,j}(\theta,q) = C_{i,1}(\theta,q) \cap C_{N,j}(\theta,q)$ and from (4.10) we have $C_{i,j}(\theta,q) \subset C_{i+1,1}(\theta,q) \cap C_{N,j}(\theta,q) = C_{i+1,j}(\theta,q)$. Analogously one derives the unstable hierarchy $C_{N,j}(\theta,q) \subset C_{N,j-1}(\theta,q)$ for $1 < j \leq N$, which, in turn, leads to $C_{i,j}(\theta,q) \subset C_{i,1}(\theta,q) \cap C_{N,j-1}(\theta,q) = C_{i,j-1}(\theta,q)$ and we are done.

(II) In order to verify the above assertions in case $j = 1$, we only have to apply Theorem 3.3(a) repeatedly. To this end, let $1 \leq i < N$ be arbitrary, but fixed and $j = 1$. Then, due to Hypothesis 4.1(i) the linear dynamic equation (2.1) satisfies Hypothesis 3.1(i) with the complementary invariant projectors $Q, P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ given by $Q(t) := P_1^i(t)$, $P(t) := P_{i+1}^N(t)$, since we have the estimates (cf. [Hil90, Theorem 7.4(i)])

$$\|\Phi_A(t,s)Q(s)\| \leq \sum_{k=1}^{i} \|\Phi_A(t,s)P_k(s)\| \overset{(4.2)}{\leq} \sum_{k=1}^{i} K_k^+ e_{a_k}(t,s) \leq K_1(i)e_{a_i}(t,s) \quad \text{for } s \preceq t$$

and similarly $\left\|\bar{\Phi}_A(t,s)P(s)\right\| \leq K_2(i)e_{b_i}(t,s)$ for $t \preceq s$. Because assumption (4.3) yields the estimate $L_1 < \frac{\lfloor b_i - a_i \rfloor}{4(K_1(i)+K_2(i))}$, Theorem 3.3(a) implies for all $\theta \in \Theta$ that the set

$$S(\theta,q) := \left\{(\tau,x_0) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot;\tau,x_0;\theta,q) \in \mathcal{X}_{\tau,c}^+ \text{ for all } c \in \Gamma_i\right\}$$

can be represented as graph of a unique mapping $s : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$. Since $S(\theta,q)$ and s obviously depend on the particular choice of i, we denote them by $C_{i,1}(\theta,q)$ and $c_{i,1}$, respectively. Then the relation (4.6), the invariance equation, as well as the assertions (a)–(d) for indices $1 \leq i < N$, $j = 1$ follow directly from Theorem 3.3(a).

(III) Analogously to step (II), and for indices $i = N$, $1 < j \leq N$, one can use Theorem 3.3(b) to see that for all $\theta \in \Theta$ the set

$$R(\theta,q) := \left\{(\tau,x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of (2.2)} \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}_{\tau,c}^- \text{ for all } c \in \Gamma_{j-1} \end{array}\right\}$$

is representable as graph of a unique mapping $r : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$. To indicate the dependence on j, we denote $R(\theta,q)$ and r as $C_{N,j}(\theta,q)$ and $c_{N,j}$, resp., and the assertions for $i = N$, $1 < j \leq N$ yield from Theorem 3.3(b).

(IV) From now on consider $1 < j \leq i < N$, since it remains to prove Theorem 4.2 for the center hierarchy. As a preparation we investigate the operator $T : \mathcal{X}^2 \times \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}^2$,

$$T(x,z;\tau,y,\theta,q) := \begin{pmatrix} c_{N,j}(\tau,y+z;\theta,q) \\ c_{i,1}(\tau,x+y;\theta,q) \end{pmatrix}. \tag{4.11}$$

According to (3.22), (3.23) and $|\theta|\, L_2 \leq L_1$, for fixed $\tau \in \mathbb{T}$, $y \in \mathcal{X}$, $\theta \in \Theta$, we have

$$\begin{aligned} &\|T(x,z;\tau,y,\theta,q) - T(\bar{x},\bar{z};\tau,y,\theta,q)\| \\ &\overset{(4.11)}{\leq} \max\{\|c_{N,j}(\tau,y+z;\theta,q) - c_{N,j}(\tau,y+\bar{z};\theta,q)\|\,, \\ &\qquad \|c_{i,1}(\tau,x+y;\theta,q) - c_{i,1}(\tau,\bar{x}+y;\theta,q)\|\} \\ &\leq \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta - 2(K_1(k)+K_2(k))L_1} \left\|\begin{pmatrix} x-\bar{x} \\ z-\bar{z} \end{pmatrix}\right\| \quad \text{for } x,\bar{x},z,\bar{z} \in \mathcal{X}. \end{aligned} \tag{4.12}$$

It evidently holds $\frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1} < 1$ for $1 \leq k < N$ (cf. (4.3)) and consequently $T(\cdot;\tau,y,\theta,q) : \mathcal{X}^2 \to \mathcal{X}^2$ is a contraction; due to (4.11) in connection with the already established special cases $i = N$ (cf. step (III)) and $j = 1$ (cf. step (II)) of (4.6), its unique fixed point $(c_{i,j}^{(1)}, c_{i,j}^{(2)})(\tau,y;\theta,q)$ satisfies the inclusions $c_{i,j}^{(1)}(\tau,y;\theta,q) \in \mathcal{X}_1^{j-1}(\tau)$ and $c_{i,j}^{(2)}(\tau,y;\theta,q) \in \mathcal{X}_{i+1}^N(\tau)$. An argument analogous to [AW96, Lemma B.4] shows that the mappings $c_{i,j}^{(1)}, c_{i,j}^{(2)} : \mathbb{T} \times \mathcal{X} \times \Theta \times \mathcal{Q} \to \mathcal{X}$ are continuous. In addition, for fixed $\tau \in \mathbb{T}$, $x, z \in \mathcal{X}$, $\theta \in \Theta$ and arbitrary $y, \bar{y} \in \mathcal{X}$, it is not difficult to derive the estimate

$$\begin{aligned} &\|T(x,z;\tau,y,\theta,q) - T(x,z;\tau,\bar{y},\theta,q)\| \\ &\leq \max_{k\in\{i,j-1\}} \frac{K_1(k)K_2(k)\,(L_1+|\theta|\,L_2)}{\delta - (K_1(k)+K_2(k))\,(L_1+|\theta|\,L_2)} \|x-\bar{x}\| \end{aligned} \tag{4.13}$$

from (3.22) and (3.23).

Now we prove the representation (4.5) of $C_{i,j}(\theta,q)$ as graph of a function $c_{i,j}$. From step (II) we know that for arbitrary $\theta\in\Theta$, a point $x_0\in\mathcal{X}$ satisfies $(\tau,x_0)\in C_{i,1}(\theta,q)$, if and only if there exists a $\xi_0\in\mathcal{X}_1^i(\tau)$ such that $x_0=\xi_0+c_{i,1}(\tau,\xi_0;\theta,q)$ and accordingly $P_1^i(\tau)x_0=\xi_0+P_1^i(\tau)c_{i,1}(\tau,x_0;\theta,q)=\xi_0$ (cf. (4.6)). This yields $(\tau,x_0)\in C_{i,1}(\theta,q)$ if and only if $x_0=P_1^i(\tau)x_0+c_{i,1}(\tau,P_1^i(\tau)x_0;\theta,q)$, and analogously from step (III) we have $(\tau,x_0)\in C_{N,j-1}(\theta,q)$ if and only if $x_0=P_j^N(\tau)x_0+c_{N,j}(\tau,P_j^N(\tau)x_0;\theta,q)$. The unique decomposition $x_0=\xi+\eta+\zeta$ into $\xi\in\mathcal{X}_1^{j-1}(\tau)$, $\eta\in\mathcal{X}_j^i(\tau)$, $\zeta\in\mathcal{X}_{i+1}^N(\tau)$ leads to the equivalence

$$\begin{aligned}(\tau,x_0)\in C_{i,j}(\theta,q)\quad&\Leftrightarrow\quad x_0=P_1^i(\tau)x_0+c_{i,1}(\tau,P_1^i(\tau)x_0;\theta,q)\text{ and}\\ &\qquad\quad x_0=P_j^N(\tau)x_0+c_{N,j}(\tau,P_j^N(\tau)x_0;\theta,q)\\ &\Leftrightarrow\quad \zeta=c_{i,1}(\tau,\xi+\eta;\theta,q)\text{ and }\xi=c_{N,j}(\tau;\eta+\zeta;\theta,q)\\ &\overset{(4.11)}{\Leftrightarrow}\quad (\xi,\zeta)=T(\xi,\zeta;\tau,\eta,\theta,q),\end{aligned}$$

i.e., the pair $(\xi,\zeta)\in\mathcal{X}_1^{j-1}(\tau)\times\mathcal{X}_{i+1}^N(\tau)$ is a fixed point of $T(\cdot;\tau,\eta,\theta,q)$; from the above considerations it is uniquely determined by $(c_{i,j}^{(1)},c_{i,j}^{(2)})(\tau,y_0;\theta,q)$. As a result, if we define $c_{i,j}(\tau,x_0;\theta,q):=c_{i,j}^{(1)}(\tau,x_0;\theta,q)+c_{i,j}^{(2)}(\tau,x_0;\theta,q)$ for $x_0\in\mathcal{X}$ and $\theta\in\Theta$, then the continuity statement for $c_{i,j}$, as well as the representation (4.5) holds. The fiber bundle $C_{i,j}(\theta,q)$ is invariant, because for $(\tau,x_0)\in C_{i,j}(\theta,q)$ we obtain $\varphi(\cdot;\tau,x_0;\theta,q)\in\mathcal{X}_{\tau,c}^+$ and the existence of a d^--quasibounded solution $\nu:\mathbb{T}\to\mathcal{X}$ of (2.2) with $\nu(\tau)=x_0$ for all $c\in\Gamma_i$, $d\in\Gamma_{j-1}$. Then the cocycle property (2.3) implies that $\varphi(\cdot;t_0,\varphi(t_0;\tau,x_0;\theta,q);\theta,q)$, $t_0\in\mathbb{T}_\tau^+$, is also c^+- and d^--quasibounded (in the sense above), and therefore $(t_0,\varphi(t_0;\tau,x_0;\theta,q))\in C_{i,j}(\theta,q)$.

(a) From step (II) and (III) (cf. Theorem 3.3$(a_1),(b_1)$) one sees that $(0,0)$ is obviously the unique fixed point of $T(\cdot;\tau,0,\theta,q)$, and this implies $c_{i,j}(\tau,0;\theta,q)=0$ for all $\theta\in\Theta$.

(b) We have to verify the Lipschitz estimates for $c_{i,j}$. To do so, we suppress the dependence on $q\in\mathcal{Q}$ from now on and abbreviate $\gamma_{i,j}:=(c_{i,j}^{(1)},c_{i,j}^{(2)})$. To that end, for fixed $\theta\in\Theta$ and $x,\bar{x}\in\mathcal{X}$, due to the fixed point identities for $T(\cdot;\tau,x,\theta)$ and $T(\cdot;\tau,\bar{x},\theta)$, respectively, we obtain from (4.12) and (4.13)

$$\begin{aligned}&\|\gamma_{i,j}(\tau,x;\theta)-\gamma_{i,j}(\tau,\bar{x};\theta)\|\\ &\overset{(4.11)}{\le}\max_{k\in\{i,j-1\}}\frac{K_1(k)K_2(k)\,(L_1+|\theta|\,L_2)}{\delta-(K_1(k)+K_2(k)+K_1(k)K_2(k))\,(L_1+|\theta|\,L_2)}\,\|x-\bar{x}\|\,,\end{aligned}\tag{4.14}$$

which easily yields the first Lipschitz estimate for $c_{i,j}$. Moreover, for fixed $\tau\in\mathbb{T}$, $x\in\mathcal{X}$ and arbitrary $\theta,\bar{\theta}\in\Theta$ we get from the assertion (c_1) that

$$\begin{aligned}&\left\|c_{i,1}(\tau,x+c_{i,j}^{(1)}(\tau,x;\theta);\theta)-c_{i,1}(\tau,x+c_{i,j}^{(1)}(\tau,x;\bar{\theta});\bar{\theta})\right\|\\ &\overset{(3.22)}{\le}\frac{2K_1(i)K_2(i)L_1}{\delta-2(K_1(i)+K_2(i))L_1}\left\|c_{i,j}^{(1)}(\tau,x;\theta)-c_{i,j}^{(1)}(\tau,x;\bar{\theta})\right\|\\ &\qquad+\left\|c_{i,1}(\tau,x+c_{i,j}^{(1)}(\tau,x;\bar{\theta});\theta)-c_{i,1}(\tau,x+c_{i,j}^{(1)}(\tau,x;\bar{\theta});\bar{\theta})\right\|\\ &\overset{(4.7)}{\le}\max_{k\in\{i,j-1\}}\frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1}\left\|\gamma_{i,j}(\tau,x;\theta)-\gamma_{i,j}(\tau,x;\bar{\theta})\right\|\\ &\qquad+\frac{\delta K_1(i)K_2(i)(K_1(i)+K_2(i))L_2}{[\delta-2(K_1(i)+K_2(i))L_1]^2}\left\|x+c_{i,j}^{(1)}(\tau,x;\bar{\theta})\right\|\\ &\le\max_{k\in\{i,j-1\}}\frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1}\left\|\gamma_{i,j}(\tau,x;\theta)-\gamma_{i,j}(\tau,x;\bar{\theta})\right\|\\ &\qquad+\frac{\delta K_1(i)K_2(i)(K_1(i)+K_2(i))L_2}{[\delta-2(K_1(i)+K_2(i))L_1]^2}\left(\|x\|+\left\|\gamma_{i,j}(\tau,x;\bar{\theta})-\gamma_{i,j}(\tau,0;\bar{\theta})\right\|\right)\\ &\overset{(4.14)}{\le}\max_{k\in\{i,j-1\}}\frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1}\left\|\gamma_{i,j}(\tau,x;\theta)-\gamma_{i,j}(\tau,x;\bar{\theta})\right\|\end{aligned}$$

$$+ L_{i,j} \max_{k\in\{i,j-1\}} \frac{\delta K_1(k)K_2(k)(K_1(k)+K_2(k))L_2}{[\delta - 2(K_1(k)+K_2(k))L_1]^2} \|x\| .$$

With the aid of the fixed point identities for $T(\cdot;\tau,x,\theta)$ and $T(\cdot;\tau,x,\bar{\theta})$, respectively, together with the corresponding estimate for $c_{N,j}$, this implies

$$\begin{aligned}\left\|\gamma_{i,j}(\tau,x;\theta) - \gamma_{i,j}(\tau,x;\bar{\theta})\right\| \le{}& \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta - 2(K_1(k)+K_2(k))L_1} \left\|\gamma_{i,j}(\tau,x;\theta) - \gamma_{i,j}(\tau,x;\bar{\theta})\right\| \\ &+ L_{i,j} \max_{k\in\{i,j-1\}} \frac{\delta K_1(k)K_2(k)(K_1(k)+K_2(k))L_2}{[\delta - 2(K_1(k)+K_2(k))L_1]^2} \|x\|\end{aligned}$$

and we readily get the second Lipschitz estimate in (c_2).

(c) The mapping $c_{i,j}$ was constructed using the fixed points of the operator T defined in (4.11). Under Hypothesis 4.1(iii) and the gap conditions (4.8), we know from step (II) and (III) (cf. Theorem 3.3$(a_3),(b_3)$) that $T(\cdot;\tau,\cdot,q) : \mathcal{X}^3 \to \mathcal{X}^2$, $\tau \in \mathbb{T}$, $q \in \mathcal{Q}$, is $m_{c_{i,j}}$-times differentiable with continuous partial derivatives $D^n_{(1,2,4)}T$ for $n \in \{1,\dots,m_{i,j}\}$. Then it is a consequence of the uniform contraction principle (cf. [CH96, p. 25, Theorem 2.2]) that $c_{i,j}$ has the claimed smoothness property. Finally, the estimates (4.9) result similarly to [Kel99, p. 80, Satz 3.7.6].

(d) Let $T > 0$ and assume $\sigma_T : \mathbb{T} \to \mathbb{T}$ satisfies $\mu(\sigma_T(t),t) \equiv T$ on $\mathbb{T}$. For a T-periodic dynamic equation (2.2) and T-periodic projectors $P_2,\dots,P_N$, we obtain from step (II) and (III) (cf. Theorem 3.3$(a_4),(b_4)$) that the identity $T(x,z;\tau,y,\theta,q) = T(x,z;\sigma_T(\tau),y,\theta,q)$ holds for all $\tau \in \mathbb{T}$, $x,y,z \in \mathcal{X}$ and parameters $\theta \in \Theta$. Then the unique fixed points $(c^{(1)}_{i,j}, c^{(2)}_{i,j})(\tau,y,\theta,q)$, $(c^{(1)}_{i,j}, c^{(2)}_{i,j})(\sigma_T(\tau),y,\theta,q) \in \mathcal{X}^2$ of the contractions $T(x,z;\tau,y,\theta,q)$ and $T(x,z;\sigma_T(\tau),y,\theta,q)$, resp., coincide and this yields the T-periodicity of $c_{i,j}(\cdot,x_0;\theta,q)$ for all $x_0 \in \mathcal{X}$.

Therefore, the proof of Theorem 4.2 is finished. □

5. Conclusion and Perspectives

This paper continued our research within the field of dynamic equations on time scales and provided the basic results for two areas of further research:

(I) A flexible (meaning, nonautonomous, pseudo-hyperbolic) and general (dependence on the parameter q) existence theorem for invariant fiber bundles is fundamental to derive a geometric theory of dynamic equations on time scales. Theorem 3.3 has been employed in [Pöt06] to derive invariant foliations of the extended state space, as well as a principle on reduced stability. Moreover, after these preparations, we have been able to address topological linearization issues and prove a generalized Hartman-Grobman result in [Pöt07a].

(II) We advocate for time scales as a useful and convenient vehicle in analytical discretization theory. The particular dependence on the parameter θ allows us to interpret (2.2), i.e.,

$$x^{\Delta} = A(t)x + F_1(t,x) + \theta F_2(t,x)$$

in three ways (we refer to [KP05] for details):

- On the time scale $\mathbb{T} = \mathbb{R}$ and for $\theta = 0$, $F_1 = F$, the above dynamic equation reduces to a nonautonomous ordinary differential equation

$$\dot{x} = A(t)x + F(t,x). \tag{5.1}$$

- On the time scale $\mathbb{T} = \mathbb{D}$ (cf. Example 2.1), for $\theta = 0$ and a specific F_1, the dynamic equation (2.2) describes the behavior of solutions for (5.1) restricted to the discrete set $\mathbb{D}$.
- On the time scale $\mathbb{T} = \mathbb{D}$, for $\theta > 0$ and a specific F_2, the dynamic equation (2.2) captures the behavior of an explicit one-step scheme with varying step sizes $t_{k+1} - t_k$ applied to numerically solve (5.1), like for example a Runge-Kutta method.

Summarizing this, the obtained perturbation results open the door to show that geometric properties of ODEs, like for instance the existence of invariant manifolds (see [KP05]), invariant foliations or topological conjugacies (see [Pöt07b]), persist under discretization. In addition, one obtains convergence for small step-sizes by letting $\theta \to 0$.

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