Monotonicity and discretization of Urysohn integral operators

Magdalena Nockowska-Rosiak [a]

[a] *Institute of Mathematics, Lodz University of Technology, al. Politechniki 10, 93-590 Łódź, Poland*

Christian Pötzsche [b]

[b] *Institut für Mathematik, Universität Klagenfurt, Universitätsstraße 65–67, 9020 Klagenfurt, Austria*

Abstract

The property that a nonlinear operator on a Banach spaces preserves an order relation, is subhomogeneous or order concave w.r.t. an order cone has profound consequences. In Nonlinear Analysis it allows to solve related equations by means of suitable fixed point or monotone iteration techniques. In Dynamical Systems the possible long term behavior of associate integrodifference equations is drastically simplified. This paper contains sufficient conditions for vector-valued Urysohn integral operators to be monotone, subhomogeneous or concave. It also provides conditions guaranteeing that these properties are preserved under spatial discretization of particularly Nyström type. This fact is crucial for numerical schemes to converge, or for simulations to reproduce the actual behavior and asymptotics.

Key words: Urysohn operator, Monotonicity, Subhomogeneity, Order concavity, Nyström method, Integrodifference equation, Monotone iteration
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1 Introduction

Urysohn operators [11, pp. 158ff] are a natural generalization of Fredholm integral operators and thus also known as nonlinear Fredholm operators [14].

Email addresses: magdalena.nockowska@p.lodz.pl (Magdalena Nockowska-Rosiak), christian.poetzsche@aau.at (Christian Pötzsche).

For the purpose of this paper they are of the form

$$\mathcal{F}(u) := \int_\Omega f(\cdot, y, u(y))\,\mathrm{d}\mu(y) \tag{F}$$

with a vector-valued kernel function f, a compact domain Ω and a measure μ fulfilling $\mu(\Omega) < \infty$. They represent an important species in the zoo of integral equations. For instance, nonlinear elliptic BVPs (provided Green's function is known, cf. [11, pp. 181ff]) can be reformulated as fixed point equations involving an Urysohn operator. Besides being theoretically relevant, their applications are wide-spread and range from fluid dynamics [5, pp. 419ff, Chap. 10] over theoretical ecology [10] to system identification [13].

A canonical approach to solve nonlinear integral equations involving Urysohn operators are monotone iteration techniques (see [14, pp. 163ff, Chap. 11] or [15, pp. 269ff, Chap. 7]). This rather simple constructive method merely requires $\mathcal{F}$ to preserve an order relation induced by a cone in their state space, which is typically a Banach space of integrable functions. Beyond preserving such an order relation, additional properties like subhomogeneity or concavity do further simplify the solution behavior of problems involving an operator $\mathcal{F}$ (see [9] or [16, pp. 43, Chap. 3]). Finally, we also remark that several fixed point results are based on related monotonicity assumptions [1].

For this reason it is a relevant task to provide conditions on the kernel functions f guaranteeing (different degrees of) monotonicity, subhomogeneity or concavity for an Urysohn operator $\mathcal{F}$. While it is clear that the monotonicity of a real-valued f in the third argument extends to $\mathcal{F}$, we do address $\mathbb{R}^d$-valued functions f and order-relations determined by arbitrary cones in $\mathbb{R}^d$. This framework is well-motivated from applications e.g. in theoretical ecology when modeling various relationships between different species.

Another important question is whether the mentioned structural properties of an Urysohn operator $\mathcal{F}$ persist in simulations or numerical computations? This aspect is crucial in simulations to capture the actual behavior. In the second part of the paper we establish that monotonicity, subhomogeneity and concavity are preserved under Nyström discretizations

$$\mathcal{F}^n(u) := \sum_{j=0}^{q_n} w_j f(\cdot, \eta_j, u(\eta_j)) \tag{F^n}$$

of $\mathcal{F}$ (see [2, Sect. 3]), where the real weights w_j and the $q_n \in \mathbb{N}$ nodes $\eta_j \in \Omega$ are determined by a numerical quadrature (or cubature) rule (cf. [4,7]). In doing so, we focus on persistence issues and refer to e.g. [6] for aspects like consistency, convergence or numerical stability.

Concerning related literature we refer to [12] focussing on linear integral operators and suitable discretizations including projection methods. On an abstract level, a projection method is a (linear) projection of the operator values $\mathcal{F}(u)$ onto a finite-dimensional space. For this reason, the results obtained in [12] also apply in our present setting and need not to be discussed again.

The paper is structured as follows: As starting point we formulate conditions ensuring the well-definedness of Urysohn operators on the space of continuous functions over the compact set Ω. Sec. 2 presents sufficient conditions on the kernel functions f to imply a monotone, subhomogeneous or order concave Urysohn operator $\mathcal{F}$, possibly in a strict or strong way. These properties actually carry over from the kernel functions f with values in $\mathbb{R}^d$ to the nonlinear integral operators $\mathcal{F}$ mapping into the continuous $\mathbb{R}^d$-valued functions. Persistence of these properties under Nyström discretization is established in Sec. 3, provided the chosen integration rules for $\mathcal{F}^n$ have positive weights. This property is satisfied for a large class of numerical methods and, besides guaranteeing well-posedness and computational stability [4,7], yields another reason for the popularity of methods with positive weights. In Sec. 4 we provide some quick information on semi-discretizations based on collocation, since the essential issues were settled in [12] already. The final Sec. 5 illustrates our results by means of monotone iteration techniques applied to Urysohn operators. First, the effect of different quadrature rules when solving a system of monotone nonlinear integral equations is studied. Moreover, we approximate periodic solutions of generalized Beverton-Holt integrodifference equations. Finally, for the convenience of the reader, an appendix collects basic and necessary results on cones in Banach spaces and monotone, subhomogeneous or order concave mappings.

Notation

We write $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals. Norms on finite-dimensional spaces are denoted by $|\cdot|$ and $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$ is the Euclidean inner product on $\mathbb{R}^d$. For subsets U of a metric space (X, d), U° is the interior and $\overline{U}$ the closure. Then we abbreviate $B_r(x) := \{y \in X : d(y, x) < r\}$ and $\bar{B}_r(x) := \{y \in X : d(y, x) \leqslant r\}$ for the open resp. closed ball in X of radius $r > 0$ and center $x \in X$. The distance of x from a set U is

$$\operatorname{dist}_U(x) := \inf_{u \in U} d(x, u)$$

and we define $B_r(U) := \{y \in X : \operatorname{dist}_U(y) < r\}$ as r-neighborhood of U.

On a Banach space X, $L_l(X)$ denotes the normed space of all bounded l-linear maps $T : X^l \to X$, $l \in \mathbb{N}$, supplemented by $L_0(X) := X$ and $L(X) := L_1(X)$ for the bounded linear maps on X. We write $N(S) := S^{-1}(\{0\})$ for the kernel of $S \in L(X)$.

Unless otherwise noted, suppose $(\Omega, \mathfrak{A}, \mu)$ is a measure space with $\mu(\Omega) < \infty$, where $\Omega \neq \emptyset$ is a compact metric space and the σ-algebra $\mathfrak{A}$ contains the Borel sets. The μ-integral of a μ-measurable function $u : \Omega \to \mathbb{R}^d$ is denoted by $\int_\Omega u(y)\,\mathrm{d}\mu(y)$ and satisfies

$$\left\langle \int_\Omega u(y)\,\mathrm{d}\mu(y), y' \right\rangle = \int_\Omega \langle u(y), y' \rangle\,\mathrm{d}\mu(y) \quad \text{for all } y' \in \mathbb{R}^d. \tag{1.1}$$

For compact subsets $\Omega \subset \mathbb{R}^\kappa$ and the κ-dimensional Lebesgue measure $\mu = \lambda_\kappa$ the resulting Lebesgue integral is abbreviated as $\int_\Omega u(y)\,\mathrm{d}y := \int_\Omega u(y)\,\mathrm{d}\lambda_\kappa(y)$.

The set $C(\Omega)^d$ of all continuous functions $u : \Omega \to \mathbb{R}^d$ is a real Banach space when equipped with the maximum norm $\|u\|_\infty := \max_{x\in\Omega} |u(x)|$. Throughout the text, $Y_+ \subset \mathbb{R}^d$ denotes a fixed cone inducing the relations $\leqslant, <$ and $\ll$ in $\mathbb{R}^d$ (cf. (A.1) and App. A for the related terminology); Y_+' is the dual cone. Last but not least, on the cone (cf. [12, Lemma 2.2])

$$C(\Omega)^d_+ := \left\{ u \in C(\Omega)^d : u(x) \in Y_+ \text{ for all } x \in \Omega \right\}$$

and for functions $u, \bar{u} \in C(\Omega)^d$ we introduce the relations

$$\begin{aligned} u \preceq \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+, \\ u \prec \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+ \setminus \{0\}, \\ u \prec\!\!\prec \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in (C(\Omega)^d_+)^\circ; \end{aligned}$$

note that $Y_+^\circ \neq \emptyset$ ensures $(C(\Omega)^d_+)^\circ \neq \emptyset$ (cf. [12, Lemma 2.2] again).

Lemma 1.1 (cf. [12, Lemma 2.3]) *If $u, \bar{u} \in C(\Omega)^d$, then:*

(a) $u \preceq \bar{u} \Leftrightarrow u(x) \leqslant \bar{u}(x)$ *for all* $x \in \Omega \Leftrightarrow \langle u(x), y' \rangle \leqslant \langle \bar{u}(x), y' \rangle$ *for all* $x \in \Omega$ *and* $y' \in Y_+'$.
(b) $u \prec \bar{u} \Leftrightarrow u(x) \leqslant \bar{u}(x)$ *for all* $x \in \Omega$ *and* $u(x_0) < \bar{u}(x_0)$ *for some* $x_0 \in \Omega$.
(c) If Y_+ *is solid, then* $u \prec\!\!\prec \bar{u} \Leftrightarrow u(x) \ll \bar{u}(x)$ *for all* $x \in \Omega \Leftrightarrow \langle u(x), y' \rangle < \langle \bar{u}(x), y' \rangle$ *for all* $x \in \Omega$, $y' \in Y_+' \setminus \{0\}$.

2 Urysohn integral operators

This section studies *Urysohn operators*

$$\mathcal{F} : U \to C(\Omega)^d, \qquad \mathcal{F}(u) := \int_\Omega f(\cdot, y, u(y))\,\mathrm{d}\mu(y), \tag{F}$$

for which we are about to establish monotonicity, subhomogeneity and order concavity in various degrees. This essentially means to demonstrate that corresponding properties of the kernel functions $f(x, y, \cdot)$ carry over to the integral operators (F). For this endeavor, we impose

Hypothesis 2.1 *Let $Z \subseteq \mathbb{R}^d$ be nonempty. Assume that $f : \Omega^2 \times Z \to \mathbb{R}^d$ fulfills the following Carathéodory conditions with $l \in \{0, 1\}$:*

(U^l) *$D_3^l f(x, \cdot, z) : \Omega \to L_l(\mathbb{R}^d)$ is μ-measurable for all $x \in \Omega$, $z \in Z$, for every $r > 0$ there exists a μ-measurable function $\beta_r^l : \Omega^2 \to \mathbb{R}_+$ satisfying*

$$\sup_{x\in\Omega} \int_\Omega \beta_r^l(x, y) \,\mathrm{d}\mu(y) < \infty,$$

such that for μ-a.a. $y \in \Omega$ it is $\left|D_3^l f(x, y, z)\right| \leqslant \beta_r^l(x, y)$ for all $x \in \Omega$, $z \in Z \cap \bar{B}_r(0)$ and $D_3^l f(\cdot, y, \cdot) : \Omega \times Z \to L_l(\mathbb{R}^d)$ exists as continuous function for μ-a.a. $y \in \Omega$. Furthermore, for every $r > 0$ there exist a μ-measurable function $\gamma_r^l : \Omega^3 \to \mathbb{R}_+$ satisfying

$$\lim_{x\to x_0} \int_\Omega \gamma_r^l(x, x_0, y) \,\mathrm{d}\mu(y) = 0 \quad \textit{for all } x_0 \in \Omega,$$

such that for μ-a.a. $y \in \Omega$ one has $\left|D_3^l f(x, y, z) - D_3^l f(\bar{x}, y, z)\right| \leqslant \gamma_r^l(x, \bar{x}, y)$ for all $x, \bar{x} \in \Omega$, $z \in Z \cap \bar{B}_r(0)$.

If we assume Hypothesis (U^0), then [11, pp. 164–165, Prop. 3.1] yields that the Urysohn operator (F) is well-defined on

$$U := \left\{u \in C(\Omega)^d : \, u(x) \in Z \text{ for all } x \in \Omega\right\}. \tag{2.1}$$

In particular, for $f : \Omega^2 \times \mathbb{R}^d \to \mathbb{R}^d$ of the form $f(x, y, z) := k(x, y)z$ with a kernel $k : \Omega^2 \to L(\mathbb{R}^d)$, then $\mathcal{F}$ becomes a Fredholm operator and Hypothesis (U^0) ensures its well-definedness and continuity (cf. [12, Hypothesis (L)]).

Remark 2.2 (differentiability on Z) *Since we imposed no openness on the set $Z \subseteq \mathbb{R}^d$, the existence of the partial derivative $D_3 f$ is understood as follows: There is an open superset $\tilde{Z} \supseteq Z$ and an extension $\tilde{f} : \Omega^2 \times \tilde{Z} \to \mathbb{R}^d$ of f whose partial derivative $D_3 \tilde{f}$ exists with restrictions $f, D_3 f$ to $\Omega^2 \times Z$ satisfying the above assumptions with $l \in \{0, 1\}$.*

For later reference we state basic properties of the domain U:

Lemma 2.3 (the set U) *Let $U \subseteq C(\Omega)^d$ be defined in (2.1).*

(a) If Z is open (or closed), then also U is open (resp. closed).
(b) If Z is Y_+-convex, then also U is $C(\Omega)^d_+$-convex.

PROOF. (a) Let $u_0 \in U$ for an open set Z. This means $u_0(x) \in Z$ for all $x \in \Omega$ and whence $Z_0 := u_0(\Omega) \subseteq Z$ is a compact subset of the open set Z. With the closed complement $F := \mathbb{R}^d \setminus Z$ consider the continuous distance function $\mathrm{dist}_F : Z_0 \to \mathbb{R}_+$. Due to $Z_0 \cap F = \emptyset$ it is always positive and thus

$\rho := \min_{\zeta\in Z_0} \operatorname{dist}_F(\zeta) > 0$ holds. If $z \in B_\rho(Z_0)$, then $|\zeta - z| < \rho \leqslant \operatorname{dist}_F(\zeta)$ is true for some $\zeta \in Z_0$, consequently $z \notin F$ and therefore $B_\rho(Z_0) \subseteq \mathbb{R}^d \setminus F = Z$. But this guarantees that $u(x) \in Z$ for all $x \in \Omega$ and $u \in B_{\rho/2}(u_0)$, i.e. u_0 is an interior point of Z. The corresponding assertion for closed Z is evident.
(b) Given $u, \bar{u} \in U$ with $u \preceq \bar{u}$, one has $u(x), \bar{u}(x) \in Z$ and $(\bar{u}-u)(x) \in Y_+$ for $x \in \Omega$. Since Z is Y_+-convex, we obtain from $u(x)+\theta(\bar{u}-u)(x) \in Z$ for $x \in \Omega$ that $0 \preceq u + \theta(\bar{u} - u)$ for all $\theta \in [0, 1]$ holds and thus U is $C(\Omega)^d_+$-convex. □

The remaining section studies in which sense structural properties of kernel functions f such as monotonicity, subhomogeneity and order concavity extend to Urysohn operators $\mathcal{F}$.

Theorem 2.4 (properties of $\mathcal{F}$) *Let Hypothesis (U^0) hold. If for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is*

(a) Y_+-monotone, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-monotone,
(b) Y_+-subhomogeneous with $Z = Y_+$, then $\mathcal{F} : C(\Omega)^d_+ \to C(\Omega)^d_+$ is $C(\Omega)^d_+$-subhomogeneous,
(c) Y_+-concave with Y_+-convex Z, then $\mathcal{F} : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-concave.

PROOF. Suppose $u, \bar{u} \in U$ are given.

(a) Let $y' \in Y'_+$ and $u \prec \bar{u}$. Our assumptions imply $f(x, y, u(y)) \leqslant f(x, y, \bar{u}(y))$ and then $\langle f(x, y, u(y)), y'\rangle \leqslant \langle f(x, y, \bar{u}(y)), y'\rangle$ for all $x \in \Omega$, μ-a.a. $y \in \Omega$ results due to Lemma A.1(a). This, with monotonicity of the integral, guarantees

$$\begin{aligned}\langle \mathcal{F}(u)(x), y'\rangle &\overset{(1.1)}{=} \int_\Omega \langle f(x, y, u(y)), y'\rangle \,\mathrm{d}\mu(y) \leqslant \int_\Omega \langle f(x, y, \bar{u}(y)), y'\rangle \,\mathrm{d}\mu(y)\\ &\overset{(1.1)}{=} \langle \mathcal{F}(\bar{u})(x), y'\rangle \quad \text{for all } x \in \Omega.\end{aligned}$$

Since $x \in \Omega$ was arbitrary, Lemma 1.1(a) yields $\mathcal{F}(u) \preceq \mathcal{F}(\bar{u})$.
(b) Let $0 \prec u$ and $y' \in Y_+$. By $f(x, y, \cdot) : Y_+ \to Y_+$ we obtain

$$\langle \mathcal{F}(u)(x), y'\rangle \overset{(1.1)}{=} \int_\Omega \langle f(x, y, u(y)), y'\rangle \,\mathrm{d}\mu(y) \geq 0 \quad \text{for all } x \in \Omega$$

and hence $\mathcal{F}(u) \in C(\Omega)^d_+$ due to Lemma 1.1(a).
Thanks to the assumptions it is $U = C(\Omega)^d_+$. If $\theta \in (0, 1)$, then Lemma A.1(a) implies $\langle \theta f(x, y, u(y)), y'\rangle \leqslant \langle f(x, y, (\theta u)(y)), y'\rangle$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ and consequently the monotonicity of the integral leads to

$$\begin{aligned}\langle \theta\mathcal{F}(u)(x), y'\rangle &\overset{(1.1)}{=} \int_\Omega \langle \theta f(x, y, u(y)), y'\rangle \,\mathrm{d}\mu(y) \leqslant \int_\Omega \langle f(x, y, \theta u(y)), y'\rangle \,\mathrm{d}\mu(y)\\ &\overset{(1.1)}{=} \langle \mathcal{F}(\theta u)(x), y'\rangle \quad \text{for all } x \in \Omega.\end{aligned}$$

Hence, Lemma 1.1(a) ensures $\theta\mathcal{F}(u) \preceq \mathcal{F}(\theta u)$, i.e. $\mathcal{F}$ is subhomogeneous.
(c) Referring to Lemma 2.3 the set U is $C(\Omega)_+^d$-convex. Let $y' \in Y'_+$, $u \prec \bar{u}$ and $\theta \in (0,1)$. Our assumption implies

$$\theta f(x,y,u(y)) + (1-\theta) f(x,y,\bar{u}(y)) \leqslant f(x,y,\theta u(y) + (1-\theta)\bar{u}(y))$$

and Lemma A.1(a) guarantees

$$\langle \theta f(x,y,u(y)) + (1-\theta) f(x,y,\bar{u}(y)), y' \rangle \leqslant \langle f(x,y,\theta u(y) + (1-\theta)\bar{u}(y)), y' \rangle$$

for all $x \in \Omega$ and μ-a.a. $y \in \Omega$. Thus, the monotonicity of the integral yields

$$\begin{aligned}
&\langle \theta\mathcal{F}(u)(x) + (1-\theta)\mathcal{F}(\bar{u})(x), y' \rangle \\
\overset{(1.1)}{=} &\int_\Omega \langle \theta f(x,y,u(y)) + (1-\theta) f(x,y,\bar{u}(y)), y' \rangle \, \mathrm{d}\mu(y) \\
\leqslant &\int_\Omega \langle f(x,y,\theta u(y) + (1-\theta)\bar{u}(y)), y' \rangle \, \mathrm{d}\mu(y) \\
\overset{(1.1)}{=} &\langle \mathcal{F}(\theta u + (1-\theta)\bar{u})(x), y' \rangle \quad \text{for all } x \in \Omega
\end{aligned}$$

and Lemma 1.1(a) leads to $\theta\mathcal{F}(u) + (1-\theta)\mathcal{F}(\bar{u}) \preceq \mathcal{F}(\theta u + (1-\theta)\bar{u})$. □

Theorem 2.5 (strict properties of $\mathcal{F}$) *Let Hypothesis (U^0) hold, nonempty, open subsets of Ω have positive measure and suppose there exists a $\bar{x} \in \Omega$ such that $f(\bar{x},\cdot)$ is continuous. If for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ a kernel function $f(x,y,\cdot) : Z \to \mathbb{R}^d$ is*

(a) Y_+-monotone and $f(\bar{x},y,\cdot) : Z \to \mathbb{R}^d$ is strictly Y_+-monotone for μ-a.a. $y \in \Omega$, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is strictly $C(\Omega)_+^d$-monotone,
(b) Y_+-subhomogeneous with $Z = Y_+$ and $f(\bar{x},y,\cdot) : Y_+ \to Y_+$ is strictly Y_+-subhomogeneous for μ-a.a. $y \in \Omega$, then $\mathcal{F} : C(\Omega)_+^d \to C(\Omega)_+^d$ is strictly $C(\Omega)_+^d$-subhomogeneous,
(c) Y_+-concave with Y_+-convex Z and $f(\bar{x},y,\cdot) : Z \to \mathbb{R}^d$ is strictly Y_+-concave for μ-a.a. $y \in \Omega$, then $\mathcal{F} : U \to C(\Omega)^d$ is strictly $C(\Omega)_+^d$-concave.

PROOF. Let $u, \bar{u} \in U$.
(a) By Thm. 2.4(a) it remains to show $\mathcal{F}(u) \neq \mathcal{F}(\bar{u})$ for $u \prec \bar{u}$. There is a $y_0 \in \Omega$ such that $u(y_0) < \bar{u}(y_0)$ and thus $f(\bar{x},y_0,u(y_0)) < f(\bar{x},y_0,\bar{u}(y_0))$. Then Lemma A.1(a) yields for some $y'_0 \in Y'_+ \setminus \{0\}$ the strict inequality

$$0 < \langle f(\bar{x},y_0,\bar{u}(y_0)) - f(\bar{x},y_0,u(y_0)), y'_0 \rangle.$$

From the continuity of $f(\bar{x},\cdot)$ we conclude that there is an open neighborhood Ω_0 of y_0 with $0 < \langle f(\bar{x},y,\bar{u}(y)) - f(\bar{x},y,u(y)), y'_0 \rangle$ for all $y \in \Omega_0$. Furthermore, the estimate $0 \leqslant \langle f(\bar{x}, y, \bar{u}(y)) - f(\bar{x}, y, u(y)), y' \rangle$ is valid for all $y' \in Y'_+$ and μ-a.a. $y \in \Omega$. Integrating over Ω_0 and the fact $\mu(\Omega_0) > 0$ yield

$$\begin{aligned} 0 &< \int_{\Omega_0} \langle f(\bar{x}, y, \bar{u}(y)) - f(\bar{x}, y, u(y)), y'_0 \rangle \, \mathrm{d}\mu(y) \\ &\leqslant \int_{\Omega} \langle f(\bar{x}, y, \bar{u}(y)) - f(\bar{x}, y, u(y)), y'_0 \rangle \, \mathrm{d}\mu(y) \stackrel{(1.1)}{=} \langle \mathcal{F}(\bar{u})(\bar{x}) - \mathcal{F}(u)(\bar{x}), y'_0 \rangle. \end{aligned}$$

Therefore we have $\mathcal{F}(u)(\bar{x}) \neq \mathcal{F}(\bar{u})(\bar{x})$ and arrive at $\mathcal{F}(u) \prec \mathcal{F}(\bar{u})$.
(b) and (c) can be established using analogous arguments. $\square$

Theorem 2.6 (strong monotonicity of $\mathcal{F}$) *Let Hypothesis (U^0) hold, nonempty, open subsets of Ω have positive measure and let Y_+ be solid. If for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is strongly Y_+-monotone, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is strongly $C(\Omega)^d_+$-monotone.*

PROOF. Suppose that $u, \bar{u} \in U$, $u \prec \bar{u}$, $x \in \Omega$ and $y' \in Y'_+ \setminus \{0\}$. Note that $\Omega_0 = \{y \in \Omega : u(y) \neq \bar{u}(y)\}$ is open subset of Ω and $\mu(\Omega_0) > 0$. The strong monotonicity of $f(x, y, \cdot)$ yields the inequality $f(x, y, u(y)) \ll f(x, y, \bar{u}(y))$ and Lemma A.1(b) implies

$$0 < \langle f(x, y, \bar{u}(y)) - f(x, y, u(y)), y' \rangle =: \phi(y) \quad \text{for } \mu\text{-a.a. } y \in \Omega_0.$$

Now at least one of the preimages $\Omega_k := \phi^{-1}((\frac{1}{k}, \infty))$ for $k \in \mathbb{N}$ has positive measure, since $\Omega_0 = \bigcup_{k \in \mathbb{N}} \Omega_k$ is of positive measure. If $\mu(\Omega_l) > 0$, then

$$0 < \frac{\mu(\Omega_l)}{l} \leqslant \int_{\Omega_l} \phi(y) \, \mathrm{d}\mu(y) \leqslant \int_{\Omega_0} \phi(y) \, \mathrm{d}\mu(y) \leqslant \int_{\Omega} \phi(y) \, \mathrm{d}\mu(y)$$

and $\langle \mathcal{F}(u)(x), y' \rangle < \langle \mathcal{F}(\bar{u})(x), y' \rangle$ and $\mathcal{F}(u) \lll \mathcal{F}(\bar{u})$ holds due to Lemma A.1. $\square$

Theorem 2.7 (strong subhomogeneity and concavity of $\mathcal{F}$) *Let Hypothesis (U^0) hold with $\mu(\Omega) > 0$ and let Y_+ be solid. If for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is strongly*

(a) Y_+-subhomogeneous with $Z = Y_+$, then an Urysohn operator $\mathcal{F} : C(\Omega)^d_+ \to C(\Omega)^d_+$ is strongly $C(\Omega)^d_+$-subhomogeneous,
(b) Y_+-concave with Y_+-convex Z, then $\mathcal{F} : U \to C(\Omega)^d$ is strongly $C(\Omega)^d_+$-concave.

PROOF. Suppose that $\theta \in (0, 1)$, $u, \bar{u} \in U$, $u \lll \bar{u}$, $x \in \Omega$ and $y' \in Y'_+ \setminus \{0\}$.
(a) By assumption $f(x, y, \cdot) : Y_+ \to Y_+$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ we deduce that $0 \leqslant \langle f(x, y, u(y)), y' \rangle$ for all $x \in \Omega$, μ-a.a $y \in \Omega$ and all $y' \in Y'_+$. This implies $\mathcal{F}(u) \in C(\Omega)^d_+$. With $\phi(y) := \langle f(x, y, \theta u(y)) - \theta f(x, y, u(y)), y' \rangle$

for $y \in \Omega$ the argument is analogous to the proof of the Thm. 2.6.
(b) Proceed as above with the function $\phi(y) := \langle f(x, y, \theta u(y) + (1-\theta)\bar{u}(y)) - \theta f(x, y, u(y)) - (1-\theta)f(x, y, (1-\theta)u(y)), y' \rangle$ for $y \in \Omega$. □

The subsequent consequence might be of independent interest:

Corollary 2.8 (monotonicity of the integral) *Suppose $u, \bar{u} : \Omega \to \mathbb{R}^d$ are μ-integrable.*

(a) If $\langle u(y), y' \rangle \leqslant \langle \bar{u}(y), y' \rangle$ for μ-a.a. $y \in \Omega$ and every $y' \in Y'_+$, then the estimate $\int_\Omega u(y)\, \mathrm{d}\mu(y) \leqslant \int_\Omega \bar{u}(y)\, \mathrm{d}\mu(y)$ holds.
(b) Let $u, \bar{u} \in C(\Omega)^d$ and assume nonempty, open subsets of Ω have positive measure. If $u \prec \bar{u}$, then $\int_\Omega u(y)\, \mathrm{d}\mu(y) < \int_\Omega \bar{u}(y)\, \mathrm{d}\mu(y)$.
(c) If $\mu(\Omega) > 0$, Y_+ is solid and $\langle u(y), y' \rangle < \langle \bar{u}(y), y' \rangle$ for μ-a.a. $y \in \Omega$ and every $y' \in Y'_+ \setminus \{0\}$, then $\int_\Omega u(y)\, \mathrm{d}\mu(y) \ll \int_\Omega \bar{u}(y)\, \mathrm{d}\mu(y)$.

Let us remark that $u \preceq \bar{u}$ or $u \prec\!\!\prec \bar{u}$ imply the assumptions in (a) resp. (c) to hold (cf. Lemma 1.1) i.e. the integral is monotone in this sense.

PROOF. (a) and (b) result from Thms. 2.4(a) resp. 2.5(a) with the kernel function $f(x, y, z) := z$, whereas the argument for (c) is similar to the proof of Thm. 2.6 using $\phi(y) := \langle \bar{u}(y) - u(y), y' \rangle$ for μ-a.a. $y \in \Omega$. □

Under the additional Hypothesis (U^1) the derivative of $\mathcal{F}$ exists as

$$D\mathcal{F}(u)v = \int_\Omega D_3 f(\cdot, y, u(y))v(y)\, \mathrm{d}\mu(y) \quad \text{for all } v \in C(\Omega)^d \tag{2.2}$$

in the Fréchet-sense (in $u \in U^\circ$); moreover $\mathcal{F}$ is of class C^1. Then it is sometimes convenient to formulate assumptions on f in terms of conditions on the partial derivatives $D_3 f$:

Corollary 2.9 (monotonicity of $\mathcal{F}$) *Let (U^l), $l \in \{0, 1\}$, hold on a Y_+-convex and open $Z \subseteq \mathbb{R}^d$. If $D_3 f(x, y, z)$ is Y_+-positive for all $x \in \Omega$, $z \in Z$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is $C(\Omega)^d_+$-monotone. In addition, if nonempty, open subsets of Ω have positive measure and moreover*

(a) there exists a $\bar{x} \in \Omega$ so that $f(\bar{x}, \cdot)$ is continuous and for μ-a.a. $y \in \Omega$ and all $z, \bar{z} \in Z$, $z < \bar{z}$ the derivative $D_3 f(\bar{x}, y, z^*)$ is Y_+-injective for all $z^* \in \overline{z, \bar{z}}$, then $\mathcal{F}$ is strictly $C(\Omega)^d_+$-monotone,
(b) Y_+ is solid and $D_3 f(x, y, z)$ is strongly Y_+-positive for all $x \in \Omega$, $z \in Z$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is strongly $C(\Omega)^d_+$-monotone.

PROOF. It follows using Lemma 2.3 that U is also open and $C(\Omega)_+^d$-convex. Let $u \in U$ and $v \in C(\Omega)_+^d$. Due to our assumptions $\mathcal{F} : U \to C(\Omega)^d$ is of class C^1 and the derivative (2.2) is a Fredholm integral operator with kernel $k(x,y) := D_3 f(x,y,u(y))$. Then this kernel $k : \Omega^2 \to L(\mathbb{R}^d)$ satisfies the assumptions of [12, Thm. 2.7] and consequently $D\mathcal{F}(u)$ is positive. Therefore, the monotonicity of $\mathcal{F}$ follows using Lemma A.2.
(a) Let $x \in \Omega$. It results from Lemma A.2(a) that each $f(x,y,\cdot)$ is monotone, while $f(\bar{x},y,\cdot)$ is even strictly monotone for μ-a.a. $y \in \Omega$. Thus, Thm. 2.5(a) yields the claim.
(b) Now [12, Thm. 2.7(b)] applies to k and thus $D\mathcal{F}(u)$ is strongly positive. Therefore, Lemma A.2(b) implies that $\mathcal{F}$ is strongly monotone. □

Corollary 2.10 (subhomogeneity of $\mathcal{F}$) *Let (U^l), $l \in \{0,1\}$, hold on $Z = Y_+$.*

- *(a) If $D_3 f(x,y,z)$ is Y_+-positive and $D_3 f(x,y,z)z \leqslant f(x,y,z)$ for all $x \in \Omega$, $z \in Y_+$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is $C(\Omega)_+^d$-subhomogeneous.*
- *(b) If nonempty, open subsets of Ω have positive measure and in addition to the assumption of (a) there exists a $\bar{x} \in \Omega$ with $D_3 f(\bar{x},y,z)z < f(\bar{x},y,z)$ for μ-a.a. $y \in \Omega$ and all $z \in Y_+ \setminus \{0\}$, then $\mathcal{F}$ is strictly $C(\Omega)_+^d$-subhomogeneous.*
- *(c) If $\mu(\Omega) > 0$, Y_+ is solid and $D_3 f(x,y,z)z \ll f(x,y,z)$ for all $x \in \Omega$, $z \in Y_+^\circ$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is strongly $C(\Omega)_+^d$-subhomogeneous.*

PROOF. Let $u \in C(\Omega)_+^d$ be given.
(a) Above all, it follows by Cor. 2.9 that $\mathcal{F}$ is monotone. Combining this with $0 \leqslant f(x,y,0)$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$, we have $\mathcal{F} : C(\Omega)_+^d \to C(\Omega)_+^d$. Let $0 \prec u$. By assumption $D_3 f(x,y,u(y))u(y) \leqslant f(x,y,u(y))$ holds and Lemma A.1(a) guarantees $\langle D_3 f(x,y,u(y))u(y), y'\rangle \leqslant \langle f(x,y,u(y)), y'\rangle$ for μ-a.a. $y \in \Omega$, all $x \in \Omega$ and all $y' \in Y_+'$, while Cor. 2.8(a) yields

$$\begin{aligned}[D\mathcal{F}(u)u](x) &\overset{(2.2)}{=} \int_\Omega D_3 f(x,y,u(y))u(y)\,\mathrm{d}\mu(y) \\ &\leqslant \int_\Omega f(x,y,u(y))\,\mathrm{d}\mu(y) \overset{(F)}{=} \mathcal{F}(u)(x) \quad \text{for all } x \in \Omega.\end{aligned}$$

Therefore, Lemma A.3 implies that $\mathcal{F}$ is subhomogeneous.
(b) Let $0 \prec u$ and $\Omega_0 = \{y \in \Omega : u(y) \neq 0\}$. By assumption for μ-a.a. $y \in \Omega_0$ there exists $y_y' \in Y_+' \setminus \{0\}$ so that $\langle D_3 f(\bar{x},y,u(y))u(y), y_y'\rangle < \langle f(\bar{x},y,u(y)), y_y'\rangle$ holds. Thanks to the continuity for μ-a.a. $y \in \Omega_0$ there exists $\varepsilon_y > 0$ such that

$$\langle D_3 f(\bar{x},\tilde{y},u(\tilde{y}))u(\tilde{y}), y_y'\rangle < \langle f(\bar{x},\tilde{y},u(\tilde{y})), y_y'\rangle \quad \text{for any } \tilde{y} \in B_{\varepsilon_y}(y).$$

The family $\{B_{\varepsilon_y}(y) : \mu\text{-a.a. } y \in \Omega_0\}$ is an open cover of $\overline{\Omega_1}$, where Ω_1 is an open set satisfying $\overline{\Omega_1} \subset \Omega_0$, the Borel-Lebesgue Theorem yields a finite subcover

$\{B_{\varepsilon_{y_i}}(y_i) : 1 \leqslant i \leqslant n\}$ of the closure $\overline{\Omega_1}$. Moreover, $\mu(\Omega_1) > 0$. If we define $\bar{y}' = \sum_{i=1}^n y'_{y_i} \in Y'_+ \setminus \{0\}$, then $\langle D_3 f(\bar{x}, y, u(y))u(y), \bar{y}' \rangle < \langle f(\bar{x}, y, u(y)), \bar{y}' \rangle$ for any $y \in \Omega_1$ leads to

$$\begin{aligned}
&\langle [D\mathcal{F}(u)u](\bar{x}) - \mathcal{F}(u)(\bar{x}), \bar{y}' \rangle \\
\overset{(2.2)}{=}&\Big\langle \int_\Omega D_3 f(\bar{x}, y, u(y))u(y) - f(\bar{x}, y, u(y))\, \mathrm{d}\mu(y), \bar{y}' \Big\rangle \\
\leqslant&\Big\langle \int_{\Omega_1} D_3 f(\bar{x}, y, u(y))u(y) - f(\bar{x}, y, u(y))\, \mathrm{d}\mu(y), \bar{y}' \Big\rangle < 0.
\end{aligned}$$

This implies that $[D\mathcal{F}(u)u](\bar{x}) \neq \mathcal{F}(u)(\bar{x})$ and also $D\mathcal{F}(u)u \prec \mathcal{F}(u)$. Therefore, Lemma A.3(b) ensures that $\mathcal{F}$ is strictly subhomogeneous.
(c) Let $0 \prec\!\!\prec u$ and $y' \in Y'_+ \setminus \{0\}$. Here, $D_3 f(x, y, u(y))u(y) \ll f(x, y, u(y))$ holds by assumption and Lemma A.1(b) yields that $\langle D_3 f(x, y, u(y))u(y), y' \rangle < \langle f(x, y, u(y)), y' \rangle$ μ-a.a. $y \in \Omega$, which implies

$$\begin{aligned}
\langle [D\mathcal{F}(u)u](x), y' \rangle &\overset{(2.2)}{=} \Big\langle \int_\Omega D_3 f(x, y, u(y))u(y)\, \mathrm{d}\mu(y), y' \Big\rangle \\
&< \Big\langle \int_\Omega f(x, y, u(y))\, \mathrm{d}\mu(y), y' \Big\rangle \overset{(F)}{=} \langle \mathcal{F}(u)(x), y' \rangle
\end{aligned}$$

for all $x \in \Omega$. Whence, Lemma 1.1(c) yields $D\mathcal{F}(u)u \prec\!\!\prec \mathcal{F}(u)$ and Lemma A.3(c) implies that $\mathcal{F}$ is strongly subhomogeneous. □

Corollary 2.11 (concavity of $\mathcal{F}$) *Let (U^l), $l \in \{0, 1\}$, hold on a Y_+-convex $Z \subseteq \mathbb{R}^d$.*

- *(a) If $D_3 f(x, y, \bar{z})(\bar{z} - z) \leqslant D_3 f(x, y, z)(\bar{z} - z)$ for all $x \in \Omega$, $z, \bar{z} \in Z$, $z < \bar{z}$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is $C(\Omega)^d_+$-concave.*
- *(b) If nonempty, open subsets of Ω have positive measure and in addition to the assumption of (a) there exists a $\bar{x} \in \Omega$ such that $D_3 f(\bar{x}, y, \bar{z})(\bar{z} - z) < D_3 f(\bar{x}, y, z)(\bar{z} - z)$ for all μ-a.a. $y \in \Omega$ and $z \in Z$, then $\mathcal{F}$ is strictly $C(\Omega)^d_+$-concave.*
- *(c) If $\mu(\Omega) > 0$, Y_+ is solid and $D_3 f(x, y, \bar{z})(\bar{z} - z) \ll D_3 f(x, y, z)(\bar{z} - z)$ for all $x \in \Omega$, $z, \bar{z} \in Z$, $z \ll \bar{z}$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is strongly $C(\Omega)^d_+$-concave.*

PROOF. The arguments are analogous to those given in the above Cor. 2.10, but one uses Lemma A.5 instead of Lemma A.3. □

Remark 2.12 (order convexity) *The previous theory also allows to deduce conditions yielding the order convexity of Urysohn operators $\mathcal{F}$. For this simply replace the kernel function f by the negative $-f$ in the above criteria.*

3 Nyström methods

A natural way to evaluate an operator involving a $\mathbb{R}^d$-valued Lebesgue integral approximately, is to replace the integral by a quadrature rule

$$\int_\Omega u(y)\,\mathrm{d}y = \sum_{j=0}^{q_n} w_j u(\eta_j) + e_n(u) \tag{Q_n}$$

with weights $w_j \in \mathbb{R}$, nodes $\eta_j \in \Omega$, a strictly increasing sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_0$ and an error term $e_n(u) \in \mathbb{R}^d$. Applied to Urysohn operators (F) (with the Lebesgue measure $\mu = \lambda_\kappa$) one arrives at the *discrete Urysohn operator*

$$\mathcal{F}^n(u) := \sum_{j=0}^{q_n} w_j f(\cdot, \eta_j, u(\eta_j)). \tag{F^n}$$

In detail, we impose

Hypothesis 3.1 *Assume that $Z \subseteq \mathbb{R}^d$ has nonempty interior and that the kernel function $f : \Omega^2 \times Z \to \mathbb{R}^d$ fulfills for $l \in \{0, 1\}$:*

(NU^l) $D_3^l f : \Omega^2 \times Z \to L_l(\mathbb{R}^d)$ *exists as continuous function.*

Note that (NU^l) implies the above Hypothesis (U^l) for $l \in \{0, 1\}$. Furthermore, the discrete operator (F^n) allows the natural domains $U \subseteq C(\Omega)^d$ from (2.1) and $U_n := \{u : \Omega_n \to Z\}$. In both cases, (NU^0) ensures that $\mathcal{F}^n$ is well-defined on U and U_n.

Remark 3.2 ($\mathcal{F}^n$ on the domain U_n) *Assume an integration rule (Q_n) has nonnegative weights. Then the results from Sec. 2 apply to the specific measure*

$$\mu_n(\Omega_n) := \sum_{j=0}^{q_n} w_j, \qquad \Omega_n := \{\eta_j \in \Omega : 0 \leqslant j \leqslant q_n\}$$

under which the general integral operator (F) becomes a discrete Urysohn operator (F^n). Thus, the corresponding properties for $\mathcal{F}^n$ literally carry over from Sec. 2 with the assumption "μ-a.a. $y \in \Omega$" replaced by "all $y \in \Omega_n$".

For this reason we focus on the domain U (rather than U_n) from now on. Concerning the quadrature rule (Q_n) two features need to be pointed out. First, for numerical stability the commonly used rules (Q_n) have positive weights [6, pp. 51–52, Sect. 4.5]. The next results provide yet another motivation to apply such methods. Second, one requires that the distance between neighboring nodes in Ω_n can be made arbitrarily small as $n \to \infty$. In detail this means that (Q_n) fulfills the *net condition*

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \; \Omega \subseteq \bigcup_{j=0}^{q_n} B_\varepsilon(\eta_j) \quad \text{for all } n \geq n_0(\varepsilon). \tag{3.1}$$

In the following we investigate how properties such as monotonicity, subhomogeneity and order concavity of (F) persist under Nyström discretizations:

Theorem 3.3 (properties of $\mathcal{F}^n$ on U) *Let Hypothesis (NU^0) hold and a quadrature rule (Q_n), $n \in \mathbb{N}$, has nonnegative weights. If for all $x \in \Omega$ and $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is*

(a) Y_+-monotone, then a discrete Urysohn operator $\mathcal{F}^n : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-monotone,
(b) Y_+-subhomogeneous with $Z = Y_+$, then $\mathcal{F}^n : C(\Omega)^d_+ \to C(\Omega)^d_+$ is $C(\Omega)^d_+$-subhomogeneous,
(c) Y_+-concave with Y_+-convex Z, then $\mathcal{F}^n : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-concave.

PROOF. Mimic the proof of Thm. 2.4 with $y \in \Omega$ replaced by $\eta \in \Omega_n$ and the abstract integral $\int_\Omega \phi(y)\,\mathrm{d}\mu(y)$ by $\sum_{\eta \in \Omega_n} w_\eta \phi(\eta)$ for corresponding real-valued functions ϕ. $\square$

While monotonicity, subhomogeneity and order concavity directly extend from the kernel function to the discrete Urysohn operators, the situation changes for corresponding strict and strong notions. This is due to the fact that functions in $C(\Omega)^d_+ \setminus \{0\}$ might vanish everywhere except from arbitrarily small balls being disjoint from a grid Ω_n.

Nevertheless, for sufficiently large $n \in \mathbb{N}$ (depending on $u, \bar{u} \in U$ yet), that is for sufficiently fine grids Ω_n, the following weaker versions of strict monotonicity and strict concavity can be established:

Theorem 3.4 (eventual strict monotonicity and concavity of $\mathcal{F}^n$ on U) *Let Hypothesis (NU^0) hold with $\Omega = \overline{\Omega^\circ}$ and quadrature rules (Q_n) satisfying the net condition (3.1) with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is*

(a) Y_+-monotone and $f(\bar{x}, \eta, \cdot)$ is strictly Y_+-monotone for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $\mathcal{F}^n(u) \prec \mathcal{F}^n(\bar{u})$,
(b) Y_+-concave and $f(\bar{x}, \eta, \cdot)$ is strictly Y_+-concave for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$ with Y_+-convex Z, then

$$\theta \mathcal{F}^n(u) + (1-\theta)\mathcal{F}^n(\bar{u}) \prec \mathcal{F}^n(\theta u + (1-\theta)\bar{u}) \quad \textit{for all } \theta \in (0,1).$$

PROOF. Let $u, \bar{u} \in U$ with $u \prec \bar{u}$. This implies $u(x) \leqslant \bar{u}(x)$ for $x \in \Omega$ and there exists a $x_0 \in \Omega$ such that $u(x_0) \neq \bar{u}(x_0)$. Consequently, $B := \{x \in \Omega : u(x) \neq \bar{u}(x)\} \neq \emptyset$. By the continuity of $u, \bar{u}$ and the assumption on Ω there exist $\varepsilon_0 > 0$, $x_1 \in \Omega$ so that $B_{2\varepsilon_0}(x_1) \subset B$. By the net condition (3.1) we find an $n_0(\varepsilon_0) \in \mathbb{N}$ such that for $n \geq n_0(\varepsilon_0)$ there exist $j_n \in \{0, \ldots, q_n\}$ such that $\eta_{j_n} \in B$. From now on, assume (Q_n) have positive weights for $n \geq n_1$ with some $n_1 \in \mathbb{N}$ and let $n \geq N := \max\{n_0(\varepsilon_0), n_1\}$:
(a) Strict monotonicity of $f(\bar{x}, \eta_{j_n}, \cdot)$ implies $f(\bar{x}, \eta_{j_n}, u(\eta_{j_n})) < f(\bar{x}, \eta_{j_n}, \bar{u}(\eta_{j_n}))$. Thus, $\mathcal{F}^n(u)(\bar{x}) \neq \mathcal{F}^n(\bar{u})(\bar{x})$ and consequently $\mathcal{F}^n(u) \prec \mathcal{F}^n(\bar{u})$.
(b) results analogously. □

In the same spirit, also strict subhomogeneity and strong monotonicity hold merely for sufficiently large $n \in \mathbb{N}$:

Theorem 3.5 (eventual strict subhomogeneity of $\mathcal{F}^n$ on U) *Let Hypothesis (NU^0) hold with $\Omega = \overline{\Omega^\circ}$ and quadrature rules (Q_n) satisfying the net condition (3.1) with eventually positive weights. For each $0 \prec u$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Y_+ \to \mathbb{R}^d$ is Y_+-subhomogeneous and $f(\bar{x}, \eta, \cdot)$ is strictly Y_+-subhomogeneous for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $\theta\mathcal{F}^n(u) \prec \mathcal{F}^n(\theta u)$ for all $\theta \in (0, 1)$.*

PROOF. Let $\theta \in (0, 1)$, $0 \prec u$ and η_{j_n} as in the proof of Thm. 3.4. This implies $0 \leqslant u(x)$ for $x \in \Omega$ and there exists a $x_0 \in \Omega$ so that $0 \neq u(x_0)$ holds. Consequently, $B := \{x \in \Omega : 0 \neq u(x)\} \neq \emptyset$. Then strict subhomogeneity of the mapping $f(\bar{x}, \eta_{j_n}, \cdot)$ results in $\theta f(\bar{x}, \eta_{j_n}, u(\eta_{j_n})) < f(\bar{x}, \eta_{j_n}, \theta u(\eta_{j_n}))$, hence $\theta\mathcal{F}^n(u)(\bar{x}) \neq \mathcal{F}^n(\theta u)(\bar{x})$ and the inequality $\theta\mathcal{F}^n(u) \prec \mathcal{F}^n(\theta u)$ holds. □

Theorem 3.6 (eventual strong monotonicity of $\mathcal{F}^n$ on U) *Let Hypothesis (NU^0) hold with $\Omega = \overline{\Omega^\circ}$, a solid Y_+ and quadrature rules (Q_n) satisfying the net condition (3.1) with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is strongly Y_+-monotone, then $\mathcal{F}^n(u) \ll \mathcal{F}^n(\bar{u})$.*

PROOF. The argument is analogous to the proof of Thm. 3.4. □

Strong subhomogeneity and concavity however transfer from the kernel functions f to an Urysohn operator $\mathcal{F}$:

Theorem 3.7 (strong subhomogeneity and concavity of $\mathcal{F}^n$ on U) *Let Hypothesis (NU^0) hold with $\Omega = \overline{\Omega^\circ}$, a solid Y_+ and quadrature rules (Q_n)*

satisfying the net condition (3.1) *with positive weights. If for all* $x \in \Omega$, $\eta \in \Omega_n$ *a kernel function* $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ *is strongly*

(a) Y_+*-subhomogeneous with* $Z = Y_+$*, then a discrete Urysohn operator* $\mathcal{F}^n : C(\Omega)_+^d \to C(\Omega)_+^d$ *is strongly* $C(\Omega)_+^d$*-subhomogeneous.*
(b) Y_+*-concave with* Y_+*-convex* Z*, then* $\mathcal{F}^n : U \to C(\Omega)_+^d$ *is strongly* $C(\Omega)_+^d$*-concave.*

PROOF. (a) Let $\theta \in (0,1)$, $0 \lll u$, $y' \in Y_+' \setminus \{0\}$, $x \in \Omega$ and $\eta_{j_n} \in \Omega_n$. Consequently, $0 \ll u(y)$ for $y \in \Omega$ and the strong subhomogeneity of $f(x, \eta_{j_n}, \cdot)$ with Lemma A.1(b) yield $\langle \theta f(x, \eta_{j_n}, u(\eta_{j_n})), y' \rangle < \langle f(x, \eta_{j_n}, \theta u(\eta_{j_n})), y' \rangle$. Whence, $\langle \theta \mathcal{F}^n(u)(x), y' \rangle < \langle \mathcal{F}^n(\theta u)(x), y' \rangle$ holds and referring to Lemma 1.1(c) this means $\theta \mathcal{F}^n(u) \lll \mathcal{F}^n(\theta u)$.
(b) can be shown analogously. □

So far we have seen in Thm. 3.3 that positive quadrature weights in (Q_n) are sufficient for monotonicity, subhomogeneity and concavity to persist. The necessity of this property is explored in the next

Theorem 3.8 (necessary condition for monotonicity of $\mathcal{F}^n$ on U_n) *Let Hypothesis* (NU^0) *hold. If a discrete Urysohn operator* $\mathcal{F}^n : U_n \to C(\Omega_n)^d$ *is strictly* $C(\Omega_n)_+^d$*-monotone on* U_n *for some* $n \in \mathbb{N}$ *and* $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ *is* Y_+*-monotone for all* $x \in \Omega$, $\eta \in \Omega_n$*, then* (Q_n) *has positive weights.*

PROOF. Let $z, \bar{z} \in Z$ with $z < \bar{z}$ and $j \in \{0, \ldots, q_n\}$. The functions

$$u_j, \bar{u}_j : \Omega_n \to Z, \qquad u_j(x) :\equiv z, \qquad \bar{u}_j(x) := \begin{cases} z & x \neq \eta_j, \\ \bar{z} & x = \eta_j \end{cases}$$

satisfy $u_j, \bar{u}_j \in U_n$ and $u_j \prec \bar{u}_j$ with $\bar{u}_j(\eta_j) = \bar{z}$. Due to the strict $C(\Omega_n)_+^d$-monotonicity of $\mathcal{F}^n$ on U_n this leads to $\mathcal{F}^n(u_j) \prec \mathcal{F}^n(\bar{u}_j)$. By Lemma 1.1(b) and Lemma A.1(a) there are $x_0 \in \Omega$ and $y_0' \in Y_+ \setminus \{0\}$ such that

$$\begin{aligned} 0 &< \langle \mathcal{F}^n(\bar{u}_j)(x_0) - \mathcal{F}^n(u_j)(x_0), y_0' \rangle \\ &= \langle \sum_{k=0}^{q_n} w_k \Big(f(x_0, \eta_k, \bar{u}_j(\eta_k)) - f(x_0, \eta_k, u_j(\eta_k)) \Big), y_0' \rangle \\ &= w_j \langle f(x_0, \eta_j, \bar{z}) - f(x_0, \eta_j, z), y_0' \rangle. \end{aligned}$$

Furthermore, the Y_+-monotonicity of $f(x_0, \eta_j, \cdot)$ and Lemma A.1(a) guarantee $0 \leqslant \langle f(x_0, \eta_j, \bar{z}) - f(x_0, \eta_j, z), y_0' \rangle$. Hence, $0 < \langle f(x_0, \eta_j, \bar{z}) - f(x_0, \eta_j, z), y_0' \rangle$ and therefore results $w_j > 0$. Because $j \in \{0, \ldots, q_n\}$ was arbitrary this implies the assertion. □

As in the framework of Sec. 2 it is convenient to provide conditions for monotonicity, subhomogeneity and concavity in terms of the derivatives

$$D\mathcal{F}^n(u)v = \sum_{j=0}^{q_n} w_j D_3 f(\cdot, \eta_j, u(\eta_j)) \quad \text{for all } u \in U^\circ,\ v \in C(\Omega)^d; \tag{3.2}$$

note that $\mathcal{F}^n$ is continuously differentiable.

Corollary 3.9 (monotonicity of $\mathcal{F}^n$ on U) *Let (NU^l) for $l \in \{0,1\}$ hold on an Y_+-convex, open $Z \subseteq \mathbb{R}^d$ and (Q_n), $n \in \mathbb{N}$, has nonnegative weights. If $D_3 f(x, \eta, z)$ is Y_+-positive for all $x \in \Omega$, $\eta \in \Omega_n$, $z \in Z$, then $\mathcal{F}^n$ is $C(\Omega)^d_+$-monotone on U. In case additionally $\Omega = \overline{\Omega^\circ}$, (Q_n) have eventually positive weights and the net condition* (3.1) *hold, then for each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$:*

- *(a) If there exists a $\bar{x} \in \Omega$ so that for all $\eta \in \Omega_n$ and $z, \bar{z} \in Z$, $z < \bar{z}$ the derivative $D_3 f(\bar{x}, \eta, z^*)$ is Y_+-injective for $z^* \in \overline{z, \bar{z}}$, then $\mathcal{F}^n(u) \prec \mathcal{F}^n(\bar{u})$.*
- *(b) If Y_+ is solid and $D_3 f(x, \eta, z)$ is strongly Y_+-positive for all $x \in \Omega$, $\eta \in \Omega_n$ and $z \in Z$, then $\mathcal{F}^n(u) \prec\!\!\prec \mathcal{F}^n(\bar{u})$.*

PROOF. Let $u \in U$ and $v \in C(\Omega)^d_+$. With Lemma 2.3 the set U is open and $C(\Omega)^d_+$-convex, while $\mathcal{F} : U \to C(\Omega)^d$ is of class C^1 and the derivative (2.2) is a Fredholm integral operator with kernel $k(x, y) = D_3 f(x, y, u(y))$. Since k satisfies the assumptions of [12, Thm. 3.4], $D\mathcal{F}^n(u)$ is positive. Hence, the monotonicity of $\mathcal{F}^n$ follows using Lemma A.2.
(a) Lemma A.2(a) shows that each $f(x, y, \cdot)$, $x \in \Omega$, is monotone, while $f(\bar{x}, y, \cdot)$ is even strictly monotone for μ-a.a. $y \in \Omega$. Therefore, Thm. 3.4(a) yields the claim.
(b) Here [12, Thm. 3.4(d)] can be applied to k. Whence, $D\mathcal{F}^n(u)$ is strongly positive and Lemma A.2(b) implies that $\mathcal{F}^n$ is strongly monotone. □

Based on arguments from the proof of the corresponding Cor. 2.10 one obtains:

Corollary 3.10 (subhomogeneity and eventual strict subhomogeneity of $\mathcal{F}^n$ on U) *Let (U^l) for $l \in \{0,1\}$ hold on $Z = Y_+$ and (Q_n), $n \in \mathbb{N}$, have nonnegative weights. If $D_3 f(x, \eta, z)$ is Y_+-positive and $D_3 f(x, \eta, z)z \leqslant f(x, \eta, z)$ for all $x \in \Omega$, $\eta \in \Omega_n$, $z \in Y_+$, then $\mathcal{F}^n$ is $C(\Omega)^d_+$-subhomogeneous. In case additionally $\Omega = \overline{\Omega^\circ}$, (Q_n) have eventually positive weights and the net condition* (3.1) *hold, then for each $0 \prec u$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If there exists a $\bar{x} \in \Omega$ such that $D_3 f(\bar{x}, \eta, z)z < f(\bar{x}, \eta, z)$ for all $\eta \in \Omega_n$ and $z \in Y_+ \setminus \{0\}$, then $\theta \mathcal{F}^n(u) \prec \mathcal{F}^n(\theta u)$ for all $\theta \in (0,1)$.*

Corollary 3.11 (strong subhomogeneity of $\mathcal{F}^n$ on U) *Let (U^l) for $l \in \{0,1\}$ hold on $Z = Y_+$ with solid Y_+ and (Q_n), $n \in \mathbb{N}$, have positive weights.*

If $D_3 f(x,\eta,z)$ is Y_+-positive, $D_3 f(x,\eta,z)z \leqslant f(x,\eta,z)$ for all $x \in \Omega$, $\eta \in \Omega_n$, $z \in Y_+$ and $D_3 f(x,\eta,z)z \ll f(x,\eta,z)$ for all $x \in \Omega$, $\eta \in \Omega_n$ and $z \in Y_+^\circ$, then $\mathcal{F}^n$ is strongly $C(\Omega)_+^d$-subhomogeneous.

Similarly, proceeding as in the proof of the corresponding Cor. 2.11 it results:

Corollary 3.12 (concavity and eventual strict concavity of $\mathcal{F}^n$ on U) *Let (U^l) for $l \in \{0,1\}$ hold on a Y_+-convex Z and (Q_n), $n \in \mathbb{N}$, have nonnegative weights. If $D_3 f(x,\eta,\bar{z})(\bar{z}-z) \leqslant D_3 f(x,\eta,z)(\bar{z}-z)$ for all $x \in \Omega$, $\eta \in \Omega_n$, $z,\bar{z} \in Z$, $z < \bar{z}$, then $\mathcal{F}^n$ is $C(\Omega)_+^d$-concave. In case additionally $\Omega = \overline{\Omega^\circ}$, (Q_n) have eventually positive weights and the net condition (3.1) hold, then for each $u,\bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If there exists a $\bar{x} \in \Omega$ such that $D_3 f(\bar{x},\eta,\bar{z})(\bar{z}-z) < D_3 f(\bar{x},\eta,z)(\bar{z}-z)$ for all $\eta \in \Omega_n$ and $z,\bar{z} \in Z$, $z < \bar{z}$, then*

$$\theta\mathcal{F}^n(u) + (1-\theta)\mathcal{F}^n(\bar{u}) \prec \mathcal{F}^n(\theta u + (1-\theta)\bar{u}) \quad \text{for all } \theta \in (0,1).$$

Corollary 3.13 (strong concavity of $\mathcal{F}^n$ on U) *Let (U^l), $l \in \{0,1\}$, hold on a Y_+-convex Z with solid Y_+ and (Q_n), $n \in \mathbb{N}$, have positive weights. If $D_3 f(x,\eta,\bar{z})(\bar{z}-z) \ll D_3 f(x,\eta,z)(\bar{z}-z)$ for all $x \in \Omega$, $\eta \in \Omega_n$ and $z,\bar{z} \in Z$, $z \ll \bar{z}$, then $\mathcal{F}^n$ is strongly $C(\Omega)_+^d$-concave.*

4 Collocation methods

Another popular approach to tackle nonlinear integral equations numerically are collocation methods [2, Sect. 2]. On an abstract level, they are based on linear projections $\Pi_n : C(\Omega)^d \to X_n^d$, where each X_n is a finite-dimensional subspace of $C(\Omega)$. This results in a finite-dimensional problem involving an operator

$$\mathcal{F}^n = \Pi_n \mathcal{F} : U \to X_n^d \tag{4.1}$$

rather than $\mathcal{F}$. Monotonicity properties of such projections $\Pi_n \in L(C(\Omega)^d)$ were studied in [12, Sect. 4].

Theorem 4.1 (properties of collocation methods) *Let $\mathcal{F} : U \to C(\Omega)^d$ be an Urysohn operator (F).*

(a) Let Π_n be $C(\Omega)_+^d$-positive. If $\mathcal{F}$ is $C(\Omega)_+^d$-monotone (-subhomogeneous or -concave), then $\mathcal{F}^n : U \to X_n^d$ is $C(\Omega)_+^d$-monotone (-subhomogeneous resp. -concave). If additionally $\Pi_n U \subseteq U$ holds, then the same is true for $\mathcal{F} \circ \Pi_n : U \to C(\Omega)^d$.

(b) Let $\Pi_n (C(\Omega)_+^d)^\circ \subseteq (C(\Omega)_+^d)^\circ$ with a solid Y_+. If $\mathcal{F}$ is strongly $C(\Omega)_+^d$-monotone (-subhomogeneous or -concave), then $\mathcal{F}^n$ is strongly $C(\Omega)_+^d$-monotone (-subhomogeneous resp. -concave). If additionally $\Pi_n U \subseteq U$, then strong $C(\Omega)_+^d$-subhomogeneity (or -concavity) of $\mathcal{F}$ extends to $\mathcal{F} \circ \Pi_n$.

PROOF. The reader directly verifies from the definition that the corresponding properties of $\mathcal{F}$ are preserved under composition with Π_n. $\square$

More detailed, [12, Thm. 4.2] contains sufficient conditions for Π_n to be positive or to satisfy $\Pi_n(C(\Omega)^d_+)^\circ \subseteq (C(\Omega)^d_+)^\circ$. These criteria can be combined with our above analysis providing related assumptions on the kernel functions f such that Thm. 4.1 applies.

5 Applications

5.1 Fixed points of a Leslie-Gower model

Assume $k : \Omega^2 \to L(\mathbb{R}^2)$ and the coefficient functions $c^1, c^2 : \Omega \to (0,\infty)$, $b^1, b^2 : \Omega \to \mathbb{R}_+$ are continuous. We consider the nonlinear integral equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \int_\Omega k(\cdot,y) \begin{pmatrix} \frac{c^1(y)u(y)}{1+u(y)+b^1(y)v(y)} \\ \frac{c^2(y)v(y)}{1+b^2(y)u(y)+v(y)} \end{pmatrix} \mathrm{d}y, \tag{5.1}$$

whose right-hand side is an Urysohn operator (F) with $Z = \mathbb{R}^2_+$, the kernel function $f(x,y,z) := k(x,y)g(y,z)$ and the measure $\mu = \lambda_\kappa$. Abbreviating

$$g : \Omega \times \mathbb{R}^2_+ \to \mathbb{R}^2_+, \qquad g(y,z) := \begin{pmatrix} \frac{c^1(y)z_1}{1+z_1+b^1(y)z_2} \\ \frac{c^2(y)z_2}{1+b^2(y)z_1+z_2} \end{pmatrix}$$

the partial derivative of f is given as $D_3 f(x,y,z) = k(x,y)D_2 g(y,z)$ with

$$D_2 g(y,z) = \begin{pmatrix} \frac{c^1(y)(1+b^1(y)z_2)}{(1+z_1+b^1(y)z_2)^2} & -\frac{b^1(y)c^1(y)z_1}{(1+z_1+b^1(y)z_2)^2} \\ -\frac{b^2(y)c^2(y)z_2}{(1+b^2(y)z_1+z_2)^2} & \frac{c^2(y)(1+b^2(y)z_1)}{(1+b^2(y)z_1+z_2)^2} \end{pmatrix}.$$

The south-east cone $Y_+ := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \leqslant 0 \leqslant x_1\}$ is generated by the linearly independent vectors $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{-1}$ and we choose $e'_1 = \binom{1}{0}$, $e'_2 = \binom{0}{-1}$ according to $\langle e_i, e'_j \rangle = \delta_{ij}$ for $1 \leqslant i,j \leqslant 2$. Thus, the inequalities

$$\langle D_2 g(y,z)e_1, e'_1 \rangle = \tfrac{c^1(y)(1+b^1(y)z_2)}{(1+z_1+b^1(y)z_2)^2} > 0, \quad \langle D_2 g(y,z)e_1, e'_2 \rangle = \tfrac{b^2(y)c^2(y)z_2}{(1+b^2(y)z_1+z_2)^2} \geq 0,$$
$$\langle D_2 g(y,z)e_2, e'_1 \rangle = \tfrac{b^1(y)c^1(y)z_1}{(1+b^2(y)z_1+z_2)^2} \geq 0, \quad \langle D_2 g(y,z)e_2, e'_2 \rangle = \tfrac{c^2(y)(1+b^2(y)z_1)}{(1+b^2(y)z_1+z_2)^2} > 0$$

guarantee that $D_3 f(x,y,z)$ is Y_+-monotone for all $x,y \in \Omega$ and $z \in \mathbb{R}^2_+$, provided $k(x,y)$ has nonnegative entries.

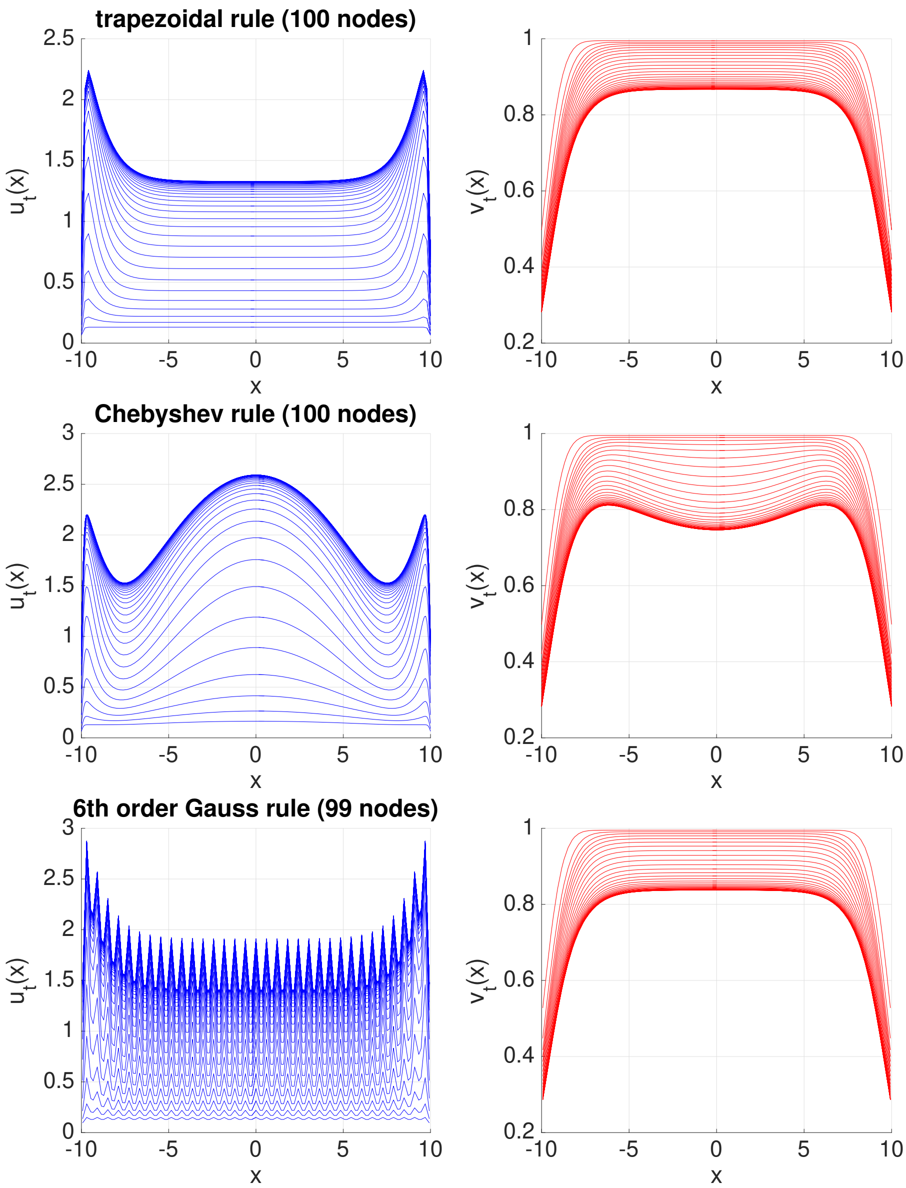

Fig. 1. Monotone convergence of solutions (u_t, v_t) for (5.2) to extremal solutions for Nyström discretizations of (5.1) based on different quadrature rules [12, App. C]

Although the monotone convergence is preserved for Nyström methods based on quadrature rules (Q_n) having positive weights, the shape of the extremal solutions varies significantly:

Example 5.1 (effect of quadrature rules) *Consider a Leslie-Gower model* (5.1) *with the habitat* $\Omega = [-10, 10]$, *a diagonal Gauß kernel*

$$k(x,y) := \begin{pmatrix} \frac{1}{\sqrt{2\pi\alpha_1^2}} \exp\left(-\frac{(x-y)^2}{2\alpha_1^2}\right) & 0 \\ 0 & \frac{1}{\sqrt{2\pi\alpha_2^2}} \exp\left(-\frac{(x-y)^2}{2\alpha_2^2}\right) \end{pmatrix}$$

having dispersal rates $\alpha_1, \alpha_2 > 0$ and constant parameters

$$\begin{aligned} &\alpha_1 = 0.1, && c^1(x) \equiv 4, && b^1(x) \equiv 2, \\ &\alpha_2 = 0.9, && c^2(x) \equiv 2, && b^2(x) \equiv 0.1 \quad \text{on } \Omega. \end{aligned}$$

The iterates (u_t, v_t) for the Nyström discretizations of the recursion

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \int_\Omega k(\cdot, y) \begin{pmatrix} \frac{c^1(y)u_t(y)}{1+u_t(y)+b^1(y)v_t(y)} \\ \frac{c^2(y)v_t(y)}{1+b^2(y)u_t(y)+v_t(y)} \end{pmatrix} \mathrm{d}y, \tag{5.2}$$

based on various quadrature methods are illustrated in Fig. 1. Since in particular the first component of the kernel function is not very 'smooth' (note that $\alpha_1 = 0.1$), the limit functions have rather different shapes. This effect is due to the comparatively small number $q_n \leqslant 100$ of nodes used in the discretization and balances out for larger values of q_n.

5.2 Integrodifference equations

Suppose that $\mathbb{I}$ is a discrete interval, i.e. the intersection of a real interval with the integers, and set $\mathbb{I}' := \{t \in \mathbb{I} : t+1 \in \mathbb{I}\}$. We are interested in nonautonomous difference equations

$$u_{t+1} = \mathcal{F}_t(u_t), \tag{I_0}$$

whose right-hand sides are Urysohn operators

$$\mathcal{F}_t : U_t \to C(\Omega)^d, \qquad \mathcal{F}_t(u) := \int_\Omega f_t(\cdot, y, u(y))\,\mathrm{d}\mu(y) \quad \text{for all } t \in \mathbb{I}'. \tag{5.3}$$

One speaks of an *integrodifference equation* (I_0) (briefly, IDE). For well-definedness of $\mathcal{F}_t$, assume throughout that $Z_t \subseteq \mathbb{R}^d$ is nonempty and all kernel functions $f_t : \Omega^2 \times Z_t \to \mathbb{R}^d$ satisfy at least the assumptions (U^l) for $l = 0$ resp. $l \in \{0, 1\}$, when derivatives are involved. Moreover, define the domain

$$U_t := \left\{ u \in C(\Omega)^d : \, u(x) \in Z_t \text{ for all } x \in \Omega \right\}$$

and suppose that $\mathcal{F}_t(U_t) \subseteq U_{t+1}$ holds for all $t \in \mathbb{I}'$. The forward solution to (I_0) starting at an initial time $\tau \in \mathbb{I}$ in the initial state $u_\tau \in U_\tau$ given by

$$\varphi(t; \tau, u_\tau) := \begin{cases} \mathcal{F}_{t-1} \circ \ldots \circ \mathcal{F}_\tau(u_\tau), & \tau < t, \\ u_\tau, & t = \tau \end{cases} \tag{5.4}$$

is the *general solution* $\varphi : \left\{ (t, \tau, u) \in \mathbb{I} \times \mathbb{I} \times C(\Omega)^d : \, \tau \leqslant t, \, u \in U_\tau \right\} \to C(\Omega)^d$ to the IDE (I_0).

		strict	strong
monotone	(M0)	(M1)	(M2)
	Thm. 2.4(a)	Thm. 2.5(a)	Thm. 2.6
	Cor. 2.9	Cor. 2.9(a)	Cor. 2.9(b)
subhomogeneous	(S0)	(S1)	(S2)
	Thm. 2.4(b)	Thm. 2.5(b)	Thm. 2.7(a)
	Cor. 2.10(a)	Cor. 2.10(b)	Cor. 2.10(c)
order concave	(C0)	(C1)	(C2)
	Thm. 2.4(c)	Thm. 2.5(c)	Thm. 2.7(b)
	Cor. 2.11(a)	Cor. 2.11(b)	Cor. 2.11(c)

Table 1
Assumptions guaranteeing various degrees of monotonicity, subhomogeneity or order concavity for IDEs (I_0)

The purpose of Tab. 1 is to encode the assumptions required in the subsequent results in a space saving manner. For instance, (M0) means that the assumptions of Thm. 2.4(a) or of Cor. 2.9 (which guarantee monotonicity) are satisfied. Given this, we obtain

Theorem 5.2 (monotonicity of IDEs) *Let $\tau < t$. If the kernel functions $f_s : \Omega^2 \times Z_s \to \mathbb{R}^d$ satisfy*

(a) (M0) for $\tau \leqslant s < t$, then the general solution $\varphi(t; \tau, \cdot) : U_\tau \to U_t$ of (I_0) is $C(\Omega)^d_+$-monotone on U_τ,
(b) (M1) for $\tau \leqslant s < t$, then $\varphi(t; \tau, \cdot)$ is strictly $C(\Omega)^d_+$-monotone on U_τ,
(c) (M2) for $s = t - 1$ and (M1) for all $\tau \leqslant s < t - 1$, then $\varphi(t; \tau, \cdot)$ is strongly $C(\Omega)^d_+$-monotone on U_τ.

PROOF. For mappings $\mathcal{G}_i : V_i \subseteq C(\Omega)^d \to C(\Omega)^d$, $i = 0, 1$, with $\mathcal{G}_0(V_0) \subseteq V_1$ we obtain directly from the definition (see App. A) that the following holds:

- If $\mathcal{G}_0, \mathcal{G}_1$ are monotone, then also $\mathcal{G}_1 \circ \mathcal{G}_0$ is monotone.
- If $\mathcal{G}_0, \mathcal{G}_1$ are strictly monotone, then also $\mathcal{G}_1 \circ \mathcal{G}_0$ is strictly monotone.
- If $\mathcal{G}_0$ is strictly monotone, while $\mathcal{G}_1$ is strongly monotone, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is strongly monotone.

Since Thms. 2.4–2.6 and Cor. 2.9 provide sufficient conditions for (strict, strong) monotonicity of $\mathcal{F}_s$, this combined with (5.4) yield the claims. □

Theorem 5.3 (subhomogeneity of IDEs) *Let $\tau < t$. If the kernel functions $f_s : \Omega^2 \times Y_+ \to Y_+$ satisfy*

(a) *(S0) for* $\tau \leqslant s < t$ *and (M0) for* $\tau < s < t$*, then the general solution* $\varphi(t;\tau,\cdot) : C(\Omega)^d_+ \to C(\Omega)^d_+$ *of* (I_0) *is* $C(\Omega)^d_+$*-subhomogeneous,*
(b) *(S1) for* $s = \tau$ *and (S0), (M1) for all* $\tau < s < t$*, or*
(S0) for $\tau \leqslant s < t-1$ *and (M0) for* $\tau < s < t$*, (S1) for* $s = t-1$ *and the inclusion* $\mathcal{F}_s(C(\Omega)^d_+ \setminus \{0\}) \subseteq C(\Omega)^d_+ \setminus \{0\}$ *hold for* $\tau \leqslant s < t-1$*,*
then $\varphi(t;\tau,\cdot)$ *is strictly* $C(\Omega)^d_+$*-subhomogeneous,*
(c) *(S2) for* $s = \tau$ *and (S0), (M1) for all* $\tau < s < t$*, or*
(S0) for $\tau \leqslant s < t-1$ *and (M0) for* $\tau < s < t$*, (S2) for* $s = t-1$ *and the inclusion* $\mathcal{F}_s((C(\Omega)^d_+)^\circ) \subseteq (C(\Omega)^d_+)^\circ$ *for* $\tau \leqslant s < t-1$ *hold,*
then $\varphi(t;\tau,\cdot)$ *is strongly* $C(\Omega)^d_+$*-subhomogeneous.*

PROOF. Let $X_+ := C(\Omega)^d_+$. For $\mathcal{G}_i : X_+ \to X_+$, $i = 0, 1$, we get by App. A:

- If $\mathcal{G}_0, \mathcal{G}_1$ are subhomogeneous and $\mathcal{G}_1$ is monotone, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is subhomogeneous.
- If $\mathcal{G}_0$ is strictly subhomogeneous and $\mathcal{G}_1$ is subhomogeneous and strictly monotone, or if $\mathcal{G}_0$ is subhomogeneous with $\mathcal{G}_0(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$ and $\mathcal{G}_1$ is monotone and strictly subhomogeneous, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is strictly subhomogeneous.
- If $\mathcal{G}_0$ is strongly subhomogeneous and $\mathcal{G}_1$ is subhomogeneous and strongly monotone, or if $\mathcal{G}_0$ is subhomogeneous with $\mathcal{G}_0(X_+^\circ) \subseteq X_+^\circ$ and $\mathcal{G}_1$ is monotone and strongly subhomogeneous, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is strongly subhomogeneous.

(a) Note that $\mathcal{G}_0 := \mathcal{F}_\tau$ is subhomogeneous and $\mathcal{G}_1 := \varphi(t;\tau+1,\cdot)$, $\tau < t$, is monotone and subhomogeneous, thus $\varphi(t;\tau,\cdot)$ is subhomogeneous.
(b) We restrict to the second set of assumptions with mappings $\mathcal{G}_1 = \mathcal{F}_{t-1}$ and $\mathcal{G}_0 := \varphi(t-1;\tau,\cdot)$. Then $\mathcal{G}_0$ is subhomogeneous by (a) and satisfies the inclusion $\mathcal{G}_0(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$, while $\mathcal{G}_1$ is monotone and strictly subhomogeneous.
(c) results analogously to (b). □

The next result guaranteeing $C(\Omega)^d_+$-concavity has simplified assumptions:

Theorem 5.4 (concavity of IDEs) *Let* $\tau < t$*. If the kernel functions* $f_s : \Omega^2 \times Z_s \to \mathbb{R}^d$ *with* Y_+*-convex* Z_s *satisfy*

(a) *(C0), (M0) for* $\tau \leqslant s < t$*, then the general solution* $\varphi(t;\tau,\cdot) : U_\tau \to U_t$ *of* (I_0) *is* $C(\Omega)^d_+$*-concave on* U_τ*,*
(b) *(C1), (M1) for* $\tau \leqslant s < t$*, then* $\varphi(t;\tau,\cdot)$ *is strictly* $C(\Omega)^d_+$*-concave on* U_τ*,*
(c) *(C2), (M2) for* $\tau \leqslant s < t$*, then* $\varphi(t;\tau,\cdot)$ *is strongly* $C(\Omega)^d_+$*-concave on* U_τ*.*

PROOF. For $\mathcal{G}_i : V_i \subseteq C(\Omega)^d \to C(\Omega)^d$, $i = 0, 1$, with $\mathcal{G}_0(V_0) \subseteq V_1$ the definitions given in App. A directly yield:

- If $\mathcal{G}_0, \mathcal{G}_1$ are X_+-concave and monotone, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is order concave.
- If $\mathcal{G}_0$ is strictly concave and strictly monotone and $\mathcal{G}_1$ is order concave and strictly monotone, or if $\mathcal{G}_0$ is order concave and strictly monotone and $\mathcal{G}_1$ is strictly order concave and monotone, then also $\mathcal{G}_1 \circ \mathcal{G}_0$ is strictly order concave.
- If $\mathcal{G}_0$ is strongly order concave and strictly monotone and $\mathcal{G}_1$ is order concave and strongly monotone, or if $\mathcal{G}_0$ is order concave and strongly monotone and $\mathcal{G}_1$ is strongly order concave and monotone, then $\mathcal{G}_1 \circ \mathcal{G}_0$ is strongly order concave.

Given this, the proof follows as above. □

5.2.1 Structure-preserving discretizations

When it comes to numerical simulations of IDEs (I_0) one has to restrict to finite-dimensional subspaces of $C(\Omega)^d$ and the integral in (5.3) can be evaluated only approximately. One achieves this by applying discretization methods for the numerical solution of integral equations (cf. [2]) to the right-hand side of (I_0) yielding a difference equation

$$u_{t+1} = \mathcal{F}_t^n(u_t) \tag{I_n}$$

on the state space $C(\Omega)^d$. This means the right-hand side $\mathcal{F}_t$ is replaced by a Nyström discretization (F^n) or collocation semi-discretization (4.1).

For Nyström discretizations we provided criteria that structural properties persist in Sec. 3. Hence, Thms. 5.2–5.4 also hold for the general solution

$$\varphi^n(t;\tau,u_\tau) := \begin{cases} \mathcal{F}_{t-1}^n \circ \ldots \circ \mathcal{F}_\tau^n(u_\tau), & \tau < t, \\ u_\tau, & t = \tau \end{cases}$$

of (I_n) with the conditions from Tab. 1 replaced by the corresponding results of Sec. 3. For collocation methods, related results are based on Thm. 4.1.

5.2.2 Periodic generalized Beverton-Holt models

We finally illustrate how a monotone iteration technique (see [14, pp. 163ff, Chap. 11] or [15, pp. 269ff, Chap. 7]) can be used to approximate periodic solutions to IDEs (I_0) and their Nyström discretizations.

Let $\theta \in \mathbb{N}$, $\delta \in (0,1]$ and $\Omega \subset \mathbb{R}^\kappa$ be compact. Suppose that $\tilde{k} : \mathbb{R}^\kappa \to \mathbb{R}_+$ is a continuous function such that there exist reals $0 < k_- \leqslant k_+$ satisfying

$$k_- \leqslant \tilde{k}(x-y) \leqslant k_+ \quad \text{for all } x \in \{y_1 - y_2 \in \mathbb{R}^\kappa : \, y_1, y_2 \in \Omega\}$$

and that $\mathbb{R}$ is equipped with the cone $Y_+ = \mathbb{R}_+$. Under these assumptions we consider the *spatial generalized Beverton-Holt model*

$$u_{t+1}(x) = \int_\Omega \tilde{k}(x-y)\frac{c_t(y)u_t(y)}{1+u_t(y)^\delta}\,\mathrm{d}\mu(y) \quad \text{for all } x \in \Omega, \tag{5.5}$$

where $c_{t+\theta} = c_t : \Omega \to \mathbb{R}_+$, $t \in \mathbb{Z}$, are continuous functions. As a result, (5.5) becomes a θ-periodic difference equation, whose right-hand side is an Urysohn operator with the kernel function

$$f : \Omega \times \Omega \times Y_+ \to Y_+, \qquad f(x,y,z) := \tilde{k}(x-y)\frac{c_t(y)z}{1+z^\delta};$$

for the Lebesgue measure $\mu = \lambda_\kappa$ one obtains an IDE as in [10]. In order to study it along with its Nyström discretizations, we consider the monotone mappings (being subhomogeneous and order concave as well)

$$\mathcal{F}_t^n : C(\Omega)_+ \to C(\Omega)_+, \qquad \mathcal{F}_t^n(u)(x) := \int_\Omega \tilde{k}(x-y)\frac{c_t(y)u(y)}{1+u(y)^\delta}\,\mathrm{d}\mu(y)$$

for the measures $\mu \in \{\lambda_\kappa, \mu_n\}$, where the weighted counting measures μ_n are introduced in Rem. 3.2. Under the assumption $1 < k_- \min_{s=0}^{\theta-1} \int_\Omega c_s(y)\,\mathrm{d}\mu(y)$ we define the reals

$$u_- := \min_{s=0}^{\theta-1}\left(k_-\int_\Omega c_s(y)\,\mathrm{d}\mu(y) - 1\right)^{1/\delta}, \quad u_+ := \max_{s=0}^{\theta-1}\left(k_+\int_\Omega c_s(y)\,\mathrm{d}\mu(y) - 1\right)^{1/\delta}$$

and obtain for all $u \in C(\Omega)$ satisfying $u_- \leqslant u(x) \leqslant u_+$ on Ω the estimates

$$\begin{aligned} u_- &\leqslant \int_\Omega k_-\frac{c_s(y)u_-}{1+u_-^\delta}\,\mathrm{d}\mu(y) \leqslant \int_\Omega k_-\frac{c_s(y)u(y)}{1+u(y)^\delta}\,\mathrm{d}\mu(y) \\ &\leqslant \mathcal{F}_s(u)(x) \\ &\leqslant \int_\Omega k_+\frac{c_s(y)u(y)}{1+u(y)^\delta}\,\mathrm{d}\mu(y) \leqslant \int_\Omega k_+\frac{c_s(y)u_+}{1+(u_+)^\delta}\,\mathrm{d}\mu(y) \leqslant u_+ \quad \text{for all } s \in \mathbb{Z} \end{aligned}$$

and $x \in \Omega$. Consequently the restrictions $\mathcal{F}_s : [u_-, u_+] \to [u_-, u_+]$, $s \in \mathbb{Z}$, to the order interval $[u_-, u_+] \subseteq C(\Omega)_+$ are well-defined. Thus, for each $\tau \in \mathbb{Z}$ also the *period maps*

$$\Pi_\tau : [u_-, u_+] \to [u_-, u_+], \qquad \Pi_\tau(u) := \varphi(\tau+\theta; \tau, u)$$

are well-defined and monotone thanks to Thm. 5.2. Furthermore, the fixed points $u_\tau \in [u_-, u_+]$ of Π_τ correspond to initial values for θ-periodic solutions of (5.5). Indeed a fixed point of (5.5) can be approximated from above and below by means of monotone iterations (see for instance [14, p. 168, Thm. 11.1]). If we define the iterates of the period map Π_0 as

$$v_t^+ := \varphi(t\theta; 0, u_+), \qquad v_t^- := \varphi(t\theta; 0, u_-) \quad \text{for all } 0 \leqslant t,$$

then the sequence $(v_t^+)_{t\geq 0}$ is decreasing to a fixed point v^+, while the sequence $(v_t^-)_{t\geq 0}$ is increasing to v_- of Π_τ as limit; one has the error estimate

$$u_- \leqslant v_t^- \leqslant v^- \leqslant v^+ \leqslant v_t^+ \leqslant u_+ \quad \text{for all } 0 \leqslant t.$$

The following concrete example illustrates the situation of a unique fixed point $v^+ = v^-$ in $C(\Omega)_+^\circ$ being approximated by a monotone iteration technique:

Example 5.5 (Gauß kernel) *On an interval $\Omega = [-\frac{L}{2}, \frac{L}{2}]$ for some $L > 0$ let us consider the Gauß kernel $\tilde{k}(x) := \frac{1}{\sqrt{2\pi\alpha^2}}\exp(-\frac{x^2}{2\alpha^2})$ and growth rates $c_t(x) := \gamma_t(2 + \sin(\frac{2\pi}{L}x)) > 0$. We choose*

$$k_- = \frac{1}{\sqrt{2\pi\alpha^2}}\exp(-\frac{L^2}{2\alpha^2}), \qquad\qquad k_+ = \frac{1}{\sqrt{2\pi\alpha^2}}$$

and thanks to

$$\ell := \int_{-L/2}^{L/2} 2 + \sin(\tfrac{2\pi}{L}y)\,\mathrm{d}\mu(y) = \begin{cases} 2L, & \mu = \lambda_1, \\ \sum_{\eta\in\Omega_n} w_\eta(2 + \sin(\frac{2\pi}{L}\eta)), & \mu = \mu_n \end{cases}$$

the above theory applies for sufficiently large coefficients γ_t. More precisely, the estimate $1 < \frac{\ell}{\sqrt{2\pi\alpha^2}}\exp(-\frac{L}{2\alpha^2})\min_{t=0}^{\theta-1}\gamma_t$ needs to hold and we choose

$$u_-(x) :\equiv \left(\frac{\ell}{\sqrt{2\pi\alpha^2}}\exp(-\frac{L^2}{2\alpha^2})\min_{s=0}^{\theta-1}\gamma_s - 1\right)^{1/\delta}, \quad u_+(x) :\equiv \left(\frac{\ell}{\sqrt{2\pi\alpha^2}}\max_{s=0}^{\theta-1}\gamma_s - 1\right)^{1/\delta}$$

on $[-\frac{L}{2}, \frac{L}{2}]$ as initial functions for the monotone iteration process. For the specific parameter values $\alpha = \delta = \frac{1}{2}$, $L = 2$ and $\gamma_t := 19 + (-1)^t$ one obtains $u_-(x) \equiv 2.72 \cdot 10^{-3}$ and $u_+(x) \equiv 3.95 \cdot 10^3$. Then Fig. 2 illustrates the convergence of upper and lower solutions to the fixed point of Π_0 for Nyström discretizations of (5.5) *with different quadrature rules.*

A Cones, monotone, subhomogeneous and concave mappings

Assume X is a real Banach space with dual space X' and the duality pairing $\langle x, x'\rangle := x'(x)$. A nonempty closed and convex subset $X_+ \subseteq X$ is called a *cone*, if $\mathbb{R}_+X_+ \subseteq X_+$ and $X_+\cap(-X_+) = \{0\}$ hold. Equipped with such a cone, X is called an *ordered Banach space*. Let us assume $X_+ \neq \{0\}$ throughout. For vectors $x, \bar{x} \in X$ we introduce the relations

$$\begin{aligned} x \leqslant \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+, \\ x < \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+ \setminus \{0\}, \\ x \ll \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+^\circ; \end{aligned} \tag{A.1}$$

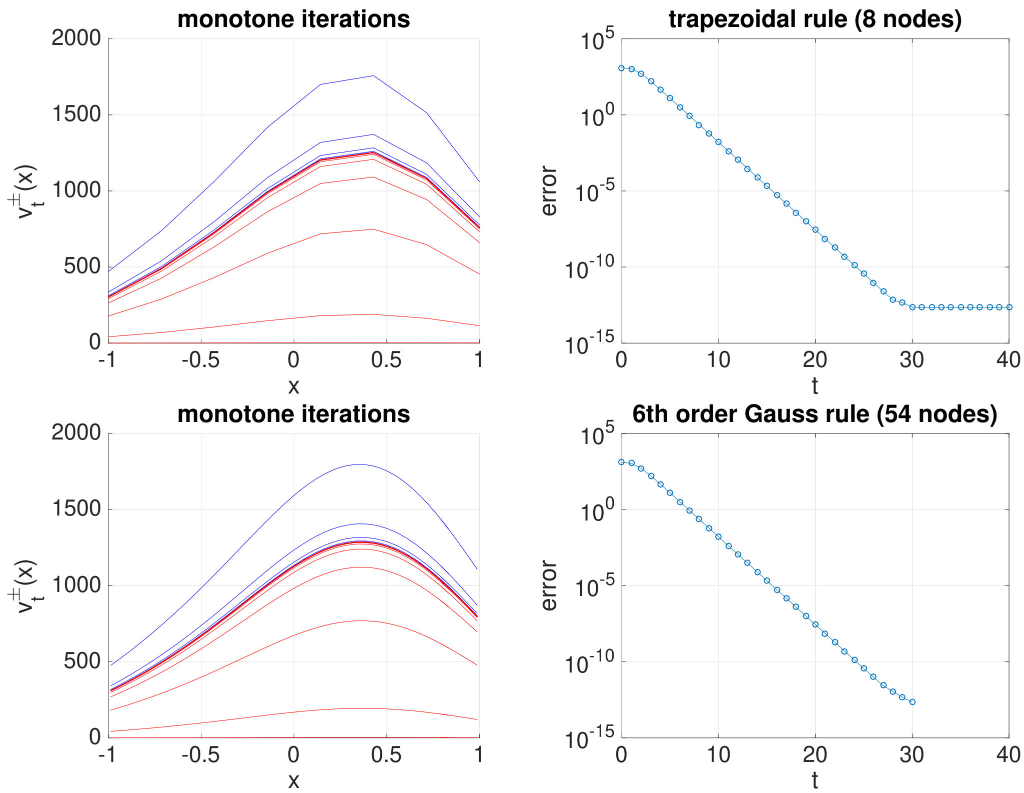

Fig. 2. Nyström discretization to approximate the 2-periodic solution to (5.5) based on the trapezoidal and the 6th order Gauß rule (see [12, App. C]).
left: Monotone iterations from below (red) and above (blue)
right: Error between the upper and lower approximations (for the Gauß rule the error vanishes for $t \geq 30$ iterations)

the latter one requires X_+ to have nonempty interior; one speaks of a *solid cone* X_+. *Order intervals* are defined as $[x, \bar{x}] := \{y \in X : x \leqslant y \leqslant \bar{x}\}$.

By means of the *dual cone* $X'_+ := \{x' \in X' : 0 \leqslant \langle x, x' \rangle \text{ for all } x \in X_+\}$ it is possible to characterize the elements of X_+ and X_+° as follows:

Lemma A.1 (cf. [12, Lemma A.1]) *(a) $X'_+ \neq \{0\}$ and for $x \in X$ holds:*

$$x \in X_+ \Leftrightarrow 0 \leqslant \langle x, x' \rangle \quad \textit{for all } x' \in X'_+,$$
$$x \in X_+ \setminus \{0\} \Rightarrow 0 < \langle x, x'_0 \rangle \quad \textit{for some } x'_0 \in X'_+ \setminus \{0\}\,.$$

(b) If X_+ is solid, then for every $x \in X$ the following holds:

$$x \in X_+^\circ \Leftrightarrow 0 < \langle x, x' \rangle \quad \textit{for all } x' \in X'_+ \setminus \{0\}\,,$$
$$x \in \partial X_+ \Rightarrow 0 = \langle x, x'_0 \rangle \quad \textit{for some } x'_0 \in X'_+ \setminus \{0\}\,.$$

One denotes a subset $U \subseteq X$ as X_+-*convex*, if for all $x, \bar{x} \in U$ satisfying $x < \bar{x}$ the inclusion $\overline{x, \bar{x}} := \{x + \theta(\bar{x} - x) \in X : \theta \in [0, 1]\} \subseteq U$ holds. For instance, the cone X_+ itself is X_+-convex, and so is every convex set U.

Let $U \subseteq X$. A mapping $F : U \to X$ is called [1]

- *monotone*, if $x < \bar{x} \Rightarrow F(x) \leqslant F(\bar{x})$,
- *strictly monotone*, if $x < \bar{x} \Rightarrow F(x) < F(\bar{x})$,
- *strongly monotone*, if $x < \bar{x} \Rightarrow F(x) \ll F(\bar{x})$ for all $x, \bar{x} \in U$.

When working with several cones, we sometimes write X_+-monotone etc., in order to refer to a specific cone and proceed similarly with our further terminology. In particular, a linear mapping $T \in L(X)$ is

- monotone (then called *positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+$,
- strictly monotone (then called *strictly positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$,
- strongly monotone (then called *strongly positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+^\circ$.

We denote $T \in L(X)$ as X_+-*injective*, if its kernel satisfies $N(T) \cap X_+ = \{0\}$. Note that T is strictly monotone, if and only if it is monotone and X_+-injective. A strongly monotone T yields the inclusion $TX_+^\circ \subseteq X_+^\circ$.

Lemma A.2 (conditions for monotonicity) *Suppose $F : U \to X$ is a C^1-mapping on a X_+-convex, open subset $U \subseteq X$. If $DF(x) \in L(X)$ is positive for all $x \in U$, then F is monotone. Moreover, the following holds with $x, \bar{x} \in U$:*

(a) If for every $x < \bar{x}$ and $x^* \in \overline{x, \bar{x}}$ the derivative $DF(x^*)$ is X_+-injective, then F is strictly monotone.
(b) If X_+ is solid and for every $x < \bar{x}$ there exists some $x^* \in \overline{x, \bar{x}}$ such that $DF(x^*)$ is strongly positive, then F is strongly monotone.

PROOF. The monotonicity of F and (b) are shown in [8, Lemma 2.2].
(a) Given $x, \bar{x} \in U$ with $x < \bar{x}$ one has $F(x) \leqslant F(\bar{x})$. We prove $F(x) \neq F(\bar{x})$ and thereto abbreviate $x_\theta := x + \theta(\bar{x} - x) \in U$ for $\theta \in [0, 1]$. By monotonicity of $DF(x_\theta)$ we derive from Lemma A.1(a) that $\langle DF(x_\theta)(\bar{x} - x), x' \rangle \geq 0$ holds for all $x' \in X'_+$. The X_+-injectivity of $DF(x_\theta)$ implies $DF(x_\theta)(\bar{x} - x) \neq 0$. From again Lemma A.1(a) we see that there exists a functional $x'_\theta \in X'_+ \setminus \{0\}$ with $\langle DF(x_\theta)(\bar{x} - x), x'_\theta \rangle > 0$. Furthermore, the continuity of DF yields that there exists an $\varepsilon_\theta > 0$ with

$$\langle DF(x_s)(\bar{x} - x), x'_\theta \rangle > 0 \quad \text{for all } s \in (\theta - \varepsilon_\theta, \theta + \varepsilon_\theta).$$

Since the family $\{(\theta - \varepsilon_\theta, \theta + \varepsilon_\theta) : \theta \in [0, 1]\}$ is an open cover of $[0, 1]$, the Borel-Lebesgue Theorem provides a finite subcover $\{(\theta_i - \varepsilon_i, \theta_i + \varepsilon_i) : 1 \leqslant i \leqslant n\}$.

[1] we implicitly assume here that U contains at least two $x, \bar{x} \in U$ such that $x < \bar{x}$

If we define $\tilde{x}' := \sum_{i=1}^{n} x'_{\theta_i} \in X'_+ \setminus \{0\}$, then $\langle DF(x_\theta)(\bar{x}-x), \tilde{x}'\rangle > 0$ and thus

$$\langle F(\bar{x}) - F(x), \tilde{x}'\rangle = \int_0^1 \langle DF(x_\theta)(\bar{x}-x), \tilde{x}'\rangle \, \mathrm{d}\theta > 0,$$

by the Mean Value Theorem, which finally guarantees that $F(\bar{x}) \neq F(x)$. □

A.2 Subhomogeneous mappings

A self-mapping $F : X_+ \to X_+$ is called [2]

- *subhomogeneous*, if $0 < x \Rightarrow \theta F(x) \leqslant F(\theta x)$,
- *strictly subhomogeneous*, if $0 < x \Rightarrow \theta F(x) < F(\theta x)$,
- *strongly subhomogeneous*, if $0 \ll x \Rightarrow \theta F(x) \ll F(\theta x)$ for all $x \in X_+$ and $\theta \in (0,1)$.

Affine-linear mappings $F(x) = Tx + y$ with positive $T \in L(X)$ and $y \in X_+$ are always subhomogeneous, while strict subhomogeneity requires an inhomogeneity $y \in X_+ \setminus \{0\}$ and strong subhomogeneity holds for $y \in X_+^\circ$.

Lemma A.3 (conditions for subhomogeneity) *Suppose $F : U \to X$ is differentiable.*

(a) F is subhomogeneous, if and only if $DF(x)x \leqslant F(x)$ for all $x \in X_+ \setminus \{0\}$.
(b) If $DF : X_+ \to L(X)$ is continuous and $DF(x)x < F(x)$ for all $x \in X_+ \setminus \{0\}$, then F is strictly subhomogeneous.
(c) If X_+ is solid and $DF(x)x \ll F(x)$ for all $x \in X_+^\circ$, then F is strongly subhomogeneous.

The differentiability of $F : X_+ \to X_+$ is to be understood so that F has a differentiable extension $\bar{F} : U \to X$ to an open superset $U \subseteq X$ of X_+.

PROOF. Given $x' \in X'_+$ and $x \in X_+ \setminus \{0\}$, our assumptions guarantee that the function $\phi_{x',x} : (0,1] \to \mathbb{R}$, $\phi_{x',x}(\theta) := \frac{1}{\theta}\langle F(\theta x), x'\rangle$ is differentiable with

$$\dot{\phi}_{x',x}(\theta) = \tfrac{1}{\theta^2}\langle \theta DF(\theta x)x - F(\theta x), x'\rangle \quad \text{for all } \theta \in (0,1]. \tag{A.2}$$

(a) Let us begin with preparations:
(I) Claim: *F is subhomogeneous, if and only if $\dot{\phi}_{x',x}(\theta) \leqslant 0$ for every $\theta \in (0,1]$,*

[2] note that the terminology is not consistent in the literature: [3, p. 112, Def. 4.1.1] speaks of sublinear instead of subhomogeneous maps and additionally assumes monotonicity. Both [3, p. 112, Def. 4.1.1] and [16, p. 52, Def. 2.3.1] require the strict subhomogeneity condition to hold merely for $0 \ll x$, while [9, p. 142, Ex. 5.1.11] assumes strong subhomogeneity to be satisfied for all $0 < x$

$x' \in X'_+ \setminus \{0\}$ *and* $x \in X_+ \setminus \{0\}$.

($\Rightarrow$) If F is subhomogeneous, then $\frac{\theta_1}{\theta_2}F(\theta_2 x) \leqslant F(\theta_1 x)$, i.e., the estimate $\frac{1}{\theta_2}F(\theta_2 x) \leqslant \frac{1}{\theta_1}F(\theta_1 x)$ for all $0 < \theta_1 \leqslant \theta_2 \leqslant 1$ and $x \in X_+ \setminus \{0\}$ results. Referring to Lemma A.1(a) this implies that $\phi_{x',x}$ is nonincreasing and hence $\dot{\phi}_{x',x}(\theta) \leqslant 0$ for all $\theta \in (0,1]$.

($\Leftarrow$) Conversely, $\dot{\phi}_{x',x}(\theta) \leqslant 0$ is equivalent to $\phi_{x',x}$ being nonincreasing on $(0,1]$. Thus, $\phi_{x',x}(1) \leqslant \phi_{x',x}(\theta)$ and since $x' \in X_+ \setminus \{0\}$ was arbitrary, we obtain that $F(x) \leqslant \frac{1}{\theta}F(\theta x)$ for all $\theta \in (0,1]$ from Lemma A.1(a), i.e. $\theta F(x) \leqslant F(\theta x)$.

(II) Claim: *F is strongly subhomogeneous, if and only if the functions $\phi_{x',x}$ are strictly decreasing on $(0,1)$ for all $x' \in X'_+ \setminus \{0\}$ and $x \in X_+^\circ$.*

This follows similarly as in step (I) using Lemma A.1(b).

(III) The relation $DF(x)x \leqslant F(x)$ is equivalent to $\theta DF(\theta x)x \leqslant F(\theta x)$ for $\theta \in (0,1]$ and $x \in X_+ \setminus \{0\}$. By Lemma A.1(a) this is rephrased as $\langle \theta DF(\theta x)x - F(\theta x), x'\rangle \leqslant 0$ which, in turn, thanks to (A.2) is necessary and sufficient for $\dot{\phi}_{x',x}(\theta) \leqslant 0$ for all $\theta \in (0,1]$. Then step (I) guarantees the claimed characterization of subhomogeneity.

(b) Let $\theta \in (0,1)$ and $x \in X_+ \setminus \{0\}$. By assumption it is $\vartheta DF(\vartheta x)x < F(\vartheta x)$ for all $\vartheta \in [\theta,1]$. Using Lemma A.1(a) for any $\vartheta \in [\theta,1]$ there is a $x'_\vartheta \in X'_+ \setminus \{0\}$ such that $\langle \vartheta DF(\vartheta x)x - F(\vartheta x), x'_\vartheta\rangle < 0$. By continuity of DF and F there exists a $\varepsilon_\vartheta > 0$ with

$$\langle sDF(sx)x - F(sx), x'_\vartheta\rangle < 0 \text{ for all } s \in (\vartheta - \varepsilon_\vartheta, \vartheta + \varepsilon_\vartheta).$$

Since $\{(\varepsilon_\vartheta - \vartheta, \varepsilon_\vartheta + \vartheta) : \vartheta \in [\theta,1]\}$ is an open cover of $[\theta,1]$, the Borel-Lebesgue Theorem yields a finite subcover $\{(\varepsilon_i - \vartheta_i, \vartheta_i + \varepsilon_i) : 1 \leqslant i \leqslant n\}$ of $[\theta,1]$. If we define $\tilde{x}'_\theta := \sum_{i=1}^n x'_{\vartheta_i} \in X'_+ \setminus \{0\}$, then $\langle \vartheta DF(\vartheta x)x - F(\vartheta x), \tilde{x}'_\theta\rangle < 0$ for all $\vartheta \in [\theta,1]$. By (A.2) this shows that $\phi_{\tilde{x}'_\theta,x}$ is strictly decreasing on $[\theta,1]$, in particular $\phi_{\tilde{x}'_\theta,x}(1) < \phi_{\tilde{x}'_\theta,x}(\theta)$ and consequently $\langle \theta F(x), \tilde{x}'_\theta\rangle < \langle F(\theta x), \tilde{x}'_\theta\rangle$. This means that $\theta F(x) \neq F(\theta x)$.

(c) Let $0 \ll x$. The inequality $DF(x)x \ll F(x)$ yields $\theta F(\theta x)x \ll F(\theta x)$ and Lemma A.1(b) implies $\langle \theta DF(\theta x)x - F(\theta x), x'\rangle < 0$ for all $\theta \in (0,1)$ and functionals $x' \in X'_+ \setminus \{0\}$. Given this, using (A.2) results $\dot{\phi}_{x',x}(\theta) < 0$ on $(0,1)$ and hence $\phi_{x',x}$ is strictly decreasing. Now step (II) yields the assertion. □

Corollary A.4 (conditions for subhomogeneity) *Suppose $F : X_+ \to X$ is monotone and differentiable. Then $DF(x)x \leqslant F(x)$ for all $x \in X_+$ if and only if F is subhomogeneous.*

PROOF. ($\Rightarrow$) We only have to prove that $F : X_+ \to X_+$. By $DF(x)x \leqslant F(x)$ for $x = 0$, we get that $0 \leqslant F(0)$. Monotonicity implies $F(0) \leqslant F(x)$ for all $0 < x$. Hence, $0 \leqslant F(x)$ for all $0 < x$ and it results $F(X_+) \subseteq X_+$.

($\Leftarrow$) $F : X_+ \to X_+$ implies that $0 \leqslant F(0)$. The rest is due to Lemma A.3. □

Let $U \subseteq X$ be X_+-convex throughout this subsection. Given this, a mapping $F : U \to X$ is called [3]

- *order* or *X_+-concave*, if $x < \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) \leqslant F(\theta x + (1-\theta)\bar{x})$,
- *strictly X_+-concave*, if $x < \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) < F(\theta x + (1-\theta)\bar{x})$,
- *strongly X_+-concave*, if $x \ll \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) \ll F(\theta x + (1-\theta)\bar{x})$ for all $x, \bar{x} \in U$ and $\theta \in (0,1)$.

Corresponding notions of *order convexity* for a mapping $F : U \to X$ result when F is replaced by $-F$ in the above definitions.

Note that (strict or strong) subhomogeneity holds for (strictly resp. strongly) X_+-concave mappings $F : X_+ \to X_+$, with the reference point $x = 0$. In this sense, the concavity concepts for $F : X_+ \to X_+$ are less general than the respective subhomogeneity notions.

Affine-linear maps $F(x) = Tx + y$ with $T \in L(X)$ and $y \in X$ are X_+-concave, but never strictly or strongly X_+-concave.

Lemma A.5 (conditions for concavity) *Suppose $F : U \to X$ is a C^1-mapping on a X_+-convex, open subset $U \subseteq X$.*

(a) F is X_+-concave, if and only if $DF(\bar{x})(\bar{x} - x) \leqslant DF(x)(\bar{x} - x)$ for all $x, \bar{x} \in U$, $x < \bar{x}$.
(b) If $DF(\bar{x})(\bar{x} - x) < DF(x)(\bar{x} - x)$ for all $x, \bar{x} \in U$, $x < \bar{x}$, then F is strictly X_+-concave.
(c) If X_+ is solid and $DF(\bar{x})(\bar{x} - x) \ll DF(x)(\bar{x} - x)$ for all $x, \bar{x} \in U$, $x \ll \bar{x}$, then F is strongly X_+-concave.

PROOF. (a)($\Rightarrow$) Let F be X_+-concave and $x, \bar{x} \in U$, $x < \bar{x}$. First, the estimate $(1-\theta)F(x) + \theta F(\bar{x}) \leqslant F((1-\theta)x + \theta\bar{x}) = F(x + \theta(\bar{x} - x))$ implies

$$\begin{aligned} &F(x + (\bar{x} - x)) - F(x) - DF(x)(\bar{x} - x) \\ &\qquad \leqslant \tfrac{1}{\theta}\left[F(x + \theta(\bar{x} - x)) - F(x) - DF(x)(\theta(\bar{x} - x))\right] \quad \text{for all } \theta \in (0,1) \end{aligned}$$

and in the limit $\theta \searrow 0$ results $F(\bar{x}) - F(x) \leqslant DF(x)(\bar{x} - x)$. Second, note that $(1-\theta)F(\bar{x}) + \theta F(x) \leqslant F((1-\theta)\bar{x} + \theta x) = F(\bar{x} - \theta(\bar{x} - x))$ leads to

$$F(\bar{x} - (\bar{x} - x)) - F(\bar{x}) + DF(\bar{x})(\bar{x} - x)$$

[3] also for the notion of concavity the terminology is not consistent throughout the literature: [3, p. 114, Def. 4.1.2] assumes a self-mapping of X_+ and requires strict concavity to hold only for all $0 \ll x$

$$\leqslant \tfrac{1}{\theta}\left[F(\bar{x}-\theta(\bar{x}-x)) - F(\bar{x}) + DF(\bar{x})(\theta(\bar{x}-x))\right] \quad \text{for all } \theta \in (0,1).$$

In the limit $\theta \searrow 0$ follows $DF(\bar{x})(\bar{x}-x) \leqslant F(\bar{x}) - F(x)$ and we conclude that $DF(\bar{x})(\bar{x}-x) \leqslant DF(x)(\bar{x}-x)$.
(a)($\Leftarrow$) Conversely, for any $x, \bar{x} \in U$ with $x < \bar{x}$ we obtain $x + t(\bar{x}-x) \in U$ for all $t \in [0,1]$, since U is assumed to be X_+-convex. This allows us to define a function $\Phi : (0,1] \to X$ as $\Phi(t) := \frac{1}{t}\left[F(x+t(\bar{x}-x)) - F(x)\right]$ and we obtain

$$\begin{aligned}
\dot{\Phi}(t) &= -\frac{1}{t^2}\left[F(x+t(\bar{x}-x)) - F(x)\right] + \frac{1}{t} DF(x+t(\bar{x}-x))(\bar{x}-x) \\
&= -\frac{1}{t}\int_0^1 [DF(x+\theta t(\bar{x}-x)) - DF(x+t(\bar{x}-x))](\bar{x}-x)\,\mathrm{d}\theta \\
&= -\frac{1}{t^2}\int_0^t DF(x+\vartheta(\bar{x}-x)) - DF(x+t(\bar{x}-x))(\bar{x}-x)\,\mathrm{d}\vartheta
\end{aligned}$$

from the Mean Value Theorem, where we used the substitution $\vartheta = t\theta$ in the last equality. Let $t_1 \in (0,1)$. Integrating this identity over $[t_1, 1]$ we obtain

$$\begin{aligned}
&t_1 F(\bar{x}) + (1-t_1)F(x) - F(x+t_1(\bar{x}-x)) \\
&= t_1 \int_{t_1}^1 \frac{1}{t^2}\int_0^t \left[DF(x+t(\bar{x}-x)) - DF(x+s(\bar{x}-x))\right](\bar{x}-x)\,\mathrm{d}s\,\mathrm{d}t \leqslant 0,
\end{aligned}$$

because for $s \in (0,t]$ we have $s(\bar{x}-x) \leqslant t(\bar{x}-x)$ and later by assumption

$$DF(x+t(\bar{x}-x))(\bar{x}-x) \leqslant DF(x+s(\bar{x}-x))(\bar{x}-x)$$

and $DF(x+t(\bar{x}-x))((t-s)(\bar{x}-x)) \leqslant DF(x+s(\bar{x}-x))((t-s)(\bar{x}-x))$.
Finally, as in (a)($\Leftarrow$) one can establish (b) and (c) by means of corresponding arguments given in the proof of Lemma A.3. □

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