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MONOTONICITY AND DISCRETIZATION OF HAMMERSTEIN INTEGRODIFFERENCE EQUATIONS

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ABSTRACT. The paper provides sufficient conditions for monotonicity, subhomogeneity and concavity of vector-valued Hammerstein integral operators over compact domains, as well as for the persistence of these properties under numerical discretizations of degenerate kernel type. This has immediate consequences on the dynamics of Hammerstein integrodifference equations and allows to deduce a local-global stability principle.

1. **Introduction.** The recent years showed an increasing interest in the dynamics and the asymptotic behavior of integrodifference equations (IDEs for short). They are recursions

$$u_{t+1}(x) = \int_\Omega k_t(x,y) g_t(y, u_t(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{1.1}$$

on spaces of continuous or integrable functions over Ω involving a Hammerstein integral operator. Their popularity stems primarily from the field of theoretical ecology as tool to describe the spatial dispersal of species over time. An advantage of IDEs compared to alternative modelling approaches based e.g. reaction-diffusion equations is their flexibility. By means of directly specifying a particular kernel k_t, various different dispersal strategies can be incorporated and fitted to given data, as illustrated for instance in [2, 10, 16, 20, 32].

In spite of these motivations the paper at hand serves different and possibly more basic purposes related to the theory of Hammerstein operators. Such operators are the composition of a linear Fredholm integral operator with a nonlinear superposition operator. We give conditions on the kernel k_t and the growth function g_t yielding that the resulting right-hand sides of (1.1) possess structural properties such as monotonicity, subhomogeneity or convexity. In this context monotonicity means that the problem preserves an order relation on a real Banach space induced by an appropriate order cone. This, as well as subhomogeneity or concavity of an operator has immediate consequences in various fields:

- For Dynamical Systems, IDEs (1.1) motivated from biological or ecological applications are often monotone. This property simplifies their long term behavior and we refer to the survey [13], the monographs [5, 18] or [31, pp. 43ff, Chap. 2] providing further information. For example, solutions to autonomous IDEs (1.1) generically

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converge to periodic solutions [29] or criteria for global stability of IDEs [10] are at hand. Moreover, for general nonautonomous equations (1.1), [23] proves a limit set trichotomy, while [26] contains information on the structure of pullback attractors under monotonicity assumptions. Consequences of (additional) subhomogeneity are discussed in [31, pp. 52–58] and implications of concavity are given in [18].

- In Analysis nonlinear elliptic BVPs (provided Green's function yielding the kernel is known, [21, pp. 181ff]) can be formulated as fixed point problems involving a Hammerstein operator. Given this, a classical constructive method to solve nonlinear problems are monotone iteration techniques (see e.g. [15], [27, pp. 167ff, Chap. 11] or [30, p. 283, Thm. 7.A] etc.), which leads to an autonomous IDE (1.1). Several fixed point results are based on related monotonicity assumptions [1]. Finally, [7] provide conditions such that Hammerstein operators can be transformed into order-preserving ones.

While it is evident how monotonicity of real-valued kernel functions in (1.1) carries over to the integral operators, our approach is more flexible. We indeed allow $\mathbb{R}^d$-valued integration kernels preserving order relations w.r.t. arbitrary cones in finite dimensions. So our setting is sufficiently general to address systems of nonlinear equations, as well as multiple species models in competitive/cooperative interaction. In particular, by our results ensuring monotonicity, methods from [1, 5, 13, 18, 31] or the papers [10, 23, 26, 29] do apply.

Our further aim is related to the numerical analysis of integral equations and to computational simulation results for IDEs (1.1). We investigate the question, whether the above structural properties of integral operators (monotonicity, subhomogeneity, concavity) are preserved under commonly used spatial discretization techniques? This for instance ensures that monotone iteration applies and yields convergent iterates. For the dynamics of IDEs, such results guarantee that simulations necessarily based on spatial discretization reflect the qualitative asymptotic behavior of the original difference equation (1.1). This paper actually focusses on degenerate kernel methods (yielding semi-discretizations). Projection methods, and in particular Collocation or Bubnov-Galerkin approximations primarily require linear techniques and were covered in [24], while Nyström methods (as full discretizations) were addressed in the more general setting of Urysohn integral operators in [25] already. Our focus is to establish the persistence of structural properties of Hammerstein integral operators and the resulting IDEs under spatial discretization. In contrast to classical numerical analysis, we do not address consistency, stability or convergence issues and refer to e.g. [3, 11] for such questions. We furthermore restrict to integral operators on the space of continuous functions over a compact domain.

Concerning the related literature, we point out that parallel results in the more general class of Urysohn operators were derived in [25]. Nevertheless our separate analysis of Hammerstein operators appears legitimate, since it allows to combine linear results from [24] with structural properties of Nemytskii operators. Moreover the method of degenerate kernels applies and we put a stronger focus on IDEs. To our surprise we were not able to find general studies dealing with monotonicity preserving spatial schemes in the area of PDEs although studies on specific problems exist, for instance, see [4, 22]. Yet, monotonicity conditions for temporal Runge-Kutta discretizations were tackled in [14] for ordinary differential equations and in [9] for delay equations.

As an application we provide a criterion for systems of IDEs guaranteeing that local exponential stability of a periodic solution implies its global asymptotic stability. This extends previous results for scalar IDEs from [10].

The paper is structured as follows: First, the necessary notation related to continuous functions with values in finite-dimensional spaces and cones is established. In Sec. 2 sufficient conditions on the growth functions are given such that the resulting Hammerstein operators are monotone, subhomogeneous or order concave. In essence these properties carry over from the growth functions having values in $\mathbb{R}^d$ to the integral operators mapping into the continuous $\mathbb{R}^d$-valued functions. Sec. 3 on spatial discretization of (1.1) tackles degenerate kernel methods, where similar problems as observed in [24] for projection methods arise, i.e. various common interpolation techniques violate monotonicity. Results on monotone Hammerstein IDEs are provided in Sec. 4, which include the promised local-global stability principle in Thm. 4.4. Finally, Sec. 5 contains concrete applications illustrating the above observations. An appendix collects the required basic results on cones in Banach spaces and monotone, subhomogeneous or concave mappings.

Notation. We abbreviate $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals, $|\cdot|$ for norms on finite-dimensional spaces and $\langle x, y\rangle := \sum_{j=1}^d x_j y_j$ for $x, y \in \mathbb{R}^d$ is the Euclidean inner product. For a matrix $S \in \mathbb{R}^{d\times d}$, the element in row i and column j is S_{ij}, while I_d is the identity.

With a Banach space $(X, \|\cdot\|)$, $L_l(X)$ means the linear space of bounded l-linear mappings $T : X^l \to X$, $l \in \mathbb{N}$, in particular $L_0(X) := X$, $L(X) := L_1(X)$ are the bounded operators and $GL(X)$ the bounded invertible operators on X. We abbreviate $N(T) := T^{-1}(\{0\})$ for the kernel and $\sigma(T) \subset \mathbb{C}$ for the spectrum of $T \in L(X)$.

On subsets Ω of a metric space (X, d), Ω° is the interior and $\overline{\Omega}$ the closure. If $\Omega \neq \emptyset$ is compact, then $C(\Omega)$ abbreviates the linear space of continuous functions $u : \Omega \to \mathbb{R}$ equipped with the maximum norm. We moreover endow Ω with the Borel σ-algebra $\mathfrak{A}$ and a measure μ such that $(\Omega, \mathfrak{A}, \mu)$ becomes a measure space satisfying $\mu(\Omega) < \infty$.

In the following, $Y_+ \subset \mathbb{R}^d$ abbreviates an order cone inducing the relations $\leq, <$ and $\ll$ on $\mathbb{R}^d$ (cf. (A.1) and App. A for the related terminology). By [24, Lemma 2.2] the set

$$C(\Omega)^d_+ := \left\{u : \Omega \to \mathbb{R}^d \,|\, u(x) \in Y_+ \text{ for all } x \in \Omega\right\}.$$

is a cone. Whence, for any $u, \bar{u} \in C(\Omega)^d$ the cone property allows us to introduce

$$u \preceq \bar{u} \quad :\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+, \qquad u \prec \bar{u} \quad :\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+ \setminus \{0\}.$$

In particular, $C(\Omega)^d_+$ is normal, since $Y_+ \subset \mathbb{R}^d$ is normal as a cone in a finite-dimensional space. If Y_+ is solid, then also $C(\Omega)^d_+$ is solid and we define

$$u \prec\!\!\prec \bar{u} \quad :\Leftrightarrow \quad \bar{u} - u \in (C(\Omega)^d_+)^\circ.$$

Finally, for the convenience of the reader we restate a characterization from [24, Lemma 2.3]:

Lemma 1.1. *The following holds for all $u, \bar{u} \in C(\Omega)^d$:*

(a) $u \preceq \bar{u} \Leftrightarrow u(x) \leq \bar{u}(x)$ for all $x \in \Omega \Leftrightarrow \langle u(x), y'\rangle \leq \langle \bar{u}(x), y'\rangle$ for all $x \in \Omega$ and $y' \in Y'_+$.

(b) If Y_+ is solid, then $u \prec\!\!\prec \bar{u} \Leftrightarrow u(x) \ll \bar{u}(x)$ for all $x \in \Omega \Leftrightarrow \langle u(x), y'\rangle < \langle \bar{u}(x), y'\rangle$ for all $x \in \Omega$, $y' \in Y'_+ \setminus \{0\}$.

2. **Hammerstein integral operators.** Given *kernels* $k : \Omega^2 \to L(\mathbb{R}^d)$ and *growth functions* $g : \Omega \times Z \to \mathbb{R}^d$ on a set $Z \subseteq \mathbb{R}^d$, this paper deals with *Hammerstein operators*

$$\mathcal{H} : U \to C(\Omega)^d, \qquad \mathcal{H}(u) := \int_\Omega k(\cdot, y) g(y, u(y))\, \mathrm{d}\mu(y) \tag{2.1}$$

canonically given on the set

$$U := \left\{u \in C(\Omega)^d : \, u(x) \in Z \text{ for all } x \in \Omega\right\}.$$

It is established in [25, Lemma 2.2] that openness or Y_+-convexity of Z extend to openness resp. $C(\Omega)_+^d$-convexity of U.

Hammerstein operators can be addressed using two approaches: First, they are a special case of Urysohn integral operators [21, pp. 164ff, Sect. V.3]. Then under appropriate assumptions on k, g the functions $f : \Omega^2 \times Z \to \mathbb{R}^d$,

$$f(x, y, z) := k(x, y)g(y, z), \qquad D_3 f(x, y, z) = k(x, y)D_2 g(y, z)$$

do fulfill [25, Hypothesis (U^l)] and the criteria for monotonicity, subhomogeneity or concavity developed in [25, Sect. 2] readily apply to (2.1).

Second, Hammerstein operators can be written as composition

$$\mathcal{H} = \mathcal{K}\mathcal{G} \tag{2.2}$$

of a linear (bounded) Fredholm integral operator

$$\mathcal{K} : C(\Omega)^d \to C(\Omega)^d, \qquad \mathcal{K}u := \int_\Omega k(\cdot, y)u(y)\,\mathrm{d}\mu(y) \tag{2.3}$$

with a nonlinear Nemytskii operator

$$\mathcal{G} : U \to C(\Omega)^d, \qquad \mathcal{G}(u)(x) := g(x, u(x)) \quad \text{for all } x \in \Omega. \tag{2.4}$$

Because a detailed analysis of Urysohn operators was given in [25] already, we follow the latter approach and assume:

Hypothesis. *With nonempty $Z \subseteq \mathbb{R}^d$, functions $k : \Omega^2 \to L(\mathbb{R}^d)$, $g : \Omega \times Z \to \mathbb{R}^d$ and $l \in \{0, 1\}$ assume:*

(L) *$k(x, \cdot) : \Omega \to L(\mathbb{R}^d)$ is μ-measurable for all $x \in \Omega$ with*

$$\sup_{x\in\Omega} \int_\Omega |k(x, y)|\,\mathrm{d}\mu(y) < \infty$$

and $\lim_{x\to x_0} \int_\Omega |k(x, y) - k(x_0, y)|\,\mathrm{d}\mu(y) = 0$ for all $x_0 \in \Omega$.

(N^l) *$D_2^l g : \Omega \times Z \to L_l(\mathbb{R}^d)$ exists as a continuous function.*

Consequently, the operators (2.3) and (2.4) are well-defined and continuous for $l = 0$, and so are the resulting Hammerstein operators (2.1).

The assumption concerning (N^l) is to be understood as follows: In case conditions on the derivative $D_2 g$ are applied, then (N^l) is supposed to hold for $l \in \{0, 1\}$, while otherwise it suffices to have (N^0). We proceed accordingly for example in Thms. 2.3, 2.6 and 2.9 below.

The following is based on several sufficient criteria for positivity of the linear Fredholm operator $\mathcal{K}$ from [24, Sect. 2]. We list their assumptions for later reference:

Hypothesis. *With $k(x, y) \in L(\mathbb{R}^d)$ assume:*

- L_0 *$k(x, y)$ is Y_+-positive for all $x \in \Omega$ and μ-a.a. $y \in \Omega$,*
- L_1 *nonempty, open subsets of Ω have positive measure, there exists a $\bar{x} \in \Omega$ so that $k(\bar{x}, \cdot)$ is continuous on Ω and $k(\bar{x}, y)$ is Y_+-injective for μ-a.a. $y \in \Omega$ (cf. App. A),*
- L_2 *nonempty, open subsets of Ω have positive measure, Y_+ is solid and $k(x, y)$ is strongly Y_+-positive for all $x \in \Omega$ and μ-almost all $y \in \Omega$,*
- L_3 *$\mu(\Omega) > 0$, Y_+ is solid and $k(x, y)Y_+^\circ \subseteq Y_+^\circ$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$.*

2.1. **Monotonicity.** Since monotonicity properties of Fredholm operators $\mathcal{K}$ were settled in [24, Thm. 2.6] already, it remains to focus on the Nemytskii operator $\mathcal{G}$. For this, we list the following set of assumptions:

Hypothesis. M_0 *$g(x,\cdot) : Z \to \mathbb{R}^d$ is Y_+-monotone for all $x \in \Omega$,*
M_1 *$g(x,\cdot) : Z \to \mathbb{R}^d$ is one-to-one for all $x \in \Omega$,*
M_2 *$g(x,\cdot) : Z \to \mathbb{R}^d$ is strongly Y_+-monotone for all $x \in \Omega$.*

Proposition 2.1 (monotonicity of $\mathcal{G}$)**.** *Let Hypothesis (N^0) hold. Then a Nemytskii operator $\mathcal{G} : U \to C(\Omega)^d$ fulfills:*
(a) $\mathcal{G}$ is $C(\Omega)^d_+$-monotone, if and only if M_0 holds.
(b) If $M_0 \wedge M_1$ hold, then $\mathcal{G}$ is strictly $C(\Omega)^d_+$-monotone.
(c) Let Y_+ be solid. If $\mathcal{G}$ is strongly $C(\Omega)^d_+$-monotone, then M_2 holds.

Proof. (a) Let $z, \bar{z} \in Z$, $z < \bar{z}$.

($\Rightarrow$) Define the constant functions $u(x) :\equiv z$, $\bar{u}(x) :\equiv \bar{z}$ on Ω. Then $u \prec \bar{u}$ and because $\mathcal{G}$ is monotone, one has $\mathcal{G}(u) \preceq \mathcal{G}(\bar{u})$. Therefore, Lemma 1.1(a) implies

$$\langle g(x,z), y'\rangle \overset{(2.4)}{=} \langle \mathcal{G}(u)(x), y'\rangle \le \langle \mathcal{G}(\bar{u})(x), y'\rangle \overset{(2.4)}{=} \langle g(x,\bar{z}), y'\rangle$$

for all $x \in \Omega$ and $y' \in Y'_+$. Referring to Lemma A.1(a) this means $g(x,z) \le g(x,\bar{z})$, i.e. the mappings $g(x,\cdot)$ are Y_+-monotone for all $x \in \Omega$.
($\Leftarrow$) Conversely, suppose that $g(x,\cdot)$ is Y_+-monotone for all $x \in \Omega$ and assume $u \prec \bar{u}$. This implies $u(x) \le \bar{u}(x)$ and consequently

$$\langle \mathcal{G}(u)(x), y'\rangle \overset{(2.4)}{=} \langle g(x,u(x)), y'\rangle \le \langle g(x,\bar{u}(x)), y'\rangle \overset{(2.4)}{=} \langle \mathcal{G}(\bar{u})(x), y'\rangle$$

holds for all $x \in \Omega$, $y' \in Y'_+$. Whence, Lemma 1.1(a) shows $\mathcal{G}(u) \preceq \mathcal{G}(\bar{u})$.

(b) Let $u \prec \bar{u}$. There exists an $x_0 \in \Omega$ such that $u(x_0) \ne \bar{u}(x_0)$, whereas (a) implies $\mathcal{G}(u) \preceq \mathcal{G}(\bar{u})$. Thus, it remains to show that $\mathcal{G}(u) \ne \mathcal{G}(\bar{u})$, which results from the injectivity assumption on $g(x,\cdot)$ yielding $g(x_0, u(x_0)) \ne g(x_0, \bar{u}(x_0))$.

(c) Let $z, \bar{z} \in Z$, $z < \bar{z}$ and $u \prec \bar{u}$ denote the constant functions defined in (a). Since $\mathcal{G}$ is strongly monotone, one has $\mathcal{G}(u) \lll \mathcal{G}(\bar{u})$ and Lemma 1.1(b) implies

$$\langle g(x,z), y'\rangle \overset{(2.4)}{=} \langle \mathcal{G}(u)(x), y'\rangle < \langle \mathcal{G}(\bar{u})(x), y'\rangle \overset{(2.4)}{=} \langle g(x,\bar{z}), y'\rangle$$

for all $y' \in Y'_+ \setminus \{0\}$ and $x \in \Omega$. Now Lemma A.1(b) yields $g(x,z) \ll g(x,\bar{z})$, i.e. $g(x,\cdot)$ is strongly Y_+-monotone for all $x \in \Omega$. □

Besides the criteria from Prop. 2.1, also the abstract Lemma A.2 allows to come up with sufficient conditions for $\mathcal{G}$ to be monotone. Under the additional Hypothesis (N^1) the Nemytskii operator $\mathcal{G}$ is of class C^1 with the derivative

$$[D\mathcal{G}(u)v](x) = D_2 g(x,u(x))v(x) \quad \text{for all } x \in \Omega,\ u \in U^\circ,\ v \in C(\Omega)^d. \tag{2.5}$$

Therefore, by the Chain Rule [21, p. 33, Lemma 4.1] Hammerstein operators (2.1) are continuously differentiable with the derivative $D\mathcal{H}(u) = \mathcal{K}D\mathcal{G}(u)$.

Hypothesis. *With open and Y_+-convex $Z \subseteq \mathbb{R}^d$ assume:*
M'_0 *$D_2 g(x,z)$ is Y_+-positive for all $x \in \Omega$ and $z \in Z$,*
M'_1 *$D_2 g(x,z)$ is Y_+-injective for all $x \in \Omega$ and $z \in Z$.*

Note that M'_0 implies the previous assumption M_0 by Lemma A.2.

Corollary 2.2. *Let Hypotheses (N^l) hold for $l \in \{0,1\}$ with open, Y_+-convex Z. If*
(a) M'_0 holds, then $\mathcal{G} : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-monotone,

(b) $M_0' \wedge M_1'$ *hold, then* $\mathcal{G}$ *is strictly* $C(\Omega)^d_+$*-monotone.*

Proof. (a) By Lemma A.2 we see that M_0' implies M_0 and Prop. 2.1(a) yields the assertion.

(b) Due to [25, Lemma 2.2] the set U is open and $C(\Omega)^d_+$-convex. Since M_0' holds, $\mathcal{G}$ is monotone by (a) and according to Lemma A.2(b) it remains to show that, given $u, \bar{u} \in U$ with $u \prec \bar{u}$, the derivative $D\mathcal{G}(u^*)$ is $C(\Omega)^d_+$-injective for all $u^* \in \overline{u, \bar{u}}$. This means

$$\{0\} = \left\{v \in C(\Omega)^d_+ : D\mathcal{G}(u^*)v = 0\right\} \overset{(2.5)}{=} \left\{v \in C(\Omega)^d_+ : D_2 g(x, u^*(x))v(x) \equiv 0 \text{ on } \Omega\right\},$$

for which M_1' is a sufficient condition. □

The fact that we gave merely a necessary condition for the strong $C(\Omega)^d_+$-monotonicity of $\mathcal{G}$ in Prop. 2.1(c) is not a serious problem:

Theorem 2.3 (monotonicity of $\mathcal{H}$)**.** *Let Hypotheses* (L) *and* (N^l) *hold with* $l \in \{0, 1\}$*. If*

(a) $L_0 \wedge (M_0 \vee M_0')$ *hold, then a Hammerstein operator* $\mathcal{H} : U \to C(\Omega)^d$ *is* $C(\Omega)^d_+$*-monotone,*

(b) $(L_0 \wedge L_1) \wedge ((M_0 \wedge M_1) \vee (M_0' \wedge M_1'))$ *hold, then* $\mathcal{H}$ *is strictly* $C(\Omega)^d_+$*-monotone,*

(c) $L_2 \wedge ((M_0 \wedge M_1) \vee (M_0' \wedge M_1'))$ *hold, then* $\mathcal{H}$ *is strongly* $C(\Omega)^d_+$*-monotone.*

Proof. (a) The operator $\mathcal{K}$ is monotone due to [24, Thm. 2.6] and so is $\mathcal{G}$ by Prop. 2.1(a). Hence, (2.2) is a composition of monotone operators and therefore monotone.

(b) Similarly, [24, Thm. 2.6(a)] guarantees that $\mathcal{K}$ is strictly monotone and it remains to provide conditions that $\mathcal{G}$ is strictly monotone. According to Prop. 2.1(b) this requires $M_0 \wedge M_1$ to hold, or, by Cor. 2.2(b) the conditions $M_0' \wedge M_1'$ must be satisfied. This guarantees that $\mathcal{H}$ is a composition of strictly monotone operators.

(c) The assumption L_2 yields that $\mathcal{K}$ is strongly positive due to [24, Thm. 2.6(b)], while $\mathcal{G}$ is strictly monotone by Prop. 2.1(b) or Cor. 2.2(b). Therefore, $\mathcal{H}$ is strongly monotone due to Cor. A.3(b). □

2.2. **Subhomogeneity.** In this subsection we assume $Z = Y_+$ and obtain $U = C(\Omega)^d_+$. It is clear that a positive Fredholm operator $\mathcal{K} \in L(C(\Omega)^d)$ is subhomogeneous. The corresponding property of a growth function g extends to $\mathcal{G}$ and in turn to Hammerstein operators.

Hypothesis. *With a function* $g : \Omega \times Y_+ \to Y_+$ *assume:*

S_0 $g(x, \cdot)$ *is* Y_+*-subhomogeneous for all* $x \in \Omega$*,*

S_1 $g(x, \cdot)$ *is strictly* Y_+*-subhomogeneous for all* $x \in \Omega$*,*

S_2 $g(x, \cdot)$ *is strongly* Y_+*-subhomogeneous for all* $x \in \Omega$*.*

Proposition 2.4 (subhomogeneity of $\mathcal{G}$)**.** *Let Hypothesis* (N^0) *hold. Then a Nemytskii operator* $\mathcal{G} : C(\Omega)^d_+ \to C(\Omega)^d_+$ *fulfills:*

(a) $\mathcal{G}$ *is* $C(\Omega)^d_+$*-subhomogeneous, if and only if* S_0 *holds.*

(b) If S_1 *holds, then* $\mathcal{G}$ *is strictly* $C(\Omega)^d_+$*-subhomogeneous.*

(c) Let Y_+ *be solid. If* $\mathcal{G}$ *is strongly* $C(\Omega)^d_+$*-subhomogeneous, then* S_2 *holds.*

Proof. (a) $(\Rightarrow)$ Let $z \in Y_+ \setminus \{0\}$. If $u(x) :\equiv z$ on Ω, then $0 \prec u$ and because $\mathcal{G}$ is subhomogeneous it results $\theta\mathcal{G}(u) \preceq \mathcal{G}(\theta u)$ for $\theta \in (0, 1)$. Thus, Lemma 1.1(a) leads to

$$\langle \theta g(x, z), y' \rangle \overset{(2.4)}{=} \langle \theta \mathcal{G}(u)(x), y' \rangle \le \langle \mathcal{G}(\theta u)(x), y' \rangle \overset{(2.4)}{=} \langle g(x, \theta z), y' \rangle \quad \text{for all } x \in \Omega$$

and all $y' \in Y_+'$. By means of Lemma A.1(a) this shows $\theta g(x, z) \le g(x, \theta z)$, i.e. the mapping $g(x, \cdot)$ is Y_+-subhomogeneous for all $x \in \Omega$.

$(\Leftarrow)$ Clearly, $\mathcal{G} : C(\Omega)^d_+ \to C(\Omega)^d_+$ is well-defined. Let $u \in C(\Omega)^d$, $0 \prec u$ and $\theta \in (0, 1)$.

In case $0 = u(x)$, by $0 \preceq \mathcal{G}(0)$ we have $\theta\mathcal{G}(u)(x) = \theta g(x,0) \le g(x,0) = \mathcal{G}(\theta u)(x)$. Moreover, in case $0 < u(x)$ by assumption

$$\langle \theta\mathcal{G}(u)(x), y'\rangle \overset{(2.4)}{=} \langle \theta g(x,u(x)), y'\rangle \le \langle g(x,\theta u(x)), y'\rangle \overset{(2.4)}{=} \langle \mathcal{G}(\theta u)(x), y'\rangle$$

holds for all $y' \in Y'_+$. Whence, $\theta\mathcal{G}(u) \preceq \mathcal{G}(\theta u)$.

(b) Let $0 \prec u$ and $\theta \in (0,1)$. There exists an $x_0 \in \Omega$ with $u(x_0) \ne 0$, while (a) implies $\theta\mathcal{G}(u) \preceq \mathcal{G}(\theta u)$. Hence, $\theta\mathcal{G}(u) \ne \mathcal{G}(\theta u)$ remains to be shown, but results as follows: Since $g(x_0,\cdot)$ is strictly subhomogeneous, it is $\theta g(x_0,u(x_0)) < g(x_0,\theta u(x_0))$ and consequently $\theta\mathcal{G}(u)(x_0) \ne \mathcal{G}(\theta u)(x_0)$ due to (2.4).

(c) Let $0 \ll z$, $0 \lll u$ be the constant function introduced in (a) and $\theta \in (0,1)$. Since $\mathcal{G}$ is strongly subhomogeneous, one has $\theta\mathcal{G}(u) \lll \mathcal{G}(\theta u)$ and Lemma 1.1(b) guarantees

$$\langle \theta g(x,z), y'\rangle \overset{(2.4)}{=} \langle \theta\mathcal{G}(u)(x), y'\rangle < \langle \mathcal{G}(\theta u)(x), y'\rangle \overset{(2.4)}{=} \langle g(x,\theta z), y'\rangle \quad \text{for all } x \in \Omega$$

and all $y' \in Y'_+ \setminus \{0\}$. Then Lemma A.1(b) implies $\theta g(x,z) \ll g(x,\theta z)$, that is $g(x,\cdot)$ is strongly subhomogeneous for all $x \in \Omega$. □

Note that alternative sufficient conditions for subhomogeneity of $g(x,\cdot)$ in terms of the partial derivative $D_2 g$ can be derived using Lemma A.4. Here the differentiability of $g(x,\cdot)$ on the cone Y_+ is to be understood such that there exists an open superset $\tilde{Z} \supset Y_+$ on which the partial derivative $D_2 g(x,\cdot)$ exists. An alternative approach is given in terms of the cone differentiability from e.g. [6, pp. 225–226].

Hypothesis. *With a function $g : \Omega \times Y_+ \to Y_+$ assume:*

- S'_0 *$D_2 g(x,z)z \le g(x,z)$ for all $x \in \Omega$ and $z \in Y_+ \setminus \{0\}$,*
- S'_1 *$D_2 g(x,z)z < g(x,z)$ for all $x \in \Omega$ and $z \in Y_+ \setminus \{0\}$,*
- S'_2 *$D_2 g(x,z)z \ll g(x,z)$ for all $x \in \Omega$ and $z \in Y_+^\circ$.*

Corollary 2.5. *Let Hypotheses (N^l) hold for $l \in \{0,1\}$. If $g(x,\cdot) : Y_+ \to Y_+$ for all $x \in \Omega$, then the following holds:*

(a) S'_0 if and only if $\mathcal{G} : C(\Omega)^d_+ \to C(\Omega)^d_+$ is $C(\Omega)^d_+$-subhomogeneous,
(b) if S'_1 holds, then $\mathcal{G}$ is strictly $C(\Omega)^d_+$-subhomogeneous,
(c) if S'_2 holds, then $\mathcal{G}$ is strongly $C(\Omega)^d_+$-subhomogeneous.

Proof. (a) Combined with S'_0 we obtain from Lemma A.4(a) that $g(x,\cdot)$ is subhomogeneous, i.e. S_0 holds. Then the claim is a consequence of Prop. 2.4(a).

(b) Proceed as in (a) using Lemma A.4(b) and Prop. 2.4(b).

(c) Similar ideas to Prop. 2.4 and Lemma A.4(c). □

Theorem 2.6 (subhomogeneity of $\mathcal{H}$)**.** *Let Hypotheses (L) and (N^l) hold with $l \in \{0,1\}$. If*

(a) $L_0 \wedge (S_0 \vee S'_0)$ holds, then a Hammerstein operator $\mathcal{H} : C(\Omega)^d_+ \to C(\Omega)^d_+$ is $C(\Omega)^d_+$-subhomogeneous,
(b) $(L_0 \wedge L_1) \wedge (S_1 \vee S'_1)$ holds, then $\mathcal{H}$ is strictly $C(\Omega)^d_+$-subhomogeneous,
(c) $L_2 \wedge (S_1 \vee S'_1)$ or $L_3 \wedge S'_2$ holds, then $\mathcal{H}$ is strongly $C(\Omega)^d_+$-subhomogeneous.

Proof. (a) By assumption L_0, the Fredholm operator $\mathcal{K}$ is positive (cf. [24, Thm. 2.6]). Moreover, $\mathcal{G}$ is subhomogeneous due to Prop. 2.4(a) or Cor. 2.5(a). Hence, Cor. A.5(a) yields that (2.2) is subhomogeneous.

(b) From [24, Thm. 2.6(a)] we conclude that $\mathcal{K}$ is strictly positive, while $\mathcal{G}$ is strictly subhomogeneous, provided S_1 (see Prop. 2.4(b)) or S'_1 (cf. Cor. 2.5(b)) holds. Then the composition (2.2) is strictly subhomogeneous due to Cor. A.5(b).

(c) First, if assumption L_2 holds, then $\mathcal{K}$ is strongly positive by [24, Thm. 2.6(b)]. As above in (b) we see that $\mathcal{G}$ is strictly subhomogeneous. Second, if L_3 holds, then $\mathcal{K}$ leaves the interior of $C(\Omega)^d_+$ invariant due to [24, Thm. 2.6(c)], while Cor. 2.5(c) ensures that $\mathcal{G}$ is strongly subhomogeneous. In both cases, Cor. A.5(c) guarantees that the composition (2.2) is strongly subhomogeneous. □

2.3. **Concavity.** Concavity of a mapping is strongly related to subhomogeneity. Therefore, the following is largely parallel to Sec. 2.2.

Hypothesis. *With a Y_+-convex set $Z \subseteq \mathbb{R}^d$ assume:*

C_0 $g(x,\cdot) : Z \to \mathbb{R}^d$ *is Y_+-concave for all $x \in \Omega$,*
C_1 $g(x,\cdot) : Z \to \mathbb{R}^d$ *is strictly Y_+-concave for all $x \in \Omega$,*
C_2 $g(x,\cdot) : Z \to \mathbb{R}^d$ *is strongly Y_+-concave for all $x \in \Omega$.*

Proposition 2.7 (concavity of $\mathcal{G}$)**.** *Let Hypothesis (N^0) hold with a Y_+-convex Z. Then a Nemytskii operator $\mathcal{G} : U \to C(\Omega)^d$ fulfills:*

(a) $\mathcal{G}$ is $C(\Omega)^d_+$-concave, if and only if C_0 holds.
(b) If C_1 holds, then $\mathcal{G}$ is strictly $C(\Omega)^d_+$-concave.
(c) Let Y_+ be solid. If $\mathcal{G}$ is strongly $C(\Omega)^d_+$-concave, then C_2 holds.

Proof. (a) ($\Rightarrow$) Let $z, \bar z \in Z$, $z < \bar z$. The constant functions $u(x) :\equiv z$ and $\bar u(x) :\equiv \bar z$ on Ω satisfy $u \prec \bar u$ and since $\mathcal{G}$ is order concave, one has $\theta\mathcal{G}(u)+(1-\theta)\mathcal{G}(\bar u) \preceq \mathcal{G}(\theta u+(1-\theta)\bar u)$ for all $\theta \in [0,1]$. Then Lemma 1.1(a) yields for all $y' \in Y'_+$ and $x \in \Omega$ that

$$\begin{aligned}\langle \theta g(x,z) + (1-\theta)g(x,\bar z), y'\rangle &\overset{(2.4)}{=} \langle \theta\mathcal{G}(u)(x) + (1-\theta)\mathcal{G}(\bar u), y'\rangle\\ &\le \langle \mathcal{G}(\theta u + (1-\theta)\bar u)(x), y'\rangle \overset{(2.4)}{=} \langle g(x, \theta z + (1-\theta)\bar z), y'\rangle.\end{aligned}$$

By Lemma A.1(a) this means $\theta g(x,z) + (1-\theta)g(x,\bar z) \le g(x, \theta z + (1-\theta)\bar z)$, that is $g(x,\cdot)$ is Y_+-concave for all $x \in \Omega$.
($\Leftarrow$) Conversely, suppose C_0 holds and let $u \prec \bar u$, $\theta \in (0,1)$. In case $u(x) = \bar u(x)$, it is

$$\begin{aligned}&\theta\mathcal{G}(u)(x) + (1-\theta)\mathcal{G}(\bar u)(x) = \theta g(x,u(x)) + (1-\theta)g(x,\bar u(x)) = g(x,u(x))\\ &= g(x, \theta u(x) + (1-\theta)\bar u(x)) = \mathcal{G}(\theta u + (1-\theta)\bar u)(x),\end{aligned}$$

while in case $u(x) < \bar u(x)$ results

$$\begin{aligned}&\langle \theta\mathcal{G}(u)(x) + (1-\theta)\mathcal{G}(\bar u)(x), y'\rangle \overset{(2.4)}{=} \langle \theta g(x,u(x)) + (1-\theta)g(x,\bar u(x)), y'\rangle\\ &\le \langle g(x, \theta u(x) + (1-\theta)\bar u(x)), y'\rangle \overset{(2.4)}{=} \langle \mathcal{G}(\theta u + (1-\theta)\bar u)(x), y'\rangle\end{aligned}$$

for all $y' \in Y'_+$. Thus, $\theta\mathcal{G}(u) + (1-\theta)\mathcal{G}(\bar u) \preceq \mathcal{G}(\theta u + (1-\theta)\bar u)$.

(b) Let $u \prec \bar u$ and $\theta \in (0,1)$. There is an $x_0 \in \Omega$ so that $u(x_0) \neq \bar u(x_0)$, while (a) implies $\theta\mathcal{G}(u) + (1-\theta)\mathcal{G}(\bar u) \preceq \mathcal{G}(\theta u + (1-\theta)\bar u)$. It remains to establish $\theta\mathcal{G}(u) + (1-\theta)\mathcal{G}(\bar u) \neq \mathcal{G}(\theta u + (1-\theta)\bar u)$, which results since $g(x_0,\cdot)$ is strictly order concave and $\theta g(x_0,u(x_0)) + (1-\theta)g(x_0,\bar u(x_0)) < g(x_0, \theta u(x_0) + (1-\theta)\bar u(x_0))$. The claim follows.

(c) Let $z, \bar z \in Y_+$, $z \ll \bar z$, $\theta \in (0,1)$. Hence $u \lll \bar u$ where $u, \bar u$ are the constant functions from (a). Since $\mathcal{G}$ is strongly order concave, $\theta\mathcal{G}(u) + (1-\theta)\mathcal{G}(\bar u) \lll \mathcal{G}(\theta u + (1-\theta)\bar u)$ and Lemma 1.1(b) yield

$$\begin{aligned}&\langle \theta g(x,z) + (1-\theta)g(x,\bar z), y'\rangle \overset{(2.4)}{=} \langle \theta\mathcal{G}(u)(x) + (1-\theta)\mathcal{G}(\bar u)(x), y'\rangle\\ &< \langle \mathcal{G}(\theta u + (1-\theta)\bar u)(x), y'\rangle \overset{(2.4)}{=} \langle g(x, \theta z + (1-\theta)\bar z), y'\rangle \quad \text{for all } x \in \Omega\end{aligned}$$

and all $y' \in Y'_+ \setminus \{0\}$. Then Lemma A.1(b) implies that $\theta g(x,z) + (1-\theta)g(x,\bar z) \ll g(x, \theta z + (1-\theta)\bar z)$ for $\theta \in (0,1)$, i.e. $g(x,\cdot)$ is strongly order concave for all $x \in \Omega$. □

Sufficient conditions for the concavity of $g(x, \cdot)$ in terms of the partial derivative $D_2 g$ can be provided using Lemma A.6.

Hypothesis. *With an open, Y_+-convex set $Z \subseteq \mathbb{R}^d$ assume:*

C_0' $D_2 g(x, \bar{z})(\bar{z} - z) \le D_2 g(x, z)(\bar{z} - z)$ *for all* $x \in \Omega$ *and* $z, \bar{z} \in U$, $z < \bar{z}$,
C_1' $D_2 g(x, \bar{z})(\bar{z} - z) < D_2 g(x, z)(\bar{z} - z)$ *for all* $x \in \Omega$ *and* $z, \bar{z} \in U$, $z < \bar{z}$,
C_2' $D_2 g(x, \bar{z})(\bar{z} - z) \ll D_2 g(x, z)(\bar{z} - z)$ *for all* $x \in \Omega$ *and* $z, \bar{z} \in U$, $z \ll \bar{z}$.

Corollary 2.8. *Let Hypotheses (N^l) hold for $l \in \{0, 1\}$ with open, Y_+-convex Z. Then*

(a) C_0' if and only if $\mathcal{G} : U \to C(\Omega)_+^d$ is $C(\Omega)_+^d$-concave,
(b) if C_1' holds, then $\mathcal{G}$ is strictly $C(\Omega)_+^d$-concave,
(c) if C_2' holds, then $\mathcal{G}$ is strongly $C(\Omega)_+^d$-concave.

Proof. The argument is analogous to the proof of Cor. 2.5 with Prop. 2.4 replaced by Prop. 2.7 and employs Lemma A.6 instead of Lemma A.4. □

Theorem 2.9 (concavity of $\mathcal{H}$)**.** *Let Hypotheses (L) and (N^l) hold with $l \in \{0, 1\}$. If*

(a) $L_0 \wedge (C_0 \vee C_0')$ holds, then a Hammerstein operator $\mathcal{H} : U \to C(\Omega)_+^d$ is $C(\Omega)_+^d$-concave,
(b) $(L_0 \wedge L_1) \wedge (C_1 \vee C_1')$ holds, then $\mathcal{H}$ is strictly $C(\Omega)_+^d$-concave,
(c) $L_2 \wedge (C_1 \vee C_1')$ or $L_3 \wedge C_2'$ holds, then $\mathcal{H}$ is strongly $C(\Omega)_+^d$-concave.

Proof. The proof is analogous to the arguments required for Thm. 2.6, with Cor. A.5 replaced by Cor. A.7. □

Remark 2.10 (order-convex mappings)**.** Corresponding sufficient criteria for order-convex Hammerstein operators result by simply applying the above conditions ensuring the concavity of the Nemytskii operator $\mathcal{G}$ to the negative growth function $-g$.

Remark 2.11 (dispersal-growth operators)**.** Again motivated by applications in theoretical ecology [2, 20], we denote compositions

$$(\mathcal{G} \circ \mathcal{K})(u)(x) = g\left(x, \int_\Omega k(x, y) u(y) \, \mathrm{d}\mu(y)\right) \quad \text{for all } x \in \Omega$$

of Nemytskii operators (2.4) with Fredholm integral operators (2.3) as *dispersal-growth operators* (dispersal preceds growth). As in the above Hammerstein case, sufficient conditions for monotonicity result by combining the criteria L_0-L_2 from [24, Thm. 2.6] with Prop. 2.1 or Cor. 2.2. Similarly, based on Cor. A.5, conditions for subhomogeneity result by Prop. 2.4 or Cor. 2.5. Finally, by means of Cor. A.7 we can deduce their concavity properties from Prop. 2.7 or Cor. 2.8.

3. Degenerate kernel discretization.

Among the techniques to solve integral equations numerically, Nyström and projection methods apply to general Urysohn integral operators [24, 25]. We therefore discuss an approach tailor-made for Hammerstein operators $\mathcal{H}$. Thereto, assume the kernels $k : \Omega^2 \to L(\mathbb{R}^d)$ can be approximated by so-called *degenerate kernels* $k^n : \Omega^2 \to L(\mathbb{R}^d)$ giving rise to spatially discretized operators of the form

$$\mathcal{K}^n u = \int_\Omega k^n(\cdot, y) u(y) \, \mathrm{d}\mu(y)$$

(cf. [11, pp. 65ff, Sect. 4.2]). More precisely, if $\{\phi_1, \ldots, \phi_n\}$ is a set of linearly independent continuous functions $\phi_i : \Omega \to \mathbb{R}$, $1 \le i \le n$, and $\kappa_i : \Omega \to L(\mathbb{R}^d)$ are continuous, then k^n is of the form

$$k^n(x,y) = \sum_{i=1}^n \phi_i(x)\kappa_i(y) \quad \text{for all } x, y \in \Omega.$$

Hence, one obtains $\mathcal{K}u \in \operatorname{span}\{\phi_1, \dots, \phi_n\}$ for all $u \in C(\Omega)^d$.

In this general setting, positivity of the degenerate kernel approximation $\mathcal{K}^n$ can be determined by means of the sign of

$$\langle \mathcal{K}^n u(x), y' \rangle = \sum_{i=1}^n \phi_i(x) \int_\Omega \langle \kappa_i(y)u(y), y' \rangle \, \mathrm{d}\mu(y) \quad \text{for all } x \in \Omega,\ y' \in Y'_+.$$

In order to obtain more feasible conditions, we restrict to degenerate kernels being determined by interpolation conditions. Thereto, suppose that $\{x_1, \dots, x_n\} \subseteq \Omega$ is a set of pairwise different points so that

$$P_n := [\phi_i(x_j)]_{i,j=1}^n \in GL(\mathbb{R}^n).$$

In case $P_n = I_n$ the set $\{\phi_1, \dots, \phi_n\}$ is called a *Lagrange basis*. The interpolation conditions $k(x_j, y) = k^n(x_j, y)$ for all $y \in \Omega$ yield that $\kappa_j(y)$ are determined by the equations

$$\sum_{i=1}^n (P_n)_{ij}\kappa_i(y) = \sum_{i=1}^n \phi_i(x_j)\kappa_i(y) = k(x_j, y) \quad \text{for all } 1 \le j \le n. \tag{3.1}$$

Theorem 3.1 (positivity of $\mathcal{K}^n$). *If all the functions*

$$\sigma_j : \Omega \to \mathbb{R}, \qquad \sigma_j(x) := \sum_{i=1}^n (P_n^{-1})_{ij}\phi_i(x) \quad \textit{for all } 1 \le j \le n \tag{3.2}$$

have nonnegative values, then the following hold:

- *(a) If $k(x,y)$ is Y_+-positive for all $x \in \Omega$ and μ-a.a. $y \in \Omega$, then $\mathcal{K}^n$ is $C(\Omega)^d_+$-positive.*
- *(b) Let Y_+ be solid. If $k(x,y)$ is strongly Y_+-positive for all $x \in \Omega$ and μ-a.a. $y \in \Omega$ with*

$$\forall x \in \Omega : \exists j_0 \in \{1, \dots, n\} : \sigma_{j_0}(x) > 0, \tag{3.3}$$

then $\mathcal{K}^n$ is strongly $C(\Omega)^d_+$-positive.

Proof. Let $u \in C(\Omega)^d$, $0 \prec u$ and $y' \in Y'_+$.

(a) Using Lemma 1.1(a) we obtain that $0 \le \langle k(x,y)u(y), y' \rangle$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$. Thus, our assumptions lead to

$$\begin{aligned}
\langle \mathcal{K}^n u(x), y' \rangle &= \sum_{i=1}^n \phi_i(x) \int_\Omega \langle \kappa_i(y)u(y), y' \rangle \, \mathrm{d}\mu(y) \\
&\stackrel{(3.1)}{=} \sum_{i=1}^n \phi_i(x) \int_\Omega \langle \sum_{j=1}^n (P_n^{-1})_{ij} k(x_j, y)u(y), y' \rangle \, \mathrm{d}\mu(y) \\
&= \sum_{j=1}^n \sum_{i=1}^n (P_n^{-1})_{ij}\phi_i(x) \int_\Omega \langle k(x_j, y)u(y), y' \rangle \, \mathrm{d}\mu(y) \\
&= \sum_{j=1}^n \sigma_j(x) \int_\Omega \langle k(x_j, y)u(y), y' \rangle \, \mathrm{d}\mu(y) \ge 0 \quad \text{for all } x \in \Omega.
\end{aligned}$$

With anew Lemma 1.1(a) this means $0 \preceq \mathcal{K}^n u$, i.e. $\mathcal{K}^n$ is positive.

(b) If $y' \neq 0$, then by means of Lemma 1.1(b) the assumption yields the strict inequality $0 < \langle k(x,y)u(x), y'\rangle$ for all $x \in \Omega$ and μ-a.a. $y \in \Omega$. As in (a) this implies

$$\langle \mathcal{K}^n u(x), y'\rangle = \sum_{j=1}^{n} \sigma_j(x) \int_\Omega \langle k(x_j,y)u(y), y'\rangle \,\mathrm{d}\mu(y) \overset{(3.3)}{>} 0 \quad \text{for all } x \in \Omega$$

and consequently Lemma 1.1(b) guarantees that $\mathcal{K}^n$ is strongly positive. □

Note that Thm. 3.1 has a counterpart [24, Thm. 4.2] in the area of collocation methods, which also requires nonnegativity of the functions $\sigma_1, \dots, \sigma_n$ defined in (3.2). There we have demonstrated that Thm. 3.1 applies when $\phi_1, \dots, \phi_n$ are the hat functions (yielding piecewise linear interpolation, see [24, Exam. 4.5]) or the Lagrange functions (leading to polynomial interpolation, cf. [24, Exam. 4.6]). Nevertheless, the subsequent example illustrates that even in case $\Omega = [a,b]$ a popular interpolation method might violate positivity.

Example 3.2 (cubic splines)**.** Let $x_i := a + i\frac{b-a}{n}$ for $0 \leq i \leq n$. Suppose that ϕ_j, $-2 \leq j \leq n$, are the cubic B-splines (cf. [12, pp. 242ff]), which consist of nonnegative functions. Beyond $n+1$ interpolation conditions they require additional conditions in order to arrive at an invertible matrix $P_n \in \mathbb{R}^{(n+3)\times(n+3)}$. For this, canonical choices are:

- Hermite boundary conditions $D_1 k^n(a,y) = D_1 k(a,y)$ and $D_1 k^n(b,y) = D_1 k(b,y)$ (see [12, p. 251]) yield the functions σ_j from Fig. 1 (top, right) with the matrix

$$P_n = \frac{1}{6}\begin{pmatrix} -\frac{3n}{b-a} & 0 & \frac{3n}{b-a} & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & -\frac{3n}{b-a} & 0 & \frac{3n}{b-a} \end{pmatrix}$$

- Natural splines [12, p. 251] are based on the boundary conditions $D_1^2 k^n(a,y) = 0$, $D_1^2 k^n(b,y) = 0$, lead to the functions σ_j from Fig. 1 (bottom, left) and the matrix

$$P_n = \frac{1}{6}\begin{pmatrix} \frac{6n^2}{(b-a)^2} & -\frac{12n^2}{(b-a)^2} & \frac{6n^2}{(b-a)^2} & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & \frac{6n^2}{(b-a)^2} & -\frac{12n^2}{(b-a)^2} & \frac{6n^2}{(b-a)^2} \end{pmatrix}$$

- For periodic splines [12, p. 252] one requires that $D_1 k^n(a,y) = D_1 k^n(b,y)$ and $D_1^2 k^n(a,y) = D_1^2 k^n(b,y)$ leading to functions σ_j from Fig. 1 (bottom, right), where

$$P_n = \frac{1}{6}\begin{pmatrix} -\frac{3n}{b-a} & 0 & \frac{3n}{b-a} & & \frac{3n}{b-a} & 0 & -\frac{3n}{b-a} \\ \frac{6n^2}{(b-a)^2} & -\frac{12n^2}{(b-a)^2} & \frac{6n^2}{(b-a)^2} & & -\frac{6n^2}{(b-a)^2} & \frac{12n^2}{(b-a)^2} & -\frac{6n^2}{(b-a)^2} \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 1 \end{pmatrix}$$

None of them results in an inverse matrix P_n^{-1} with nonnegative entries. Indeed, the minimal elements $c_n := \min_{i,j=1}^{n+3}(P_n^{-1})_{ij}$ are illustrated in Fig. 1 (top, left) for $3 \leq n \leq 10$ as they become stationary for $n \to \infty$. Hence, Thm. 3.1 does not apply.

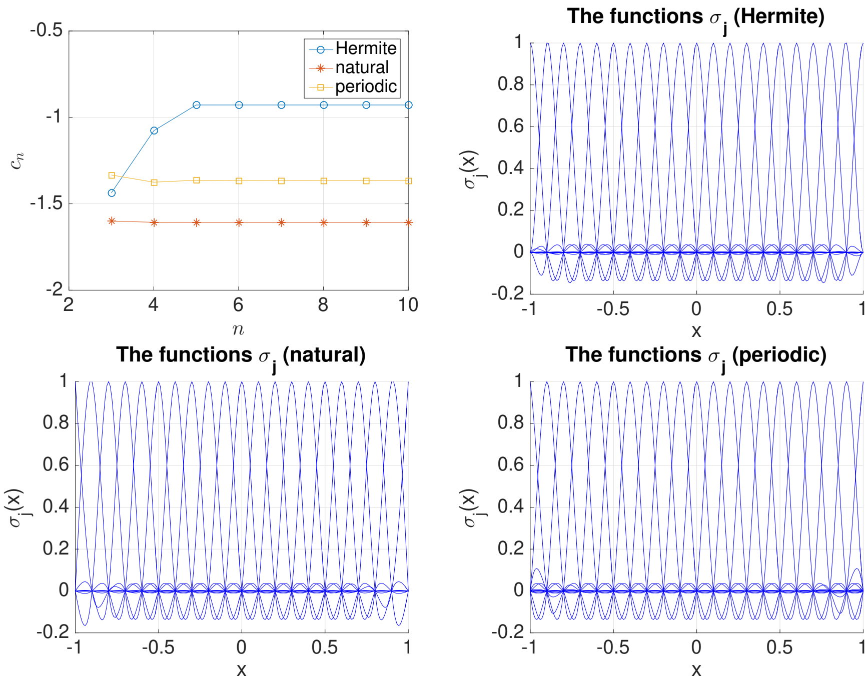

FIGURE 1. Top left: Minimal entries c_n of the inverse collocation matrices P_n^{-1} for cubic splines depending on n (for $a = -1, b = 1$) Functions $\sigma_j : [-1, 1] \to \mathbb{R}$, $-2 \le j \le n$, for $n = 20$ and different types of boundary conditions

Remark 3.3 (discrete degenerate kernel methods)**.** Evaluating the operators $\mathcal{K}^n\mathcal{G}$ still requires to approximate integrals yielding *discrete degenerate kernel methods*. For this purpose, it was shown in [24, Thm. 3.3] and [25, Sect. 2] that quadrature rules with positive weights preserve monotonicity, as well as their strict and strong versions.

4. **Integrodifference equations.** Let $\mathbb{I}$ be a discrete interval, i.e. the intersection of a real interval with the integers, and $\mathbb{I}' := \{t \in \mathbb{I} : t+1 \in \mathbb{I}\}$. We study nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{H}_t(u_t),} \tag{I_0}$$

whose right-hand sides are Hammerstein integral operators

$$\mathcal{H}_t : U_t \to C(\Omega)^d, \qquad \mathcal{H}_t(u) := \int_\Omega k_t(\cdot, y) g_t(y, u(y))\, \mathrm{d}\mu(y) \tag{4.1}$$

on domains $U_t := \{u \in C(\Omega)^d : u(x) \in Z_t \text{ for all } x \in \Omega\}$ for every $t \in \mathbb{I}'$; one speaks of *(Hammerstein) integrodifference equations*. For well-definedness of $\mathcal{H}_t$, we assume throughout this section that all kernels $k_t : \Omega^2 \to L(\mathbb{R}^d)$ satisfy Hypothesis (L), while all growth functions $g_t : \Omega \times Z_t \to \mathbb{R}^d$ with $Z_t \subseteq \mathbb{R}^d$ fulfill the Hypothesis (N^l) for $l = 0$ resp. $l \in \{0, 1\}$, when derivatives are involved.

In case $\mathbb{I}$ is unbounded above and there exists a $\theta \in \mathbb{N}$ such that $\mathcal{H}_{t+\theta} = \mathcal{H}_t$ holds for all $t \in \mathbb{I}$, then (I_0) is called *θ-periodic*.

If $\mathcal{H}_t(U_t) \subseteq U_{t+1}$ holds for $t \in \mathbb{I}'$, then the forward solution to (I_0) starting at an initial time $\tau \in \mathbb{I}$ in the initial state $u_\tau \in U_\tau$ is given by the compositions

$$\varphi(t;\tau,u_\tau) := \begin{cases} \mathcal{H}_{t-1} \circ \ldots \circ \mathcal{H}_\tau(u_\tau), & \tau < t, \\ u_\tau, & t = \tau. \end{cases} \tag{4.2}$$

We denote the resulting function $\varphi : \left\{(t,\tau,u) \times \mathbb{I}^2 \times C(\Omega)^d : \tau \le t,\ u \in U_\tau\right\} \to C(\Omega)^d$ as *general solution* of (I_0).

Structural properties of $\varphi(t;\tau,\cdot) : U_\tau \to U_t$ can be deduced as follows: Relying on the conditions given in Thm. 2.3 we obtain that (strict) monotonicity of all $\mathcal{H}_s$, $\tau \le s < t$, extends to $\varphi(t;\tau,\cdot)$ due to (4.2), while conditions for strong monotonicity are based on Cor. A.3. Similarly, subhomogeneity or concavity of $\varphi(t;\tau,\cdot)$ can be tackled via Thm. 2.6 and Cor. A.5 resp. Thm. 2.9 and Cor. A.7. Of particular importance are cone mappings:

Theorem 4.1 (cone mappings)**.** *Let $Z = Y_+$ and $\tau \le t$. If the kernels $k_t : \Omega^2 \to L(\mathbb{R}^d)$ satisfy L_0, while the growth functions $g_t : \Omega \times Y_+ \to Y_+$ fulfill M_0 and S_0 for all $t \in \mathbb{I}'$, then $\varphi(t;\tau,\cdot) : C(\Omega)^d_+ \to C(\Omega)^d_+$ is a cone mapping.*

Proof. Let $s \in \mathbb{I}'$. By assumption, one has $U_s = C(\Omega)^d_+$. From Hypotheses L_0 and M_0 we conclude using Thm. 2.3(a) that $\mathcal{H}_s$ is monotone. With L_0 and S_0 it is a consequence of Thm. 2.6(a) that $\mathcal{H}_s$ is also subhomogeneous. If $\tau \le t$, then (4.2) implies that $\varphi(t;\tau,\cdot)$ is monotone and subhomogeneous. Thus, Rem. A.8 yields the claim. □

Remark 4.2 (dispersal-growth equations)**.** Based on Rem. 2.11 we denote nonautonomous difference equations of the form

$$u_{t+1}(x) = g_{t+1}\left(x, \int_\Omega k_t(x,y)u_t(y)\,\mathrm{d}\mu(y)\right) \quad \text{for all } x \in \Omega \tag{4.3}$$

as *dispersal-growth equations*. Their right-hand side is the composition $\mathcal{G}_{t+1} \circ \mathcal{K}_t$ of a Nemytskii operator $\mathcal{G}_{t+1}$ with a Fredholm operator $\mathcal{K}_t$, $t \in \mathbb{I}'$. The general solutions φ to an IDE (I_0) and $\hat{\varphi}$ to (4.3) are semi-conjugated via

$$\varphi(t;\tau,u_\tau) = \mathcal{K}_{t-1} \circ \hat{\varphi}(t-1;\tau,\mathcal{G}_\tau(u_\tau)) \quad \text{for all } \tau < t,\ u_\tau \in U_\tau.$$

Monotonicity, subhomogeneity and concavity of the general solution $\hat{\varphi}$ result analogously to the Hammerstein case.

4.1. **Linear integrodifference equations.** Linearly homogeneous IDEs

$$u_{t+1} = \mathcal{K}_t u_t, \qquad \mathcal{K}_t u := \int_\Omega k_t(\cdot,y)u(y)\,\mathrm{d}\mu(y) \tag{4.4}$$

have the *evolution operator*

$$\Phi(t,\tau) := \begin{cases} \mathcal{K}_{t-1}\cdots\mathcal{K}_\tau, & \tau < t, \\ I_{C(\Omega)^d}, & \tau = t \end{cases} \tag{4.5}$$

and the general solution of (4.4) is given by $\varphi(t;\tau,u_\tau) = \Phi(t,\tau)u_\tau$ for all $\tau \le t$ and initial values $u_\tau \in C(\Omega)^d$. Moreover, given $\tau < t$, if the products

$$\begin{gathered}(y_1,\ldots,y_{t-\tau-1}) \mapsto k_{t-1}(x,y_{t-\tau-1})\cdots k_{\tau+1}(y_2,y_1)k_\tau(y_1,y) \\ \text{are integrable for all } x \in \Omega \text{ and } \mu\text{-a.a. } y \in \Omega,\end{gathered} \tag{4.6}$$

then we define the *iterated kernels* $k_\tau^t : \Omega^2 \to L(\mathbb{R}^d)$,

$$k_\tau^t(x,y) := \begin{cases} \int_\Omega \cdots \int_\Omega k_{t-1}(x, y_{t-\tau-1}) \cdots k_{\tau+1}(y_2, y_1) k_\tau(y_1, y) \, \mathrm{d}\mu(y_1) \cdots \mathrm{d}\mu(y_{t-\tau-1}), \\ \qquad t > \tau + 1, \\ k_\tau(x,y), \qquad t = \tau + 1, \end{cases}$$

and thanks to Fubini's theorem [28, p. 224, 7.3.4 Thm.] the evolution operator $\Phi(t,\tau)$ can be represented as single Fredholm operator

$$\Phi(t,\tau)u = \int_\Omega k_\tau^t(\cdot, y) u(y) \, \mathrm{d}\mu(y) \quad \text{for all } \tau < t, \ u \in C(\Omega)^d. \tag{4.7}$$

Assuming positivity of an iterated kernel is weaker than imposing conditions on each individual kernel:

Theorem 4.3 (positivity of linear IDEs)**.** *Let $\tau < t$ and suppose (4.6) holds. If the iterated kernels $k_\tau^t : \Omega^2 \to L(\mathbb{R}^d)$ satisfy*

(a) L_0, then $\Phi(t,\tau)$ is $C(\Omega)_+^d$-positive,
(b) L_1, then $\Phi(t,\tau)$ is strictly $C(\Omega)_+^d$-positive,
(c) L_2, then $\Phi(t,\tau)$ is strongly $C(\Omega)_+^d$-positive.

Proof. The assertion results from (4.7), if we apply [24, Thm. 2.6] to the kernel k_τ^t. □

4.2. **Global asymptotic stability.** Since we are interested in asymptotic behavior of IDEs (I_0) now, suppose that $\mathbb{I}$ is unbounded above. Under these premises we arrive at

Theorem 4.4 (local-global stability principle)**.** *Let $\theta_0, \theta_1 \in \mathbb{N}$, $\theta := \operatorname{lcm}\{\theta_0, \theta_1\}$ and let the cone Y_+ be solid. Assume that the kernels $k_t : \Omega^2 \to L(\mathbb{R}^d)$ satisfy L_0 and the growth functions $g_t : \Omega \times Y_+ \to Y_+$ fulfill M_0 and S_0, as well as*

(i) $k_t = k_{t+\theta_0}$, $g_t = g_{t+\theta_0}$,
(ii) $\mathcal{H}_t((C(\Omega)_+^d)^\circ) \subseteq (C(\Omega)_+^d)^\circ$

for all $t \in \mathbb{I}$. If a θ_1-periodic solution $\phi^* = (\phi_t^*)_{t\in\mathbb{I}}$ of (I_0) in $(C(\Omega)_+^d)^\circ$ satisfies

$$|\lambda| < 1 \quad \text{for all } \lambda \in \sigma(D\mathcal{H}_{\tau+\theta-1}(\phi_{\tau+\theta-1}) \cdots D\mathcal{H}_\tau(\phi_\tau)) \tag{4.8}$$

and one $\tau \in \mathbb{I}$, then ϕ^* is globally asymptotically stable w.r.t. $(C(\Omega)_+^d)^\circ$.

Proof. Above all, $C(\Omega)_+^d$ defines a solid and normal cone due to [24, Lemma 2.2]. Now let $\tau \in \mathbb{I}$. Since (I_0) is a θ_0-periodic IDE by (i) and the solution ϕ^* is θ_1-periodic, the period mapping $\pi_\tau := \varphi(\tau+\theta; \tau, \cdot)$ possesses the fixed point $\phi_\tau^* \in (C(\Omega)_+^d)^\circ$. First, it results from Thm. 4.1 that $\pi_\tau : C(\Omega)_+^d \to C(\Omega)_+^d$ is a cone mapping, which leaves the interior of the cone $C(\Omega)_+^d$ invariant due to (ii). Second, π_τ is of class C^1 with the derivative

$$D\pi_\tau(\phi_\tau^*) = D\mathcal{H}_{\tau+\theta-1}(\phi_{\tau+\theta-1}) \cdots D\mathcal{H}_\tau(\phi_\tau).$$

Whence, it follows from (4.8) that $\sigma(D\pi_\tau(\phi_\tau^*))$ is contained in the open unit disc in $\mathbb{C}$. Then [8, Thm. 1] implies that ϕ_τ^* is an exponentially stable fixed point of π_τ. In particular ϕ_τ^* is locally attractive and thus Thm. A.10 applies to the period map π_τ. Therefore, ϕ_τ^* is globally attractive, and being exponentially stable, it is even globally asymptotically stable w.r.t. π_τ. This property extends to the θ_1-periodic solution ϕ^* of (I_0). □

4.3. **Order-preserving discretizations.** When simulating the dynamics of IDEs (1.1) the integrals need to be approximated by a quadrature rule or one has to replace the state space $C(\Omega)^d$ by a finite-dimensional subspace (cf. [3, 11]). This means the right-hand side of (1.1) is substituted by mappings $\mathcal{H}_t^n : U_t \to C(\Omega)^d$ yielding a difference equation

$$\boxed{u_{t+1} = \mathcal{H}_t^n(u_t).} \tag{I_n}$$

In essence, the right-hand sides $\mathcal{H}_t^n$ in (I_n) are of the form

$$\mathcal{H}_t^n = \mathcal{K}_t^n \mathcal{G} \quad \text{for all } t \in \mathbb{I}',$$

where the structure of the linear operator $\mathcal{K}_t^n$ depends on the discretization method:

For Nyström methods (cf. [24, Sec. 3]), the integral in (1.1) is replaced by a numerical integration rule (involving weights $w_\eta > 0$ and a set of nodes $\eta \in \Omega_n := \{\eta_1, \dots, \eta_n\}$), which implies the discretization

$$\mathcal{K}_t^n(u) := \sum_{j=1}^n k_t(\cdot, \eta) u(\eta).$$

Such mappings can be realized as special case of the general Hammerstein operators (2.1), if we choose the weighted counting measure $\mu(\Omega') := \sum_{\eta \in \Omega'} w_\eta$ on the family of discrete subsets $\Omega' \subseteq \mathbb{R}^\kappa$ and $\Omega = \Omega_n$.

For degenerate kernel methods (cf. [24, Sec. 3]) on has

$$\mathcal{K}_t^n u := \sum_{j=1}^n \phi_j(\cdot) \int_\Omega \kappa_{j,t}(y) u(y) \,\mathrm{d}\mu(y)$$

and monotonicity can be determined in terms of the criteria from Sec. 3.

In conclusion, it remains to study monotonicity properties of $\mathcal{K}_t^n$, because $\mathcal{H}_t^n$ inherits its subhomogeneity or concavity from the Nemytskii operator $\mathcal{G}$ derived in Prop. 2.4 and Cor. 2.5 resp. Prop. 2.7 and Cor. 2.8.

5. **Applications.** Let us comment on applications from theoretical ecology [19], where we restrict to the time-invariant case for simplicity.

First, various real-valued functions $k : \Omega^2 \to \mathbb{R}_+$ are used as entries in matrix-valued kernels for (2.3), in order to describe the dispersal stage in ecological models [19, 32]; therefore one speaks of *dispersal kernels*. Often such functions are of convolution form $k(x, y) = \tilde{k}(|x - y|)$ for all $x, y \in \Omega$ with some function $\tilde{k} : \mathbb{R}_+ \to \mathbb{R}_+$. Further details were given in [24, Sec. 5.1].

Second, the sedentary state is described via certain (extensively studied) types of (nonlinear) *growth functions* $g : \Omega \times Z \to \mathbb{R}_+$ in (2.1). Given the solid cone $Y_+ = \mathbb{R}_+$, a *growth rate* $c : \Omega \to \mathbb{R}_+$ and a parameter $\delta > 0$, typical examples are:

- *Generalized Beverton-Holt function*: $g(y, z) := \frac{c(y)z}{1+z^\delta}$ with $Z = \mathbb{R}_+$ is monotone and subhomogeneous for $\delta \in (0, 1]$. In case $c(x) > 0$ for all $x \in \Omega$ the generalized Beverton-Holt model is strongly monotone and strongly subhomogeneous. The case $\delta = 1$ is the *Beverton-Holt* model and satisfies $0 \le g(y, z) \le c(y)$ for all $y \in \Omega$.
- *Hassell function*: $g(y, z) := \frac{c(y)z}{(1+z)^\delta}$ with $Z = \mathbb{R}_+$ is monotone and subhomogeneous for $\delta \in (0, 1]$. In case $c(x) > 0$ for all $x \in \Omega$ the Hassell model is strongly monotone and strongly subhomogeneous. Again, $\delta = 1$ yields the Beverton-Holt model.
- *Allee growth function*: $g(y, z) := \frac{c(y)z^2}{\delta+z^2}$ with $Z = \mathbb{R}_+$ satisfies $0 \le g(y, z) \le c(y)$ for all $y \in \Omega$, $0 \le z$. It is monotone, but not subhomogeneous. In case $c(x) > 0$ for all $x \in \Omega$ it is even strongly monotone.

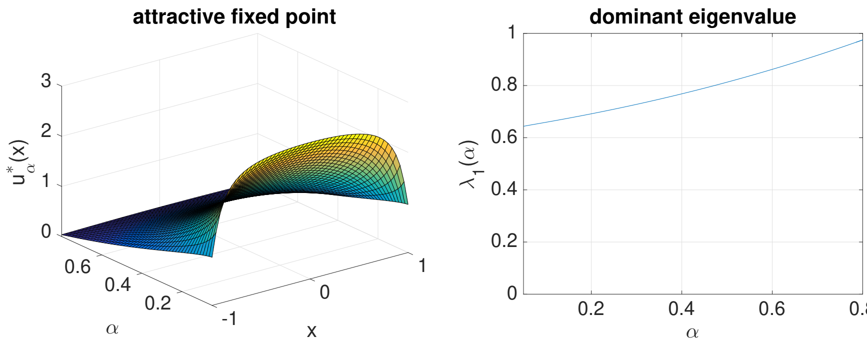

FIGURE 2. Dependence of the asymptotically stable fixed point u^*_α of (5.1) (left) and of the dominant eigenvalue on the dispersal rate $\alpha > 0$ (right), where $\gamma = 3$, $\delta = 0.5$ and $\Omega = [-1, 1]$. The positive function u^*_α exists for $\alpha \in (0, \alpha^*)$ with $\alpha^* \approx 0.84$

- *Ricker function*: $g(y, z) := c(y)ze^{-\delta z}$ with $Z = [0, \frac{1}{\delta}]$ is monotone and fulfills the inequality $0 \leq g(y, z) \leq e^{-1}c(y)$ for all $y \in \Omega$, $z \in Z$. Furthermore, strong monotonicity holds in case $c(x) > 0$, $x \in \Omega$.

Consequently, Thm. 4.4 can be applied to IDEs (1.1) involving the generalized Beverton-Holt or Hassell function for $\delta \in (0, 1)$. In addition, the Allee growth and the Ricker function allow monotone iteration techniques (cf. [27, pp. 163ff, Chapt. 11] and [30, p. 283, Thm. 7.A]); in the Ricker case at least on order intervals.

5.1. **Spatial Hassell model.** In order to illustrate Thm. 4.4, we are interested in the nontrivial fixed point of the autonomous Hassell equation

$$u_{t+1}(x) = \int_\Omega k(x, y)\frac{c(y)u_t(y)}{(1 + u_t(y))^\delta}\,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{5.1}$$

with $t \in \mathbb{Z}$, continuous functions $k : \Omega^2 \to \mathbb{R}_+$, $c : \Omega \to \mathbb{R}_+$ and a parameter $\delta > 0$. The right-hand side of (5.1) is monotone for $\delta \in (0, 1]$ due to Thm. 2.3(a) and subhomogeneous for parameters $\delta \in (0, 1]$ by Thm. 2.6(a).

Example 5.1 (Cauchy kernel)**.** We equip (5.1) with the Cauchy kernel

$$k(x, y) := \frac{\alpha}{\pi(\alpha^2 + |x - y|^2)}$$

having dispersal rate $\alpha > 0$ and a constant growth rate $c(y) :\equiv \gamma$. The right-hand side of (5.1) satisfies $\mathcal{H}(u)(x) > 0$ for all $x \in \Omega$ and $u \in C(\Omega)^d_+ \setminus \{0\}$ yielding the assumption (ii) of Thm. 4.4. The resulting Hassell IDE (5.1) depends on the three parameters $\alpha, \gamma, \delta > 0$. Combining Thm. 4.4 with the local criterion [8, Thm. 1] one obtains:

- For $\alpha \in (0, \alpha^*)$ there exists a globally asymptotically stable positive solution u^*_α until it transfers its stability to the trivial solution. Hence, an increase in the dispersal rate destabilizes u^*_α (see Fig. 2).
- The trivial solution looses its exponential stability at a growth rate γ^* in terms of a transcritical bifurcation. For $\gamma > \gamma^*$ there exists a globally asymptotically stable positive fixed point u^*_γ and we refer to Fig. 3 for an illustration.
- The positive solution u^*_δ exists and is globally asymptotically stable for $\delta \in (0, 1)$. Although monotonicity of the right-hand side is lost for $\delta > 1$, the nonzero fixed point remains (at least) locally asymptotically stable for $\delta > 1$ (see Fig. 4).

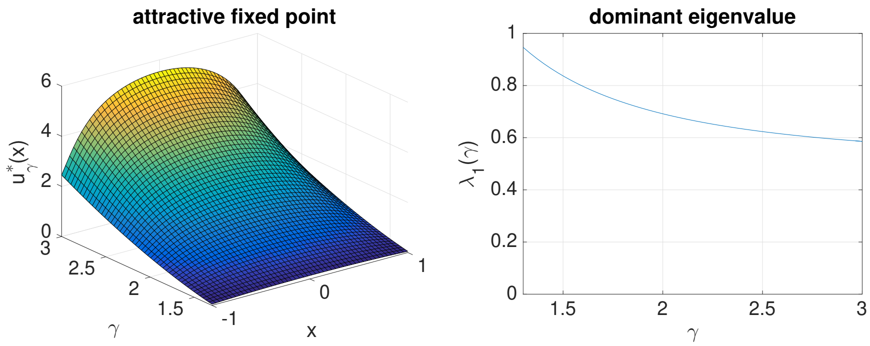

FIGURE 3. Dependence of the asymptotically stable fixed point u^*_γ of (5.1) (left) and of the dominant eigenvalue on the growth rate $\gamma > 0$ (right), where $\alpha = 0.2$, $\delta = 0.5$ and $\Omega = [-1, 1]$. The positive function u^*_γ exists for $\gamma > \gamma^* \approx 1.228$

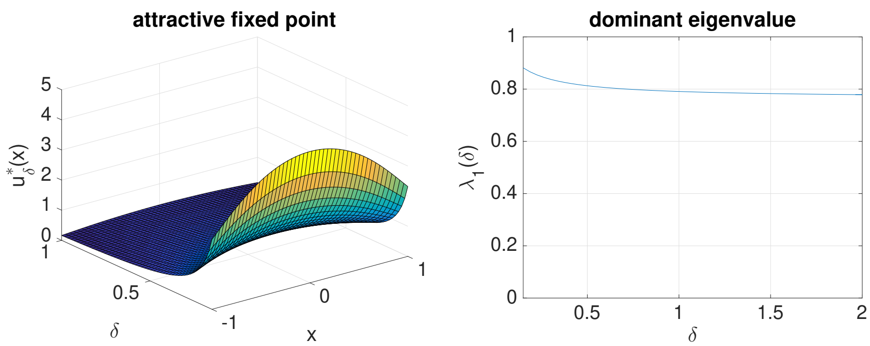

FIGURE 4. Dependence of the asymptotically stable fixed point u^*_δ of (5.1) (left) and of the dominant eigenvalue on the parameter $\delta > 0$ (right), where $\alpha = 0.2$, $\gamma = 3$ and $\Omega = [-1, 1]$

In order to obtain these results we have discretized (5.1) using the Nyström method based on the trapezoidal rule with $n = 50$ nodes. The nontrivial fixed points were computed using a Newton solver in Matlab and we relied on the Matlab function `eig` in order to compute the dominant eigenvalue of the linearization (which is real by the Krein-Rutman theorem). The fixed-point and eigenvalue branches from Figs. 2–4 were obtained by continuation using a predictor-corrector method.

A more general class of IDEs to which the local-global stability principle Thm. 4.4 applies on the north-east cone $Y_+ := \mathbb{R}^2_+$ are systems of spatial Hassell equations

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \int_\Omega \begin{pmatrix} k_{11}(\cdot, y)\frac{c_{11}(y)u_t(y)}{(1+u_t(y))^{\delta_{11}}} + k_{12}(\cdot, y)\frac{c_{12}(y)v_t(y)}{(1+v_t(y))^{\delta_{12}}} \\ k_{21}(\cdot, y)\frac{c_{21}(y)u_t(y)}{(1+u_t(y))^{\delta_{21}}} + k_{22}(\cdot, y)\frac{c_{22}(y)v_t(y)}{(1+v_t(y))^{\delta_{22}}} \end{pmatrix} \mathrm{d}y,$$

with continuous functions $k_{ij} : \Omega^2 \to (0, \infty)$, $c_{ij} : \Omega \to (0, \infty)$ and $\delta_{ij} \in (0, 1)$ for indices $1 \le i, j \le 2$, that is, the right-hand side is monotone and subhomogeneous.

5.2. **Nonautonomous spatial Leslie-Gower model.** A monotone iteration technique for nonautonomous integrodifference equations was developed in [26]. It allows to determine maximal and minimal solutions contained in the pullback attractor of dissipative IDEs.

Let $\mathbb{I} = \mathbb{Z}$ and assume that $k : \Omega^2 \to L(\mathbb{R}^2)$, as well as the coefficient functions $c_t^1, c_t^2 : \Omega \to (0,\infty)$, $b_t^1, b_t^2 : \Omega \to \mathbb{R}_+$ are continuous. Consider the *Leslie-Gower model*

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \int_\Omega k(\cdot, y) \begin{pmatrix} \frac{c_t^1(y)u_t(y)}{1+u_t(y)+b_t^1(y)v_t(y)} \\ \frac{c_t^2(y)v_t(y)}{1+b_t^2(y)u_t(y)+v_t(y)} \end{pmatrix} \mathrm{d}y, \tag{5.2}$$

whose right-hand side is a Hammerstein operator (2.1) with $Z = \mathbb{R}_+^2$ and the continuously differentiable growth functions

$$g_t : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}_+^2, \qquad g_t(x,u,v) := \begin{pmatrix} \frac{c_t^1(x)u}{1+u+b_t^1(x)v} \\ \frac{c_t^2(x)v}{1+b_t^2(x)u+v} \end{pmatrix}$$

with the partial derivatives

$$D_{(2,3)}g_t(x,u,v) = \begin{pmatrix} \frac{c_t^1(x)(1+b_t^1(x)v)}{(1+u+b_t^1(x)v)^2} & -\frac{b_t^1(x)c_t^1(x)u}{(1+u+b_t^1(x)v)^2} \\ -\frac{b_t^2(x)c_t^2(x)v}{(1+b_t^2(x)u+v)^2} & \frac{c_t^2(x)(1+b_t^2(x)u)}{(1+b_t^2(x)u+v)^2} \end{pmatrix}.$$

The south-east cone $Y_+ := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \le 0 \le x_1\right\}$ is generated by the linearly independent vectors $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{-1}$ and we choose $e_1' = \binom{1}{0}$, $e_2' = \binom{0}{-1}$ according to $\langle e_i, e_j' \rangle = \delta_{ij}$ for $1 \le i,j \le 2$. Thus,

$$\begin{aligned}
\langle D_{(2,3)}g_t(x,u,v)e_1, e_1' \rangle &= \tfrac{c_t^1(x)(1+b_t^1(x)v)}{(1+u+b_t^1(x)v)^2} > 0, \\
\langle D_{(2,3)}g_t(x,u,v)e_1, e_2' \rangle &= \tfrac{b_t^2(x)c_t^2(x)v}{(1+b_t^2(x)u+v)^2} \ge 0, \\
\langle D_{(2,3)}g_t(x,u,v)e_2, e_1' \rangle &= \tfrac{b_t^1(x)c_t^1(x)u}{(1+b_t^2(x)u+v)^2} \ge 0, \\
\langle D_{(2,3)}g_t(x,u,v)e_2, e_2' \rangle &= \tfrac{c_t^2(x)(1+b_t^2(x)u)}{(1+b_t^2(x)u+v)^2} > 0
\end{aligned}$$

and hence [24, Sect. 5.1] implies that $D_{(2,3)}g_t(x,u,v)$ is Y_+-monotone for all $t \in \mathbb{Z}$, $x \in \Omega$ and $u,v \in \mathbb{R}_+$.

As shown in [26, Thm. 4], the nonautonomous IDE (5.2) has a pullback attractor $\mathcal{A}^*$ and due to the monotonicity of the right-hand side, $\mathcal{A}^*$ is bounded below by the trivial solution and bounded above by an extremal solution $(u^+, v^+) = (u_t^+, v_t^+)_{t \in \mathbb{I}} \in \mathcal{A}^*$.

Example 5.2 (pullback attractor)**.** Consider a nonautonomous Leslie-Gower model (5.2) with $\Omega = [-10, 10]$, a diagonal matrix $k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ with Gauß kernels

$$k_i(x,y) := \tfrac{1}{\sqrt{2\pi\alpha_i^2}} \exp\left(-\tfrac{1}{2\alpha_i^2}|x-y|^2\right) \quad \text{for all } i = 1,2$$

having dispersal rates α_1, α_2 in the diagonal. The extremal solutions bounding the pullback attractor $\mathcal{A}^*$ for the Nyström discretizations are illustrated in Fig. 5. Here, we illustrated the case of 20-periodic driving using the parameters

$$\begin{aligned}
&\alpha_1 = 0.3, && c_t^1(x) \equiv 4, && b_t^1(x) \equiv 2 - \sin\tfrac{2\pi t}{20}, \\
&\alpha_2 = 0.7, && c_t^2(x) \equiv 2(1 + 0.1\sin\tfrac{2\pi t}{20}), && b_t^2(x) \equiv 0.1,
\end{aligned}$$

as well as an asymptotically autonomous situation with parameters

$$\begin{aligned}
&\alpha_1 = 0.3, && c_t^1(x) \equiv 4, && b_t^1(x) \equiv 2 - \arctan t, \\
&\alpha_2 = 0.7, && c_t^2(x) \equiv 2(1 + 0.1\arctan t), && b_t^2(x) \equiv 0.1.
\end{aligned}$$

In a periodic case, extremal solutions (and the pullback attractor) are periodic, too.

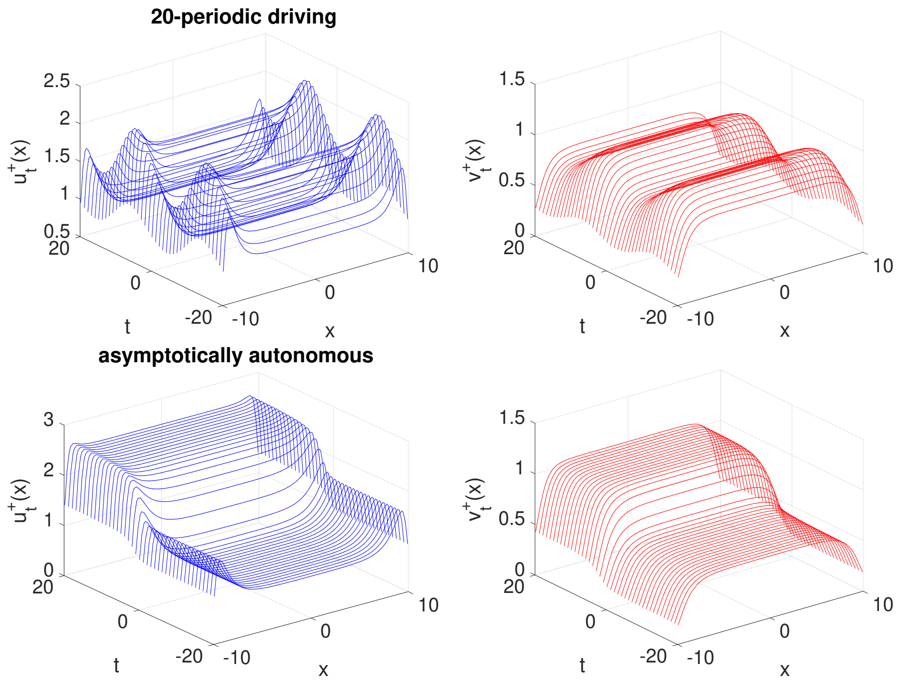

FIGURE 5. Extremal solutions $u_t^+, v_t^+ : [-10, 10] \to \mathbb{R}_+$ of the pullback attractor for Nyström discretizations (6th order Gauß with 99 nodes) of the nonautonomous Leslie-Gower model (5.2) for 20-periodic driving (top) and asymptotically autonomous driving (bottom)

Appendix A. **Cones, monotone, subhomogeneous and concave mappings.** Let $(X, \|\cdot\|)$ be a real Banach space with dual space X' and the duality pairing $\langle x, x'\rangle := x'(x)$. A nonempty, closed, convex subset $X_+ \subseteq X$ is denoted as a *cone*, if $\mathbb{R}_+ X_+ \subseteq X_+$ and $X_+ \cap (-X_+) = \{0\}$ hold. Equipped with such a cone, X is called an *ordered Banach space* X; throughout, it is assumed that $X_+ \neq \{0\}$. For elements $x, \bar{x} \in X$ we write

$$\begin{aligned} x \le \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+, \\ x < \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+ \setminus \{0\}, \\ x \ll \bar{x} \quad &:\Leftrightarrow \quad \bar{x} - x \in X_+^\circ; \end{aligned} \tag{A.1}$$

the latter relation requires X_+ to have interior $X_+^\circ \neq \emptyset$; in this case one speaks of a *solid cone*. For a *normal* cone X_+ there exists a $c \ge 0$ such that $x \le \bar{x}$ implies $\|x\| \le c\|\bar{x}\|$ for all $x, \bar{x} \in X_+$. Finally, *order intervals* are defined as $[x, \bar{x}] := \{y \in X : x \le y \le \bar{x}\}$.

By means of the *dual cone* $X'_+ := \{x' \in X' : 0 \le \langle x, x'\rangle \text{ for all } x \in X_+\}$ it is possible to characterize the elements of X_+ and X_+° as follows:

Lemma A.1 (cf. [24, Lemma A.1]). *(a) $X'_+ \neq \{0\}$ and for every $x \in X$ one has:*

$$x \in X_+ \Leftrightarrow 0 \le \langle x, x'\rangle \quad \text{for all } x' \in X'_+,$$
$$x \in X_+ \setminus \{0\} \Rightarrow 0 < \langle x, x'_0\rangle \quad \text{for some } x'_0 \in X'_+ \setminus \{0\}.$$

(b) If X_+ is solid, then for every $x \in X$ one has:

$$x \in X_+^\circ \Leftrightarrow 0 < \langle x, x'\rangle \quad \text{for all } x' \in X'_+ \setminus \{0\},$$

$$x \in \partial X_+ \Rightarrow 0 = \langle x, x_0' \rangle \quad \text{for some } x_0' \in X_+' \setminus \{0\}.$$

One denotes a subset $U \subseteq X$ as X_+-*convex*, if for all $x, \bar{x} \in U$ satisfying $x < \bar{x}$ the inclusion $\overline{x, \bar{x}} := \{x + \theta(\bar{x} - x) \in X : \theta \in [0,1]\} \subseteq U$ holds. For instance, the cone X_+ itself is X_+-convex, and so is every convex set.

A.1. **Monotone mappings.** Let $U \subseteq X$. A mapping $F : U \to X$ is called[1]

- *monotone*, if $x < \bar{x} \Rightarrow F(x) \leq F(\bar{x})$,
- *strictly monotone*, if $x < \bar{x} \Rightarrow F(x) < F(\bar{x})$,
- *strongly monotone*, if $x < \bar{x} \Rightarrow F(x) \ll F(\bar{x})$ for all $x, \bar{x} \in U$.

When working with several cones, we write X_+-monotone etc. to refer to a specific one and proceed similarly in our further terminology. In particular, linear maps $T \in L(X)$ are

- monotone (then called *positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+$,
- strictly monotone (then called *strictly positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$,
- strongly monotone (then called *strongly positive*), if $T(X_+ \setminus \{0\}) \subseteq X_+^\circ$.

We denote $T \in L(X)$ as X_+-*injective*, provided its kernel satisfies $N(T) \cap X_+ = \{0\}$. Then T is strictly monotone, if and only if it is monotone and X_+-injective. Furthermore, a strongly monotone T yields the inclusion $TX_+^\circ \subseteq X_+^\circ$.

Lemma A.2 (conditions for monotonicity)**.** *Suppose $F : U \to X$ is a C^1-mapping on a X_+-convex, open subset $U \subseteq X$. If $DF(x) \in L(X)$ is positive for all $x \in U$, then F is monotone. Moreover, the following holds with $x, \bar{x} \in U$:*

(a) If for every $x < \bar{x}$ and $x^* \in \overline{x, \bar{x}}$ the derivative $DF(x^*)$ is X_+-injective, then F is strictly monotone.

(b) If X_+ is solid and for every $x < \bar{x}$ there exists some $x^* \in \overline{x, \bar{x}}$ such that $DF(x^*)$ is strongly positive, then F is strongly monotone.

Proof. The monotonicity of F and (b) are shown in [13, Lemma 2.2], while for (a) we refer to [25, Lemma A.2(a)]. □

Note that respective monotonicity properties are preserved under composition. Beyond that, and for slightly weaker assumptions, the following holds:

Corollary A.3. *Let X_+ be a solid cone, $V \subseteq X$ and $F : U \to X$.*

(a) If F is strongly monotone and $T \in L(X)$ satisfies $TX_+^\circ \subseteq X_+^\circ$, then the composition $TF : U \to X$ is strongly monotone.

(b) If F is strictly monotone with $F(U) \subseteq V$ and $G : V \to X$ is strongly monotone, then $G \circ F : U \to X$ is strongly monotone.

Proof. Let $x, \bar{x} \in U$, $x < \bar{x}$.

(a) By the strong monotonicity of F it is $F(x) \ll F(\bar{x})$, i.e. $F(\bar{x}) - F(x) \in X_+^\circ$. Hence, $T(F(\bar{x}) - F(x)) \in X_+^\circ$ and the linearity of T implies that $TF(x) \ll TF(\bar{x})$.

(b) is immediate by definition. □

A.2. **Subhomogeneous mappings.** A self-mapping $F : X_+ \to X_+$ is called[2]

- *subhomogeneous*, if $0 < x \Rightarrow \theta F(x) \leq F(\theta x)$,
- *strictly subhomogeneous*, if $0 < x \Rightarrow \theta F(x) < F(\theta x)$,

---

[1] we implicitly assume here that U contains at least two $x, \bar{x} \in U$ such that that $x < \bar{x}$

[2] there is some inconsistency in the literature related to these notions: [5, pp. 112–113, Def. 4.1.1] and [13] speak of sublinearity instead of subhomogeneity and [5] requires subhomogeneous mappings to be monotone. Moreover, in [5] and [31, p. 52, Def. 2.3.1] strict subhomogeneity is required to hold only for $0 \ll x$, whereas [18, p. 142, Exam. 5.1.11] requires strong subhomogeneity to hold for $0 < x$.

- *strongly subhomogeneous*, if $0 \ll x \Rightarrow \theta F(x) \ll F(\theta x)$ for all $\theta \in (0,1)$.

Affine-linear mappings $F(x) = Tx + y$ with positive $T \in L(X)$ and $y \in X_+$ are always subhomogeneous, while strict subhomogeneity requires an inhomogeneity $y \in X_+ \setminus \{0\}$ and strong subhomogeneity holds for $y \in X_+^\circ$.

Lemma A.4 (conditions for subhomogeneity, cf. [25, Lemma A.4])**.** *If $F : X_+ \to X_+$ is differentiable, then the following holds:*

(a) F is subhomogeneous, if and only if $DF(x)x \le F(x)$ for all $x \in X_+ \setminus \{0\}$.
(b) If $DF : X_+ \to L(X)$ is continuous and $DF(x)x < F(x)$ for all $x \in X_+ \setminus \{0\}$, then F is strictly subhomogeneous.
(c) If X_+ is solid and $DF(x)x \ll F(x)$ for all $x \in X_+^\circ$, then F is strongly subhomogeneous.

The above differentiability of $F : X_+ \to X_+$ is to be understood so that the mapping F has a differentiable extension $\bar{F} : U \to X$ to an open superset $U \subseteq X$ of X_+.

Corollary A.5. *Let $T \in L(X)$ and $F : X_+ \to X_+$.*

(a) If T is positive and F is subhomogeneous, then $TF, F \circ T : X_+ \to X_+$ are subhomogeneous.
(b) If T is strictly positive and F is strictly subhomogeneous, then $TF, F \circ T$ are strictly subhomogeneous.
(c) If X_+ is solid, T is positive with $TX_+^\circ \subseteq X_+^\circ$ and F is strongly subhomogeneous, then TF is strongly subhomogeneous.
(d) If X_+ is solid, T is strongly positive and F is strictly subhomogeneous, then TF is strongly subhomogeneous.

Proof. (a) and (b) are immediate from the definition.

(c) Note that $TF : X_+ \to X_+$. Let $0 \ll x$ and $\theta \in (0,1)$. Provided F is strongly subhomogeneous, then the inequality $0 \ll F(\theta x) - \theta F(x)$ and the forward T-invariance of X_+° readily show that $\theta TF(x) = T(\theta F(x)) \ll TF(\theta x)$.

(d) Let $0 \ll x$ and $\theta \in (0,1)$. If F is strictly subhomogeneous, then $\theta F(x) < F(\theta x)$ and the strong monotonicity of T yields $\theta TF(x) = T(\theta F(x)) \ll TF(\theta x)$. □

A.3. **Concave mappings.** Let $U \subseteq X$ be a X_+-convex set. A map $F : U \to X$ is called[3]

- *order-* or *X_+-concave*, if $x < \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) \le F(\theta x + (1-\theta)\bar{x})$,
- *strictly X_+-concave*, if $x < \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) < F(\theta x + (1-\theta)\bar{x})$,
- *strongly X_+-concave*, if $x \ll \bar{x} \Rightarrow \theta F(x) + (1-\theta)F(\bar{x}) \ll F(\theta x + (1-\theta)\bar{x})$ for all $x, \bar{x} \in U$ and $\theta \in (0,1)$.

Note that (strict or strong) subhomogeneity holds for (strictly resp. strongly) X_+-concave mappings $F : X_+ \to X_+$, with the reference point $x = 0$. In this sense, the concavity concepts for $F : X_+ \to X_+$ are less general than the respective subhomogeneity notions.

Affine-linear mappings $F(x) = Tx + y$ with $T \in L(X)$ and $y \in X$ are X_+-concave, but never strictly or strongly X_+-concave.

Lemma A.6 (conditions for concavity, cf. [25, Lemma A.6])**.** *If $F : U \to X$ is a C^1-mapping on a X_+-convex, open set $U \subseteq X$, then the following holds:*

(a) F is X_+-concave, if and only if $DF(\bar{x})(\bar{x} - x) \le DF(x)(\bar{x} - x)$ for all $x, \bar{x} \in U$, $x < \bar{x}$.

[3] as in the concept of subhomogeneity, also the notions of strict and strong concavity are not consistently used throughout the literature: [1] demands strong concavity to hold for all $x < \bar{x}$, while [5, pp. 114–115, Def. 4.1.2] assumes $x \ll \bar{x}$ to define strict concavity.

(b) *If $DF(\bar{x})(\bar{x}-x) < DF(x)(\bar{x}-x)$ for all $x, \bar{x} \in U$, $x < \bar{x}$, then F is strictly X_+-concave.*

(c) *If X_+ is solid and $DF(\bar{x})(\bar{x}-x) \ll DF(x)(\bar{x}-x)$ for all $x, \bar{x} \in U$, $x \ll \bar{x}$, then F is strongly X_+-concave.*

Corollary A.7. *Let $U \subseteq X$ be X_+-convex.*

(a) *If $T \in L(X)$ is positive and $F : U \to X$ is X_+-concave, then $TF : U \to X$ is X_+-concave. In case additionally $TU \subseteq U$ also $F \circ T : U \to X$ is X_+-concave.*

(b) *If T is strictly positive and F is strictly X_+-concave, then TF is strictly X_+-concave. In case additionally $TU \subseteq U$ also $F \circ T : U \to X$ is strictly X_+-concave.*

(c) *If X_+ is solid, $TX_+^\circ \subseteq X_+^\circ$ and F is strongly X_+-concave, or T is strongly positive and F is strictly X_+-concave, then TF is strongly X_+-concave.*

Proof. The argument is analogous to the proof of Cor. A.5. □

A dual theory holds for *order-convex* mappings with F replaced by the negative $-F$.

A.4. **Cone mappings.** One speaks of a *cone mapping* $F : X_+ \to X_+$, provided the inclusion $F\left([\theta x, \frac{1}{\theta}x]\right) \subseteq \left[\theta F(x), \frac{1}{\theta}F(x)\right]$ for all $\theta \in (0,1)$, $x \in X_+$.

Remark A.8. A monotone $F : X_+ \to X_+$ is a cone mapping, if and only if it is subhomogeneous (cf. [17, Prop. 3.1(iii)]).

Proposition A.9. *Let $F : X_+ \to X$ be X_+-concave and monotone. If $0 \leq F(0)$, then F is subhomogeneous and a cone mapping.*

Proof. Let $\bar{x} \in X_+$. The monotonicity yields $0 \leq F(0) \leq F(\bar{x})$, i.e. $F(X_+) \subseteq X_+$. Furthermore, the concavity of F and $0 \leq F(0)$ imply for all $\theta \in (0,1)$ that

$$(1-\theta)F(\bar{x}) \leq \theta F(0) + (1-\theta)F(\bar{x}) \leq F(\theta 0 + (1-\theta)\bar{x}) = F((1-\theta)\bar{x}).$$

In conclusion, F is subhomogeneous and due to Rem. A.8 also a cone mapping. □

Finally, for monotone and subhomogeneous self-mappings of a solid cone it is possible to conclude the global asymptotic stability of a fixed point from its local asymptotic stability (provided e.g. by linearization).

Theorem A.10 (local-global stability principle, cf. [17, Thm. 4.1])**.** *Let X_+ be a solid and normal cone. If an iterate $F^{\circ k}$ of $F : X_+ \to X_+$ is a cone mapping for some $k \in \mathbb{N}$ leaving X_+° invariant, then any locally attractive fixed point of F in X_+° is globally attractive.*

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