CORRIGENDUM ON: A NOTE ON THE DICHOTOMY SPECTRUM

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My paper "A note on the dichotomy spectrum" (J. Difference Equ. Appl. **15**, no. 10, 1021–1025 (2009)) contains a serious error. In fact, [Pöt09, Lemma 7] is wrong with the consequence that also the final [Pöt09, Thm. 8] on the ℓ_0-roughness of exponential dichotomies (EDs for short) for

$$(L) \qquad x_{k+1} = A_k x_k \quad \text{for all } k \in \mathbb{Z}$$

does not hold.

Throughout this corrigendum, suppose $A_k, B_k \in L(X)$, $k \in \mathbb{Z}$, are bounded sequences of bounded linear operators on a Banach space X and we borrow our further notation and terminology from [Pöt09]. The faulty [Pöt09, Thm. 8] states that the dichotomy spectra $\Sigma(A)$ for the linear difference equation (L) and $\Sigma(A+B)$ for the linear-homogeneously perturbed difference equation

$$(P) \qquad x_{k+1} = [A_k + B_k]x_k \quad \text{for all } k \in \mathbb{Z}$$

are the same, provided the respective transition operators Φ of both (L) and (P) satisfy the injectivity assumption $N(\Phi(k, k-n)) = \{0\}$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$, and

$$\lim_{k \to \pm\infty} \|B_k\| = 0.$$

The dichotomy spectrum yielding the appropriate hyperbolicity concept for nonautonomous problems dates back to the pioneering work of Sacker & Sell [SS78]. Using a flexible perturbation result for linear skew-product flows, [SS78, Sects. 5–6] shows that $\Sigma(A)$ depends upper-semicontinuously on perturbations of the right-hand side in (L). Furthermore, in a finite-dimensional situation, the claimed invariance of $\Sigma(A)$ under perturbations $B_k \in \mathbb{R}^{d \times d}$ decaying to 0 is known for difference equations defined on semi-lines $\mathbb{Z}_\kappa^+ := \{\kappa, \kappa+1, \dots\}$ or $\mathbb{Z}_\kappa^- := \{\dots, \kappa-1, \kappa\}$ (we refer to [BG93, Thm. 2.3] for invertible coefficient matrices $A_k \in \mathbb{R}^{d \times d}$). In this sense, the dichotomy spectrum on semi-lines is essential spectrum.

When dealing with problems (L) on the full line $\mathbb{Z}$, nevertheless, this statement is not necessarily true. Indeed, the author realized the faultiness of [Pöt09, Thm. 8] while becoming aware of [Hen81, p. 235, Thm. 7.6.9]; the latter result precisely indicates that when passing over from (L) to the perturbed equation (P), point spectrum might occur. To explicitly falsify [Pöt09, Thm. 8] we need the subsequent characterization for EDs on $\mathbb{Z}$:

Lemma 1. *Let $\kappa \in \mathbb{Z}$. Equation (L) has an ED on $\mathbb{Z}$ if and only if it admits EDs on both semi-lines $\mathbb{Z}_\kappa^+$ and $\mathbb{Z}_\kappa^-$ with corresponding projectors P_k^+, P_k^- satisfying*

$$(1) \qquad R(P_\kappa^+) \oplus N(P_\kappa^-) = X.$$

Proof. Referring to [Bas00, Cor. 2 1)] and [Hen81, p. 230, Thm. 7.6.5], EDs on both semi-lines $\mathbb{Z}_\kappa^+$ and $\mathbb{Z}_\kappa^-$ extend to the whole line $\mathbb{Z}$ under (1). Conversely, for an ED on $\mathbb{Z}$ the projector P_k is uniquely determined and clearly fulfills (1). □

From this we obtain the following counterexample to [Pöt09, Thm. 8]:

Example 1. In $\mathbb{R}^2$ we consider a piecewise constant difference equation (L) with

$$A_k := \begin{pmatrix} a_k & 0 \\ 0 & a_k^{-1} \end{pmatrix}, \qquad a_k := \begin{cases} 2, & k \geq 0, \\ \frac{1}{2}, & k < 0. \end{cases}$$

Given $\gamma > 0$, its scaled counterpart

$$(L_\gamma) \qquad x_{k+1} = \gamma^{-1} A_k x_k \quad \text{for all } k \in \mathbb{Z}$$

has the following dichotomy properties: If $\gamma \notin \left\{\frac{1}{2}, 2\right\}$ then we have EDs on both semi-lines $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ with constant projectors P_+ resp. P_-. They are given by

- $\gamma < \frac{1}{2}$: $P_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_- = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$
- $\frac{1}{2} < \gamma < 2$: $P_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $P_- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$
- $2 < \gamma$: $P_+ = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $P_- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$

which yields the relations

$$R(P_+) \cap N(P_-) = \begin{cases} \{0\}, & \gamma < \frac{1}{2} \text{ or } 2 < \gamma, \\ \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \frac{1}{2} < \gamma < 2, \end{cases}$$

$$R(P_+) + N(P_-) = \begin{cases} \mathbb{R}^2, & \gamma < \frac{1}{2} \text{ or } 2 < \gamma, \\ \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \frac{1}{2} < \gamma < 2. \end{cases}$$

With Lemma 1 this shows that (L_γ) admits an ED on $\mathbb{Z}$, if and only if $\gamma \notin [\frac{1}{2}, 2]$, i.e. (L) has the dichotomy spectrum $\Sigma(A) = [\frac{1}{2}, 2]$. We perturb (L) with the matrix

$$B_k := \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix}, \qquad b_k := \begin{cases} \left(\frac{1}{2}\right)^k, & k \geq 0, \\ 0, & k < 0 \end{cases}$$

satisfying $\lim_{k\to\pm\infty} \|B_k\| = 0$ even exponentially. Due to [BG93, Thm. 2.3] the scaled perturbed difference equation

$$(P_\gamma) \qquad x_{k+1} = \gamma^{-1}[A_k + B_k] x_k \quad \text{for all } k \in \mathbb{Z}$$

admits an ED on the semi-line $\mathbb{Z}_0^+$ if and only if (L_γ) has the same property. Using the general forward solution

$$\varphi_\gamma(k; 0, \xi, \eta) = \begin{pmatrix} \left(\frac{2}{\gamma}\right)^k \left(\xi + \frac{4}{7}\eta\right) - \frac{4}{7}\eta \left(\frac{1}{4\gamma}\right)^k \\ \left(\frac{1}{2\gamma}\right)^k \eta \end{pmatrix} \quad \text{for all } k \in \mathbb{Z}_0^+,\ \xi, \eta \in \mathbb{R}$$

of equation (P_γ), the corresponding projector $\bar{P}_k^+$ for the ED of (P_γ) on $\mathbb{Z}_0^+$ satisfies the relation (cf. [Pal88, Prop. 2.3(i)])

$$R(\bar{P}_0^+) = \left\{(\xi, \eta) \in \mathbb{R}^2 : \sup_{k \geq 0} \|\varphi_\gamma(k; 0, \xi, \eta)\| < \infty\right\} = \begin{cases} \{0\}, & \gamma < \frac{1}{2}, \\ \mathbb{R}\left(\begin{smallmatrix} 4 \\ -7 \end{smallmatrix}\right), & \frac{1}{2} < \gamma < 2, \\ \mathbb{R}^2, & 2 < \gamma. \end{cases}$$

Both difference equations (L_γ) and (P_γ) coincide on $\mathbb{Z}_{-1}^-$ and therefore the perturbed projector $\bar{P}_k^-$ for the ED of (P_γ) on $\mathbb{Z}_{-1}^-$ satisfies $N(\bar{P}_{-1}^-) = N(P_{-1}^-)$. This

ED extends to the semi-line $\mathbb{Z}_0^-$ and the invariance $N(\bar{P}_0^-) = A_{-1}N(P_-)$ implies $R(\bar{P}_0^+) \oplus N(\bar{P}_0^-) = \mathbb{R}^2$ for all $\gamma \notin \{\frac{1}{2}, 2\}$. Using Lemma 1 we arrive at

$$\Sigma(A + B) = \{\tfrac{1}{2}, 2\} \neq \Sigma(A).$$

We point out that our ℓ_0-robustness result [Pöt09, Thm. 8] fails due to the preparatory but erroneous [Pöt09, Lemma 7]. Its proof relies on the abstract [Kat80, p. 243, Thm. 5.33], where the essential spectrum is assumed to be at most countable — this is typically not satisfied for the crucial weighted shift $T_A \in L(\ell^\infty)$, $(T_A\phi)_k := A_{k-1}\phi_{k-1}$. Yet, a correct version of [Pöt09, Lemma 7] reads as

Lemma 2. *If every $A_k \in L(X)$, $k \in \mathbb{Z}$, is invertible with $\sup_{k\in\mathbb{Z}} \left\|A_k^{-1}\right\| < \infty$, then the essential spectrum $\sigma_{\text{ess}}(T_A)$ of T_A satisfies $\partial\sigma(T_A) \subseteq \sigma_{\text{ess}}(T_A) \subseteq \sigma(T_A)$.*

Proof. Since $\sigma(T_A)$ is rotationally symmetric w.r.t. $0 \in \mathbb{C}$, its only possible isolated spectral point is 0. Yet, our assumptions guarantee that $T_A \in L(\ell^\infty)$ is invertible with bounded inverse $(T_A^{-1}\psi)_k = A_k^{-1}\psi_{k+1}$ and in particular $0 \notin \sigma(T_A)$. This yields $\operatorname{iso}\sigma(T_A) = \emptyset$ and using [Har88, p. 371, Thm. 9.8.4] it follows $\partial\sigma(T_A)\setminus\sigma_{\text{ess}}(T_A) = \emptyset$, i.e. one has $\partial\sigma(T_A) \subseteq \sigma_{\text{ess}}(T_A)$. The inclusion $\sigma_{\text{ess}}(T_A) \subseteq \sigma(T_A)$ holds trivially. □

Lemma 3. *If $B_k \in L(X)$, $k \in \mathbb{Z}$, is a sequence of compact operators satisfying $\lim_{k\to\pm\infty} \|B_k\| = 0$, then also $T_B \in L(\ell^\infty)$ is compact with $\sigma(T_B) = \{0\}$.*

Proof. For every $n \in \mathbb{N}$ we define compact operators $T_B^n \in L(\ell^\infty)$,

$$(T_B^n\phi)_k := \begin{cases} B_{k-1}\phi_{k-1}, & |k-1| \leq n, \\ 0, & |k-1| > n. \end{cases}$$

Thus, thanks to $\|T_B - T_B^n\|_{L(\ell^\infty)} \leq \sup_{|k|>n} \|B_k\| \xrightarrow[n\to\infty]{} 0$ also the uniform limit T_B is compact (cf. [Yos80, p. 278, Thm. (iii)]). Since the spectrum of compact operators consists of isolated points with zero as the only possible accumulation point (see [Yos80, p. 284, Thm. 2]), the rotational invariance of $\sigma(T_B)$ implies $\sigma(T_B) = \{0\}$. □

Using Lemma 2 and 3 we can establish an accurate counterpart to [Pöt09, Thm. 8] under essentially two additional assumptions: First, (L) is supposed to have discrete dichotomy spectrum, which e.g. occurs for autonomous or periodic equations. Second, the coefficient operator of (L) and the perturbation sequence B_k need to commute. Precisely, we have

Theorem 1. *(ℓ_0-roughness) Under the assumptions*

(i) *every $A_k \in L(X)$, $k \in \mathbb{Z}$, is invertible with $\sup_{k\in\mathbb{Z}} \left\|A_k^{-1}\right\| < \infty$,*
(ii) *every $B_k \in L(X)$, $k \in \mathbb{Z}$, is compact with $\lim_{k\to\pm\infty} \|B_k\| = 0$*

and $\partial\Sigma(A) = \Sigma(A)$ the following holds:

(a) *$\Sigma(A) \subseteq \Sigma(A + B)$,*
(b) *if $B_{k+1}A_k = A_{k+1}B_k$, $k \in \mathbb{Z}$, then $\Sigma(A) = \Sigma(A + B)$.*

Proof. (a) Referring to [Pöt09, Thm. 1] we have $\sigma(T_A) = \partial\sigma(T_A)$. Consequently, the above Lemma 2 guarantees

$$\sigma(T_A) = \sigma_{\text{ess}}(T_A) = \sigma_{\text{ess}}(T_A + T_B) \subseteq \sigma(T_A + T_B) = \sigma(T_{A+B}),$$

since T_B is compact due to Lemma 3 and compact perturbations leave the essential spectrum invariant (cf. [Kat80, p. 244, Thm. 5.35]). Then again our [Pöt09, Thm. 1] implies the claimed inclusion.

(b) Our assumption ensures that T_A and T_B commute. Hence, we obtain the inclusion $\sigma(T_{A+B}) = \sigma(T_A + T_B) \subseteq \sigma(T_A) + \sigma(T_B)$ (cf. [ARS94, Thm. 7.2]) and by means of Lemma 3 this in turn yields $\sigma(T_{A+B}) \subseteq \sigma(T_A)$. With [Pöt09, Thm. 1] we conclude $\Sigma(A + B) \subseteq \Sigma(A)$ and a combination with assertion (a) implies our claim. □

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