NONAUTONOMOUS CONTINUATION OF BOUNDED SOLUTIONS

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ABSTRACT. We show the persistence of hyperbolic bounded solutions to nonautonomous difference and retarded functional differential equations under parameter perturbation, where hyperbolicity is given in terms of an exponential dichotomy in variation. Our functional-analytical approach is based on a formulation of dynamical systems as operator equations in ambient sequence or function spaces. This yields short proofs, in particular of the stable manifold theorem.

As an ad hoc application, the behavior of hyperbolic solutions and stable manifolds for ODEs under numerical discretization with varying step-sizes is studied.

1. MOTIVATION, INTRODUCTION, AND PRELIMINARIES

The classical principle of Poincaré continuation states that hyperbolic periodic orbits of autonomous ordinary differential equations (ODEs) are robust under parameter variation. More precisely, suppose for a fixed parameter value λ^* an ODE

$$\dot{u} = f(u, \lambda) \tag{1.1}$$

admits a T^*-periodic orbit Γ^* with all Floquet multipliers (except one) off the unit circle. Then also for parameter values λ from a neighborhood of λ^* there exists a $T(\lambda)$-periodic orbit $\Gamma(\lambda)$ of (1.1), whose period $T(\lambda)$ depends smoothly on λ and $\Gamma(\lambda)$ is a continuation in the sense of $\lim_{\lambda\to\lambda^*}\Gamma(\lambda) = \Gamma^*$ w.r.t. the Hausdorff distance. Modern proofs of this result are based on the implicit function theorem (cf., e.g., [Ama90, pp. 352ff] or [Chi06, p. 382, Theorem 5.7]) and extensions into various directions can be found, for instance, in [Kie04, pp. 84ff] (autonomous evolutionary differential equations), [HW04] (delay differential equations) or [Per97] (for infinite systems of second order ODEs).

In general, continuation problems deal with the question of finding conditions, yielding that a solution or a more general invariant set of an evolutionary equation persists under varying system parameters, without a change of stability properties. This is strongly related to the concept of structural stability implying that hyperbolic equilibria, orbits or further objects are robust under perturbations (cf. [HS74, pp. 304ff], [SY02, pp. 481ff]). In particular, the possibility of continuation excludes bifurcation or branching phenomena. Consequently, continuation techniques are frequently used in numerical analysis to approximate various robust invariant objects and we refer to, e.g., [AG90, KOGV07] for a survey.

Whereas the above references deal with autonomous (or periodic) problems, the recent years showed an increasing interest in nonautonomous evolutionary equations. They are capable to describe models under the influence of temporally aperiodic external factors, as regulation or control effects. For example, in concrete models this is realized in a way that

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constant parameters are replaced by time-dependent functions (*parametric perturbations*). As opposed to the classical situation, for such equations with a general time dependence, it is not generic to possess constant or periodic solutions. Indeed, in various contexts one made the observation (cf., for instance, [Hül08, BS08, CLRS06] or Theorems 2.11, 3.8) that

Equilibria of autonomous equations generically persist as complete bounded solutions under small parametric perturbations.

For this reason, we suggest to investigate bounded globally defined solutions as appropriate replacement for equilibria in general nonautonomous continuation and bifurcation problems. Following this leitmotiv, the idea behind our overall strategy is to rephrase evolutionary equations as operator equations in suitable sequence or function spaces (cf. the respective Subsection 2.1 or 3.1). Clearly, in such a functional-analytical approach, ambient spaces are indispensable in order to apply tools like the (surjective) implicit function theorem. While we focus on a general time-dependence, for the particular cases of asymptotically constant, almost periodic or periodic equations, the suitable space for persisting solutions might be the set of heteroclinic or homoclinic, almost periodic or periodic functions, respectively.

The present paper deals with nonautonomous difference and retarded functional differential equations (including ODEs) in a parallel manner consecutively in the two respective Sections 2 and 3. Understanding such problems as operator equations in the space of bounded or zero sequences (or functions) requires to deduce certain differentiability properties of substitution operators. Their derivatives are weighted difference (respective differential) operators, whose invertibility is guaranteed by exponential dichotomy assumptions for the variational equation along fixed reference solutions. Having this at hand, using the implicit function theorem we can show that hyperbolic bounded solutions persist under perturbations, providing a nonautonomous version of the classical Poincaré continuation. In detail, such solutions are robust for two-sided time, while whole manifolds of bounded solutions persist for one-sided time, yielding a nonautonomous local stable manifold theorem. Compared to well-established proofs of stable manifold results using the implicit function theorem, we tackle difference resp. differential equations directly without a detour over e.g., Lyapunov-Perron operators. We may mention that, while preparing this paper, we learned that the basics of our Theorem 2.11 treating two-sided time and difference equations are contained in [Hül08, Lemma 2]. However, our setting is a bit wider, and beyond parametric perturbations, it allows an elegant application to analytical discretization theory: In the concluding Section 4, we discuss consistent variable time-step discretizations of ODEs and establish persistence and convergence properties of hyperbolic solutions and stable sets. The proof is remarkably short and an immediate consequence of our main continuation results in Theorem 2.11 and Corollary 2.16. For the reader's convenience, an appendix contains quantitative and surjective versions of the implicit function theorem.

For related work we refer to the references [Hag04, BM03, JSW03]. In [Hag04] a shadowing-type question is addressed, wether hyperbolic trajectories of discretizations guarantee the existence of a hyperbolic solution for the continuous flow, while we deal with the inverse situation. The robustness of solutions to ODEs w.r.t. time-varying perturbations is studied in [BM03], with the intention to obtain optimal bounds. For nonautonomous ODEs $\dot{u} = f(t, u)$ it is investigated in [JSW03] that solutions $u(t)$ of the algebraic equation $f(t, u) = 0$, $t \in \mathbb{R}$ understood as parameter, are perturbations of hyperbolic complete solutions.

Finally, the persistence of equilibria as bounded complete solutions motivates the possibility to study them as bifurcating objects. In such a spirit, we remark that this paper is intended to be the first one in a series of articles dealing with nonautonomous continuation and bifurcation theory using functional-analytical tools.

Notation: Throughout the paper, Banach spaces are denoted by X, Y and equipped with norm $|\cdot|$. We write Ω° for the interior of a set $\Omega \subseteq X$ and $B_\varepsilon(x)$ for the open ε-ball centered in $x \in X$. The space of bounded linear operators between X and Y is $L(X, Y)$, $L(X) := L(X, X)$ and for the corresponding toplinear isomorphisms we write $GL(X, Y)$; moreover, $L_j(X, Y)$ consists of j-linear bounded operators and, $L_0(X, Y) := Y$. Given $T \in L(X, Y)$, we write $R(T) := TX$ for the *range* and $N(T) := T^{-1}(0)$ for the *kernel*.

Furthermore, in this paper $\Lambda \subseteq Y$ denotes a nonempty open convex subset.

2. Difference equations

As usual, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ are the positive integers and a *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$; we introduce the shifted interval $\mathbb{I}' := \{k \in \mathbb{I} : k + 1 \in \mathbb{I}\}$. Given an integer $\kappa \in \mathbb{Z}$ we define the discrete intervals $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \le k\}$ and $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : \kappa \ge k\}$.

Suppose throughout that $\Omega \subseteq X$ is nonempty open and convex. In case $\mathbb{I}$ is unbounded above, we denote the set of bounded sequences $\phi = (\phi_k)_{k\in\mathbb{I}}$ with $\phi_k \in \Omega$ by $\ell^\infty(\Omega)$ and in case $0 \in \Omega$ we write $\ell_0(\Omega)$ for the space of all such sequences converging to 0; note that the two-sided limit $k \to \pm\infty$ is meant for $\mathbb{I} = \mathbb{Z}$. It is convenient to abbreviate $\ell^\infty := \ell^\infty(X)$, $\ell_0 := \ell_0(X)$. Both are Banach spaces equipped with norm

$$\|\phi\| := \sup_{k\in\mathbb{I}} |\phi_k| \, .$$

Convexity of the set $\Omega \subseteq X$ carries over to the sequence spaces $\ell^\infty(\Omega), \ell_0(\Omega)$. Nonetheless, it is easy to see that $\ell_0(\Omega)$ is open, whereas $\ell^\infty(\Omega)$ is not open in general.

As center of our interest, we consider functions $f_k : \Omega \times \Lambda \to X$, $k \in \mathbb{Z}$, which are the right-hand sides of nonautonomous parameter-dependent difference equations

$$(\Delta)_\lambda \qquad \boxed{x_{k+1} = f_k(x_k, \lambda).}$$

For a fixed parameter value $\lambda \in \Lambda$, a *solution* of the difference equation $(\Delta)_\lambda$ is a sequence $\phi = (\phi_k)_{k\in\mathbb{I}}$ with $\phi_k \in \Omega$ satisfying the recursion $(\Delta)_\lambda$ on a discrete interval $\mathbb{I}'$. In order to emphasize the dependence on λ, we may write $\phi(\lambda)$. Under the condition

$$\inf_{k\in\mathbb{I}} \operatorname{dist}(\phi_k, \Omega) > 0$$

we speak of a *permanent solution*. In case $0 \in \Omega$, solutions in $\ell_0(\Omega)$ are permanent. Moreover, a *complete* or *globally defined solution* solves $(\Delta)_\lambda$ on the whole axis $\mathbb{Z}$. *Homoclinic solutions* are complete solutions in ℓ_0. Note that complete solutions might not exist, since we impose no invertibility assumptions on $f_k(\cdot, \lambda)$. The *general solution* $\varphi(\cdot; \kappa, \xi, \lambda)$ is the unique forward solution of $(\Delta)_\lambda$ satisfying the initial condition $x_\kappa = \xi$. Note that $\varphi(\cdot; \kappa, \xi, \lambda)$ needs not to exist on the whole semiaxis $\mathbb{Z}_\kappa^+$, since $f_k(\cdot, \lambda)$ is not supposed to leave Ω invariant.

2.1. Substitution operators.

For the purpose of detecting backward or complete solutions, it is reasonable to convert a nonautonomous difference equation $(\Delta)_\lambda$ into an operator equation in a sequence space. Here, it is convenient to make use of the following convention: The symbol ℓ stands for exactly one of the symbols ℓ^∞ or ℓ_0. In the following, this allows a compact notation of results valid for both of the spaces ℓ^∞ or ℓ_0.

Proposition 2.1. *The* left shift operator $S : \ell \to \ell$, $(S\phi)_k := \phi_{k+1}$ *is linear with* $|S| \leq 1$, *satisfies* $S\ell(\Omega) \subseteq \ell(\Omega)$ *and for* $\mathbb{I} = \mathbb{Z}$ *it is an onto linear isometry.*

Proof. Let $\phi \in \ell$ and from $|(S\phi)_k| = |\phi_{k+1}| \leq \|\phi\|$ for all $k \in \mathbb{I}$ we obtain $\|S\| \leq 1$ with equality in case $\mathbb{I} = \mathbb{Z}$. The remaining assertions are clear. □

The following assumptions hold for C^m-smooth right-hand sides of equation $(\Delta)_\lambda$, whose derivatives map bounded into bounded sets uniformly in time. Throughout, differentiability is always meant in the Fréchet-sense and D denotes the derivative operator.

Hypothesis. *Let* $m \in \mathbb{N}$ *and suppose each* $f_k : \Omega \times \Lambda \to X$, $k \in \mathbb{Z}$, *is a* C^m*-function such that the following holds for* $0 \leq j \leq m$*:*

(H_0) *For all bounded* $B \subseteq \Omega$ *one has*

$$\sup_{k \in \mathbb{Z}} \sup_{x \in B} \left| D^j f_k(x, \lambda) \right| < \infty \quad \text{for all } \lambda \in \Lambda$$

(well-definedness) *and for all* $\lambda_0 \in \Lambda$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with*

$$|x - y| < \delta \quad \Rightarrow \quad \sup_{k \in \mathbb{Z}} \left| D^j f_k(x, \lambda) - D^j f_k(y, \lambda_0) \right| < \varepsilon \tag{2.1}$$

for all $x, y \in \Omega$ *and* $\lambda \in B_\delta(\lambda_0)$ (uniform continuity).

(H_1) *We have* $0 \in \Omega$ *and the limit relation* $\lim_{k \to \pm\infty} f_k(0, \lambda) = 0$ *for all* $\lambda \in \Lambda$.

(H_2) *There exist functions* $\omega_0 : \mathbb{R} \to [0, \infty)$, $\omega_1 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, *nondecreasing in each argument, such that for all* $k \in \mathbb{Z}$, $x, \bar{x} \in \Omega$ *and* $\lambda, \bar{\lambda} \in \Lambda$ *one has*

$$|f_k(x, \lambda) - f_k(\bar{x}, \lambda)| \leq \omega_0(|x - \bar{x}|),$$
$$\left| D_1 f_k(x, \lambda) - D_1 f_k(\bar{x}, \bar{\lambda}) \right| \leq \omega_1(|x - \bar{x}|, \left|\lambda - \bar{\lambda}\right|).$$

An application with infinite dimensional parameter spaces Λ is given in Section 4 and

Example 2.1 (parametric perturbation). Autonomous difference equations under parametric perturbations nicely fit in the above framework. Indeed, consider

$$x_{k+1} = g(x_k, p)$$

with right-hand side $g : \Omega \times P \to X$, where $P \subseteq Y$ is a nonempty open and convex set. Now the parameter $p \in P$ is replaced by a sequence $(p_k)_{k \in \mathbb{Z}} \in \ell^\infty(P)^\circ$ and we arrive at a nonautonomous equation $x_{k+1} = g(x_k, p_k)$. For the parameter space $\Lambda = \ell^\infty(P)^\circ$ the mapping $f_k(x, \lambda) := g(x, p_k)$ with $\lambda = (p_k)_{k \in \mathbb{Z}}$ satisfies (H_0), provided one has $g \in C^m(\Omega \times P, X)$, the derivatives are uniformly continuous in $p \in P$ and map bounded sets into bounded sets.

Under the above assumptions we formally introduce various *substitution operators* derived from the functions f_k. They are pointwise defined as

$$F(\phi, \lambda)_k := f_k(\phi_k, \lambda), \qquad F^j(\phi, \lambda)_k := D^j f_k(\phi_k, \lambda),$$
$$F^\upsilon(\phi, \lambda)_k := D_1^{\upsilon_1} D_2^{\upsilon_2} f_k(\phi_k, \lambda)$$

for all $k \in \mathbb{I}$. Here, $0 \leq j \leq m$, $\upsilon = (\upsilon_1, \upsilon_2) \in \mathbb{N}_0^2$ is a pair with $\upsilon_1 + \upsilon_2 \leq m$ and D_i denotes the partial Fréchet-derivative w.r.t. the ith argument, $i = 1, 2$.

Lemma 2.2. *Under* (H_0) *the operators* $F^j : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty(L_j(X \times Y, X))$ *and* $F^\upsilon : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty\big(L_{\upsilon_1}(X, L_{\upsilon_2}(Y, X))\big)$ *are well-defined and continuous.*

Proof. Let $\lambda \in \Lambda$, $\phi \in \ell^\infty(\Omega)$ and $0 \le j \le m$ be given. Due to (H_0) the derivatives $D^j f_k(\cdot, \lambda)$ map bounded sets into bounded sets uniformly in $k \in \mathbb{Z}$. Consequently, also the sequence $(D^j f_k(\phi_k, \lambda))_{k\in\mathbb{I}}$ is bounded and the mapping F^j has values in $\ell^\infty(L_j(X \times Y, X))$. In order to establish its continuity, we arbitrarily choose $\lambda_0 \in \Lambda$ and $\phi^* \in \ell^\infty(\Omega)$. For every $\varepsilon > 0$ we know from (H_0) that there exists a $\delta > 0$ such that (2.1) holds. In particular, for sequences $\phi \in B_\delta(\phi^*) \cap \ell^\infty(\Omega)$ and $\lambda \in B_\delta(\lambda_0)$ one has

$$|\phi_k - \phi_k^*| \le \|\phi - \phi^*\| < \delta \quad \text{for all } k \in \mathbb{I}$$

and therefore $\left|D^j f_k(\phi_k, \lambda) - D^j f_k(\phi_k^*, \lambda_0)\right| < \varepsilon$ for all $k \in \mathbb{I}$. Passing to the least upper bound over k in this inequality we arrive at $\left\|F^j(\phi, \lambda) - F^j(\phi^*, \lambda_0)\right\| \le \varepsilon$ and this proves the continuity of F^j.

These properties carry over from the mapping F^j given by the Frechét derivatives of f_k to the mapping F^υ defined via partial derivatives. □

Proposition 2.3. *Under (H_0) the operator $F : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ is well-defined and m-times continuously differentiable on $\ell^\infty(\Omega)^\circ \times \Lambda$ with partial derivatives*

$$D^\upsilon F(\phi, \lambda) = F^\upsilon(\phi, \lambda) \quad \textit{for all } \phi \in \ell^\infty(\Omega)^\circ,\ \lambda \in \Lambda,\ \upsilon_1 + \upsilon_2 \le m.$$

If (H_0)–(H_1) are satisfied, then the same holds for $F : \ell_0(\Omega) \times \Lambda \to \ell_0$.

Proof. It follows immediately from Lemma 2.2 with $j = 0$ that $F : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ is well-defined. Concerning the smoothness assertion, we only establish $D^j F = F^j$. Thereto, let $\phi^* \in \ell^\infty(\Omega)^\circ$, $\lambda_0 \in \Lambda$ and $\phi \in \ell^\infty$, $\lambda \in Y$ sufficiently small that $\lambda_0 + \lambda \in \Lambda$, $\phi_k^* + \phi_k \in \Omega$ for all $k \in \mathbb{I}$. For every $0 \le j < m$ we define the real-valued remainder

$$r_j(\phi, \lambda) := \sup_{s\in[0,1]} \left\|F^{j+1}(\phi^* + h\phi, \lambda_0 + h\lambda) - F^{j+1}(\phi^*, \lambda_0)\right\|$$

and the continuity for F^{j+1} from Lemma 2.2 guarantees $\lim_{(\phi,\lambda)\to(0,0)} r_j(\phi, \lambda) = 0$. After these preparations, the mean value theorem (cf. [Lan93, p. 341, Theorem 4.2]) implies

$$\begin{aligned}
&\left|D^j f_k(\phi_k^* + \phi_k, \lambda_0 + \lambda) - D^j f_k(\phi_k^*, \lambda) - D^{j+1} f_k(\phi_k^*, \lambda_0) \begin{pmatrix} \phi_k \\ \lambda \end{pmatrix}\right| \\
&\le \int_0^1 \left|D^{j+1} f_k(\phi_k^* + h\phi_k, \lambda_0 + h\lambda) - D^{j+1} f_k(\phi_k^*, \lambda_0)\right| dh \left|\begin{pmatrix} \phi_k \\ \lambda \end{pmatrix}\right| \\
&\le r_j(\phi, \lambda) \max\{\|\phi\|, |\lambda|\} \quad \text{for all } k \in \mathbb{I}
\end{aligned}$$

and passing over to the least upper bound for $k \in \mathbb{I}$ yields

$$\left\|F^j(\phi^* + \phi, \lambda_0 + \lambda) - F^j(\phi^*, \lambda_0) - F^{j+1}(\phi^*, \lambda_0) \begin{pmatrix} \phi \\ \lambda \end{pmatrix}\right\| \le r_j(\phi, \lambda) \max\{\|\phi\|, |\lambda|\}.$$

Since $\phi^* \in \ell^\infty(\Omega)^\circ$ was arbitrary, F^j is differentiable on $\ell^\infty(\Omega)^\circ \times \Omega$ with derivative F^{j+1} and mathematical induction implies that F is m-times differentiable with $D^j F = F^j$ for $0 \le j \le m$. Finally, from the above Lemma 2.2 we obtain that $D^m F$ is continuous.

It remains to show the assertion when F is defined on $\ell_0(\Omega) \times \Lambda$. Given a sequence $\phi^* \in \ell_0(\Omega)$, we remark that ϕ^* is an interior point of $\ell_0(\Omega)$. We deduce from (H_1) that

$$C_\lambda := \sup_{k\in\mathbb{Z}} \sup_{s\in[0,1]} |D_1 f_k(s\phi_k^*, \lambda)| < \infty \quad \text{for all } \lambda \in \Lambda.$$

Therefore, the mean value estimate (cf. [Lan93, p. 342, Corollary 4.3]) yields

$$|f_k(\phi_k^*, \lambda)| = |f_k(\phi_k^*, \lambda) - f_k(0, \lambda)| + |f_k(0, \lambda)| \le C_\lambda |\phi_k^*| + |f_k(0, \lambda)| \quad \text{for all } \lambda \in \Lambda;$$

thus, the right-hand side of this estimate tends to 0 as $k \to \pm\infty$. Hence, the substitution operator $F : \ell_0(\Omega) \times \Lambda \to \ell_0(\Omega)$ is well-defined and the corresponding smoothness assertions for F follow as above. $\square$

Having this available, the crucial tool for our whole analysis is given in

Theorem 2.4. *For every parameter $\lambda \in \Lambda$, a sequence ϕ in Ω is a solution of the difference equation $(\Delta)_\lambda$, if and only if ϕ solves the nonlinear equation*

$$G(\phi, \lambda) = 0 \tag{2.2}$$

with a formal operator $G(\phi, \lambda) := S\phi - F(\phi, \lambda)$. Moreover, $G : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ and $G : \ell_0(\Omega) \times \Lambda \to \ell_0$ are well-defined, provided (H_0) or (H_0)–(H_1) holds, respectively.

Proof. By definition of S and F, equation (2.2) explicitly reads as $\phi_{k+1} - f_k(\phi_k, \lambda) \equiv 0$ on $\mathbb{I}'$ and this is the solution identity for $(\Delta)_\lambda$. The well-definedness assertions for G follow from Proposition 2.1 and 2.3. $\square$

2.2. **Linear difference equations.** As natural robustness concept to study nonautonomous continuation properties, we employ exponential dichotomies [AHO98, Hen81, Kal94] and the associated dichotomy spectrum [BAG91, AS01, Pöt09]. In this paper, the purpose of the latter notion is to establish an analogous object to the set of eigenvalues in a classical autonomous situation.

Let $\mathbb{I}$ be a discrete interval. For a given operator sequence $A_k \in L(X)$, $k \in \mathbb{I}'$, linear difference equations are of the form

$$\boxed{x_{k+1} = A_k x_k} \tag{$L\Delta$}$$

with associated *transition operator* $\Phi(k, l) \in L(X)$, $k, l \in \mathbb{I}$, defined by

$$\Phi(k, l) := \begin{cases} I_X & \text{for } k = l \\ A_{k-1} \cdots A_l & \text{for } k > l; \end{cases}$$

if every A_k is invertible, we additionally set $\Phi(k, l) := A_k^{-1} \cdots A_{l-1}^{-1}$ for $k < l$.

An exponential dichotomy means that the *extended state space* $\mathbb{I} \times X$ of $(L\Delta)$ allows a hyperbolic splitting, i.e., it splits into invariant vector bundles consisting of solutions with a specific asymptotic behavior; these vector bundles are described using projectors. We say a sequence of projections $P_k \in L(X)$, $k \in \mathbb{I}$, is an *invariant projector*, provided

$$A_k P_k = P_{k+1} A_k \quad \text{for all } k \in \mathbb{I}' \tag{2.3}$$

and speak of a *regular projector*, if the restriction $A_k : N(P_k) \to N(P_{k+1})$, $k \in \mathbb{I}'$, is an isomorphism. Thus, the restricted transition operator $\Phi(k, l) : N(P_l) \to N(P_k)$, $k \le l$, exists with bounded inverse $\Phi(l, k)$ and we can introduce *Green's function*

$$\Gamma_P(k, l) = \begin{cases} \Phi(k, l) P_l & \text{for } l \le k \\ -\Phi(k, l)[I - P_l] & \text{for } k < l. \end{cases}$$

Having these notion at hand, a linear nonautonomous difference equation $(L\Delta)$ is said to have an *exponential dichotomy* (ED for short) on $\mathbb{I}$, if the following holds:

(i) There exists a regular invariant projector P_k,
(ii) there exist reals $K \ge 1$, $\alpha \in (0, 1)$ such that

$$|\Phi(k, l) P_l| \le K\alpha^{k-l} \quad \text{for all } l \le k, \qquad |\Phi(k, l)[I - P_l]| \le K\alpha^{l-k} \quad \text{for all } k \le l.$$

As mentioned, an exponential dichotomy means that the extended state space of $(L\Delta)$ splits into invariant vector bundles, namely

- the *stable bundle* consisting of exponentially decaying forward solutions of $(L\Delta)$ and given by the ranges $R(P_k)$ (if $\mathbb{I}$ is unbounded above),
- the *unstable bundle* consisting of solutions which exist in backward time and are exponentially decaying, given by the kernels $N(P_k)$ (if $\mathbb{I}$ is unbounded below).

Our up-coming results allow an elegant formulation extending the classical autonomous situation using the dichotomy spectrum. Thereto, for $\gamma > 0$ consider the scaled equation

$$(L)_\gamma \qquad \boxed{x_{k+1} = \gamma^{-1} A_k x_k}$$

and define the *dichotomy spectrum* of a linear system $(L\Delta)$ as

$$\Sigma_{\mathbb{I}}(A) := \{\gamma > 0 : (L)_\gamma \text{ has no ED on } \mathbb{I}\}.$$

We observe that $1 \notin \Sigma_{\mathbb{I}}(A)$ holds, if and only if $(L\Delta)$ admits an ED on $\mathbb{I}$. Under reasonable assumptions (cf., (2.5) below) the dichotomy spectrum is a bounded subset of $(0, \infty)$ and the union of so-called *spectral intervals*.

In order to illustrate these notions, we quote and benefit from a combination of results in [BAG91, Section 4], [AVM96] and [Pöt09] to deduce the following examples in which $\mathbb{I} = \mathbb{Z}$. They easily extend to more general situations since, as shown in [Pöt09, Theorem 8], the dichotomy spectrum is invariant under linearly homogenous perturbations $B_k \in L(X)$ with $\lim_{k\to\pm\infty} B_k = 0$, if $\dim X < \infty$ and $A_k + B_k \in GL(X)$, $k \in \mathbb{Z}$.

Example 2.2 (scalar equations, cf. Theorem 4.6 in [BAG91]). For scalar difference equations $x_{k+1} = a_k x_k$ with coefficients $a_k \in \mathbb{R} \setminus \{0\}$ satisfying $\sup_{k\in\mathbb{Z}} \left\{|a_k|, \left|a_k^{-1}\right|\right\} < \infty$, the dichotomy spectrum is related to *lower* and *upper Bohl exponent*

$$\beta_- = \lim_{j\to\infty} \sqrt[j]{\inf_{n\in\mathbb{Z}} \prod_{k=n}^{n+j-1} |a_k|}, \qquad \beta_+ = \lim_{j\to\infty} \sqrt[j]{\sup_{n\in\mathbb{Z}} \prod_{k=n}^{n+j-1} |a_k|},$$

respectively, in terms of $\Sigma_{\mathbb{Z}}(A) = [\beta_-, \beta_+]$. In particular, for the special case $a_k = b$ for $k \geq \kappa$ and $a_k = c$ for $k < \kappa$, $b, c \in \mathbb{R} \setminus \{0\}$, one deduces

$$\beta_- = \min\{|b|, |c|\}, \qquad \beta_+ = \max\{|b|, |c|\}.$$

The dichotomy spectrum extends the autonomous situation, where moduli of spectral points determine stability properties:

Example 2.3 (autonomous equations). In the situation of autonomous equations $(L\Delta)$ with coefficient operator $A_k \equiv A \in L(X)$ on $\mathbb{Z}$, one has $\Sigma_{\mathbb{Z}}(A) = \{|\lambda| > 0 : \lambda \in \sigma(A)\}$, which can be seen using [Ioo79, p. 6, Technical lemma 1].

Example 2.4 (periodic equations, cf. Theorem 4.1 in [BAG91]). Let $\theta \in \mathbb{N}$ be given. For a given θ-periodic difference equation $(L\Delta)$ with monodromy operator $M = \Phi(\theta, 0) \in L(X)$ one has $\Sigma_{\mathbb{Z}}(A) = \left\{\sqrt[\theta]{|\lambda|} : \lambda \in \sigma(M)\right\}$.

The next example is useful when linearizing autonomous difference equations along heteroclinic solutions:

Example 2.5 (cf. Theorem 4.8 in [BAG91]). Suppose $B, C \in GL(\mathbb{C}^d)$, $\kappa \in \mathbb{Z}$ and denote by $N(C, \rho)$ (resp. $R(B, \rho)$) the kernel (resp. range) of the Riesz projection associated to $\{z \in \mathbb{C} : |z| \leq \rho\}$, $\rho > 0$. For a difference equation $(L\Delta)$ with $A_k = C$ for $k < \kappa$ and $A_k = B$ for $k \geq \kappa$ we suppose $\sigma(B) \cup \sigma(C) = \{\lambda_1, \ldots, \lambda_{2d}\}$, where the $\lambda_i \in \mathbb{C}$ are ordered according to

$$|\lambda_1| = \ldots = |\lambda_{n_1}| < |\lambda_{n_1+1}| = \ldots = |\lambda_{n_k}| < |\lambda_{n_k+1}| = \ldots = \left|\lambda_{n_{k+1}}\right|,$$

i.e., the indices $n_1 < \ldots < n_k$ indicate one of the $k < 2d$ jumps in the moduli of the elements of $\sigma(B) \cup \sigma(C)$, and we set $n_{k+1} := 2d$. Moreover, choose $i_1 < \ldots < i_{l-1}$ in $\{1, \ldots, k\}$ such that $N(C, |\lambda_{n_{i_m}}|) \oplus R(B, |\lambda_{n_{i_m}}|) = \mathbb{R}^d$ holds for $0 \leq m < l$. This guarantees $l \leq d+1$ and, with $i_0 = 0$, $i_l = k+1$, a difference equation $(L\Delta)$ admits the dichotomy spectrum

$$\Sigma_{\mathbb{Z}}(A) = \bigcup_{m=0}^{l-1} \left[\left|\lambda_{n_{i_m+1}}\right|, \left|\lambda_{n_{i_{m+1}}}\right| \right].$$

Now suppose $\mathbb{I}$ is unbounded above. We study invertibility properties of the weighted difference operator (cf. [BK97, Bas00])

$$L : \ell \to \ell, \qquad (L\phi)_k := \phi_{k+1} - A_k \phi_k \quad \text{for all } k \in \mathbb{I}; \tag{2.4}$$

it is easily seen to be well-defined and bounded under the assumption

$$\sup_{k \in \mathbb{I}} |A_k| < \infty, \tag{2.5}$$

which we impose throughout this subsection.

Proposition 2.5. *Let $\mathbb{I} = \mathbb{Z}$. A linear equation $(L\Delta)$ admits an ED on $\mathbb{Z}$, if and only if $L \in GL(\ell)$. Moreover, the inverse of L is given by*

$$(L^{-1}\psi)_k = \sum_{j \in \mathbb{Z}} \Gamma_P(k, j+1)\psi_j \quad \textit{for all } k \in \mathbb{Z}.$$

Proof. We refer to [Hen81, p. 230, Theorem 7.6.5] (for the case $\ell = \ell^\infty$) and to [AVM96, Corollary 3], when dealing with limit zero sequences $\ell = \ell_0$. □

The next observation particularly addresses the autonomous and periodic case:

Corollary 2.6. *Let $\kappa \in \mathbb{Z}$. If a linear equation $(L\Delta)$ is almost periodic and admits an ED on a semiaxis $\mathbb{Z}_\kappa^+$ or $\mathbb{Z}_\kappa^-$, then $L \in GL(\ell)$.*

Proof. From [AHO98, Proposition 3.2] we know that for almost periodic equations $(L\Delta)$, an ED on a semiaxis extends to an ED on the whole axis $\mathbb{Z}$. Then the claim follows from the above Proposition 2.5. □

A different situation occurs for general nonautonomous equations with one-sided time:

Proposition 2.7. *Let $\kappa \in \mathbb{Z}$. If a linear equation $(L\Delta)$ admits an ED on $\mathbb{Z}_\kappa^+$, then $L \in L(\ell)$ has a complemented kernel $N(L) \subseteq \ell$ and satisfies*

$$N(L) = \{\Phi(\cdot, \kappa)\xi \in \ell : \xi \in R(P_\kappa)\}, \qquad R(L) = \ell.$$

Proof. Firstly, we suppose $\ell = \ell^\infty$. For a sequence $\psi \in \ell$ and arbitrary $x_0 \in X$ it is shown in [Kal94, pp. 34–34, Satz 3.1.2(ii)] that the inhomogeneous system $x_{k+1} = A_k x_k + \psi_k$ has a unique solution $\phi \in \ell^\infty$ satisfying $P_\kappa \phi_\kappa = P_\kappa x_0$; hence, $L : \ell^\infty \to \ell^\infty$ is onto. On the other hand, applying the result quoted above with inhomogeneity $\psi = 0$, immediately yields $\{\xi \in X : \Phi(\cdot, \kappa)\xi \in \ell^\infty\} = R(P_\kappa)$. Since P_k is an invariant projector for $(L\Delta)$, the mapping $P : \ell^\infty \to \ell^\infty$, $(P\phi)_k := P_k \phi_k$ is a bounded projector onto $N(L)$ and thus the space $N(L)$ is complemented.

In order to show the remaining assertion that $L : \ell_0 \to \ell_0$ is surjective, one proceeds analogously to [Bas00, Lemma 2]. □

For the sake of completeness and the reader's convenience, we finally summarize some auxiliary results from [Bas00] guaranteeing the invertibility of L.

Proposition 2.8. *Let $\underline{\kappa}, \overline{\kappa} \in \mathbb{Z}$ with $\underline{\kappa} < \overline{\kappa}$ and $\mathbb{I} = \mathbb{Z}$. Suppose a linear equation $(L\Delta)$ admits an ED both on $\mathbb{Z}^+_{\overline{\kappa}}$ (with projector P^+_k) and on $\mathbb{Z}^-_{\underline{\kappa}}$ (with projector P^-_k). Then $L \in GL(\ell)$ holds, if and only if*

$$(I - P^+_{\overline{\kappa}})\Phi(\overline{\kappa}, \underline{\kappa})(I - P^-_{\underline{\kappa}}) \in GL(N(P^-_{\underline{\kappa}}), N(P^+_{\overline{\kappa}})).$$

Proof. See [Bas00, Theorem 8]. □

Corollary 2.9. *If $X = \Phi(\overline{\kappa}, \underline{\kappa})N(P^-_{\underline{\kappa}}) \oplus R(P^+_{\overline{\kappa}})$, then $L \in GL(\ell)$.*

Proof. See [Bas00, Corollary 1]. □

In a parallel fashion to Proposition 2.8 we now consider the situation where $(L\Delta)$ admits EDs on positive and negative semiaxes with nonempty intersection. For the corresponding finite-dimensional situation we refer to [BK97, Boi01].

Proposition 2.10. *Let $\kappa \in \mathbb{Z}$ and $\mathbb{I} = \mathbb{Z}$. Suppose a linear equation $(L\Delta)$ admits an ED both on $\mathbb{Z}^+_\kappa$ (with projector P^+_k) and on $\mathbb{Z}^-_\kappa$ (with projector P^-_k). Then $L \in GL(\ell)$ holds, if one of the following conditions is satisfied:*

$$X = N(P^-_\kappa) \oplus R(P^+_\kappa), \qquad I - P^-_\kappa - P^+_\kappa \in GL(X).$$

Proof. See [Bas00, Corollary 2]. □

For the sake of having a future reference at hand, we illustrate the above results using a simple 2-dimensional example. It demonstrates explicitly when dichotomies on semiaxes extend to the whole discrete line $\mathbb{Z}$.

Example 2.6. Let $\gamma_-, \beta_-, \gamma_+, \beta_+ \in \mathbb{R} \setminus \{0\}$ be given and suppose $X = \mathbb{R}^2$. We define a piecewise constant coefficient matrix for $(L\Delta)$ by

$$A_k := \begin{pmatrix} a_{11}(k) & 0 \\ 0 & a_{22}(k) \end{pmatrix}, \quad a_{11}(k) := \begin{cases} \gamma_-, & k < 0, \\ \beta_-, & k \geq 0 \end{cases} \qquad a_{22}(k) := \begin{cases} \gamma_+, & k < 0, \\ \beta_+, & k \geq 0 \end{cases}$$

and arrive at the transition matrix

$$\Phi(k,l) := \begin{cases} \operatorname{diag}(\beta_+^{k-l}, \gamma_+^{k-l}), & k \geq l \geq 0, \\ \operatorname{diag}(\beta_+^k \beta_-^{-l}, \gamma_+^k \gamma_-^{-l}), & k \geq 0 > l, \\ \operatorname{diag}(\beta_-^{k-l}, \gamma_-^{k-l}), & 0 > k \geq l; \end{cases}$$

due to the invertibility of A_k we can define $\Phi(k,l) := \Phi(l,k)^{-1}$ for $k < l$. We distinguish several cases in order to describe the dichotomy properties of $(L\Delta)$. In each case, $(L\Delta)$ admits an ED on $\mathbb{Z}^+_0$ and $\mathbb{Z}^-_{-1}$ with constant projectors P^-_k resp. P^+_k; it is easy to see that the ED on $\mathbb{Z}^-_{-1}$ extends to $\mathbb{Z}^-_0$. With the help of Proposition 2.8 and 2.10 we summarize, for which parameter constellations $(L\Delta)$ admits an ED on the whole axis $\mathbb{Z}$.

(a) $|\beta_+|, |\gamma_+| < 1$: $P^+_k = I$
  (a_1) $|\beta_-|, |\gamma_-| < 1$: $P^-_k \equiv I$,
(b) $|\beta_+| < 1 < |\gamma_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}^+_0$ with $P^+_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
  (b_2) $|\beta_-| < 1 < |\gamma_-|$: $P^-_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,
(c) $|\gamma_+| < 1 < |\beta_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}^+_0$ with $P^+_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
  (c_3) $|\gamma_-| < 1 < |\beta_-|$: $P^-_k \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,
(d) $1 < |\beta_+|, |\gamma_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}^+_0$ with $P^+_k = 0$
  (d_4) $1 < |\beta_-|, |\gamma_-|$: $P^-_k \equiv 0$.

2.3. **Poincaré continuation.** For parameters $\lambda^* \in \Lambda$ we consider the *variational equation*

$$x_{k+1} = D_1 f_k(\phi_k^*, \lambda^*) x_k \tag{2.6}$$

along a fixed reference sequence $\phi^* = (\phi_k^*)_{k \in \mathbb{I}}$ in Ω. Typically, ϕ^* is a solution to the difference equation $(\Delta)_{\lambda^*}$, like for instance an equilibrium, a periodic solution or of homo- resp. heteroclinic type. Such a sequence is said to be *hyperbolic*, if (2.6) has an ED on $\mathbb{I}$. We define the *dichotomy spectrum* of ϕ^* by

$$\Sigma_{\mathbb{I}}(\phi^*, \lambda^*) := \Sigma_{\mathbb{I}}(A) \quad \text{with } A_k := D_1 f_k(\phi_k^*, \lambda^*)$$

and ϕ^* is hyperbolic, precisely if $1 \notin \Sigma_{\mathbb{I}}(\phi^*, \lambda^*)$.

Now we have collected the preparations in order to deduce a nonautonomous and discrete version of the Poincaré continuation lemma. It deals with the question, under which conditions a bounded solution ϕ^* of the nonlinear difference equation $(\Delta)_{\lambda^*}$ persists, when the parameter λ is varied near λ^*.

Theorem 2.11 (hyperbolic solutions on $\mathbb{Z}$)**.** *Let $\lambda^* \in \Lambda$, $\mathbb{I} = \mathbb{Z}$ and suppose (H_0) holds. If $\ell = \ell^\infty$ and $\phi^* \in \ell(\Omega)$ is a complete permanent solution of $(\Delta)_{\lambda^*}$ such that*

$$1 \notin \Sigma_{\mathbb{Z}}(\phi^*, \lambda^*),$$

then there exist $\rho, \varepsilon > 0$ and a C^m-function $\phi : B_\rho(\lambda^*) \to B_\varepsilon(\phi^*) \subseteq \ell(\Omega)$ with:

(a) $\phi(\lambda^*) = \phi^*$,
(b) *$\phi(\lambda)$ is the unique bounded complete solution of equation $(\Delta)_\lambda$ in $B_\varepsilon(\phi^*) \times B_\rho(\lambda^*)$,*
(c) *$\phi(\lambda)$ is hyperbolic and permanent.*

If (H_0)–(H_1) are satisfied, then the same holds with $\ell = \ell_0$.

Remark 2.1. As result of Theorem 2.11(c) the saddle point structure consisting of stable and unstable fiber bundles (or manifolds in the autonomous case, cf. [PR05]) associated to the hyperbolic complete solution ϕ^* persists under variation of $\lambda \in B_\rho(\lambda^*)$.

Proof of Theorem 2.11. (a) and (b) Suppose $\ell = \ell^\infty$ and due to the assumed permanence, ϕ^* is an interior point of $\ell^\infty(\Omega)$. Using Theorem A.1 we solve

$$G(\psi, \lambda) = 0 \quad \text{for } (\psi, \lambda) \in \ell^\infty(\Omega) \times \Lambda.$$

Above all, note that $G : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ is well-defined due to Proposition 2.3. Since ϕ^* is a solution of $(\Delta)_{\lambda^*}$ we know from Theorem 2.4 that $G(\phi^*, \lambda^*) = 0$ holds and we have to show that the partial derivative $D_1 G(\phi^*, \lambda^*) \in L(\ell^\infty)$, which exists by Proposition 2.3, is a toplinear isomorphism. With $A_k = D_1 f_k(\phi_k^*, \lambda^*)$ and the difference operator L defined in (2.4), we have the identity

$$D_1 G(\phi^*, \lambda^*)\phi = S\phi - F^{(1,0)}(\phi^*, \lambda^*)\phi = L\phi. \tag{2.7}$$

Due to $1 \notin \Sigma_{\mathbb{Z}}(\phi^*, \lambda^*)$ and the resulting ED of (2.6) on $\mathbb{Z}$ we deduce from Proposition 2.5 that $L = D_1 G(\phi^*, \lambda^*)$ is invertible and Theorem A.1 yields the existence of a $\rho > 0$ and of a unique C^m-function $\phi : B_\rho(\lambda^*) \to \ell^\infty(\Omega)$ such that $G(\phi(\lambda), \lambda) \equiv 0$ on $B_\rho(\lambda^*)$.

(c) It remains to show the hyperbolicity of $\phi(\lambda)$. To establish this, note that the variational difference equation $x_{k+1} = D_1 f_k(\phi(\lambda)_k, \lambda) x_k$ can be written as

$$x_{k+1} = D_1 f_k(\phi_k^*, \lambda^*) x_k + \left[D_1 f_k(\phi(\lambda)_k, \lambda) - D_1 f_k(\phi_k^*, \lambda^*)\right] x_k. \tag{2.8}$$

Suppose the data for the ED of (2.6) is K, α. The continuity of ϕ and $D_1 f_k$ guarantee that for sufficiently small $\rho > 0$ one has

$$|D_1 f_k(\phi(\lambda)_k, \lambda) - D_1 f_k(\phi_k^*, \lambda^*)| < \tfrac{1-\alpha}{8K} \quad \text{for all } k \in \mathbb{Z},\ \lambda \in B_\rho(\lambda^*)$$

and the ℓ^∞-roughness of EDs (see [Hen81, p. 232, Theorem 7.6.7]) implies that also (2.8) admits an exponential dichotomy on $\mathbb{Z}$. Thus, $\phi(\lambda)$ is hyperbolic. By scaling down $\rho > 0$, it is possible to choose $\varepsilon > 0$ arbitrarily small and therefore also $\phi(\lambda)$ is permanent.

Now assume also (H_1) and that $\phi^* \in \ell_0(\Omega)$ holds. Then the above arguments including Proposition 2.5 remain true, when ℓ^∞ is replaced by ℓ_0. □

Corollary 2.12. *If additionally (H_2) holds and the dichotomy data associated to the variational equation* (2.6) *are K, α, then the constants $\rho, \varepsilon > 0$ can be obtained from*

$$\tfrac{2K}{\alpha}\omega_1(\rho,\varepsilon) \le \omega < 1, \qquad \tfrac{2K}{\alpha}(\rho + \omega_0(\rho)) \le \varepsilon(1-\omega).$$

Proof. The explicit form of L^{-1} given in Proposition 2.5 yields $\left|L^{-1}\right| \le \frac{2K}{\alpha}$. Using our hypothesis (H_2), the claim follows from Corollary A.2 with $\omega_2(r) = r + \omega_0(r)$. □

Due to the differentiable dependence of the perturbed solution $\phi(\lambda)$ on the parameter $\lambda \in B_\rho(\lambda^*)$, one can approximate $\phi(\lambda)$ using a finite Taylor series in λ. Here, a phenomenon typical for nonautonomous equations occurs: Algebraic problems in an autonomous setting become dynamical problems, i.e., instead of solving algebraic equations, one has to find bounded solutions of a nonautonomous difference equation.

As a result of Taylor's theorem (cf., e.g., [Lan93, p. 350]) we can write

$$\phi(\lambda) = \phi^* + \sum_{n=1}^{m} \frac{1}{n!} D^n\phi(\lambda^*)(\lambda - \lambda^*)^{(n)} + R_m(\lambda) \tag{2.9}$$

with coefficients $D^n\phi(\lambda^*) \in L_n(Y, X)$ and remainder R_m satisfying $\lim_{\lambda\to 0} \frac{R_m(\lambda)}{|\lambda|^m} = 0$. For $1 \le n \le m$ we apply the higher order chain rule (see [PR05, Lemma 4.1] for a reference in our notation) to the solution identity

$$\phi(\lambda)_{k+1} \equiv f_k(\phi(\lambda)_k, \lambda) \quad \text{on } B_\rho(\lambda^*)$$

for all $k \in \mathbb{Z}$. This yields the relation

$$D^n\phi(\lambda)_{k+1} y_1 \cdot \ldots \cdot y_n = D_1 f_k(\phi(\lambda)_k, \lambda) D^n\phi(\lambda)_k y_1 \cdot \ldots \cdot y_n$$
$$+ \sum_{j=2}^{n} \sum_{(N_1,\ldots,N_j)\in P_j^<(l)} D^j f_k(\phi(\lambda)_k, \lambda) g_k^{\#N_1}(\lambda) y_{N_1} \cdots g_k^{\#N_j}(\lambda) y_{N_j}$$

for all $y_1, \ldots, y_n \in Y$, where we abbreviate $g_k^{\#N_1}(\lambda) := \frac{d^{\#N_1}(\phi(\lambda)_k,\lambda)}{d\lambda^{\#N_1}}$. Setting $\lambda = \lambda^*$ in this relation yields that the Taylor coefficients $D^n\phi(\lambda^*) \in L_n(Y, \ell^\infty) \cong \ell^\infty(L_n(Y, X))$ fulfill the linearly inhomogeneous difference equation

$$X_{k+1} = D_1 f_k(\phi_k^*, \lambda^*) X_k + H_n(k) \tag{$(I)_n$}$$

in $L_n(Y, X)$, where the inhomogeneity $H_n : \mathbb{Z} \to L_n(Y, X)$ reads as

$$H_n(k) y_1 \cdot \ldots \cdot y_n := \sum_{j=2}^{n} \sum_{(N_1,\ldots,N_j)\in P_j^<(l)} D^j f_k(\phi_k^*, \lambda^*) g_k^{\#N_1}(\lambda^*) y_{N_1} \cdots g_k^{\#N_j}(\lambda^*) y_{N_j},$$

and in particular $H_1(k) = D_2 f_k(\phi_k^*, \lambda^*)$. Having these preparations at hand, we deduce

Corollary 2.13. *The coefficients $D^n\phi(\lambda^*) : \mathbb{Z} \to L_n(Y, X)$, $1 \le n \le m$, in the Taylor expansion* (2.9) *can be determined recursively from the Lyapunov-Perron sums*

$$D^n\phi(\lambda^*)_k = \sum_{l\in\mathbb{Z}} \Gamma_P(k, l+1) H_n(l) \quad \textit{for all } 1 \le n \le m,$$

where Γ_P is the Green's function associated to (2.6)*.*

Proof. We prove the assertion by mathematical induction. For $n = 1$ the inhomogeneity H_1 is bounded due to (H_0). Since we want to approximate bounded solutions $\phi(\lambda)$ to equation $(\Delta)_\lambda$, it is reasonable to look for the Taylor coefficient $D\phi(\lambda^*)$ as bounded complete solution of $(I)_1$. Arguing as in Proposition 2.5, the ED of (2.6) guarantees that equation $(I)_1$ has a unique complete bounded solution, which is given by the Lyapunov-Perron sum $D\phi(\lambda^*)_k = \sum_{l\in\mathbb{Z}} \Gamma_P(k, l + 1)H_1(l)$.

In the induction step $n \to n+1$ we see from the induction hypothesis that the sequences $D\phi(\lambda^*), \dots, D^n\phi(\lambda^*)$ are bounded, which implies that also H_{n+1} is a bounded sequence. Therefore, due to the above argument, the linear equations $(I)_{n+1}$ has a unique bounded solution given by a Lyapunov-Perron sum as claimed in our corollary. □

The situation of Theorem 2.11 changes drastically for one-sided time, where hyperbolic solutions are embedded in, and persist as, families of solutions parametrized over the stable bundle associated with the linearization (2.6).

Theorem 2.14 (hyperbolic solutions on semiaxes)**.** *Let $\lambda^* \in \Lambda$, $\kappa \in \mathbb{Z}$, $\mathbb{I} = \mathbb{Z}^+_\kappa$ and suppose (H_0) holds. If $\ell = \ell^\infty$ and $\phi^* \in \ell(\Omega)$ is a permanent solution of $(\Delta)_{\lambda^*}$ on $\mathbb{Z}^+_\kappa$ such that*

$$1 \notin \Sigma_{\mathbb{Z}^+_\kappa}(\phi^*, \lambda^*) \quad \textit{and associated invariant projector } P_k, \tag{2.10}$$

then there exist $\delta, \rho > 0$ and a unique C^m-function $\psi : B_\rho(0, \lambda^*) \subseteq R(P_\kappa) \times \Lambda \to B_\delta(\phi^*) \subseteq \ell(\Omega)$ such that one has for all $(\xi, \lambda) \in B_\rho(0, \lambda^*)$:

(a) *$\psi(0, \lambda^*) = \phi^*$,*
(b) *$\psi(\xi, \lambda)$ is a hyperbolic solution of equation $(\Delta)_\lambda$.*

If (H_0)–(H_1) are satisfied, then the same holds with $\ell = \ell_0$.

Proof. We present the proof only for $\ell^\infty(\Omega)$ and set $L := D_1G(\phi^*, \lambda^*)$.

(a) Our assumption and Proposition 2.7 imply that $N(L) \cong R(P_\kappa)$ is complemented, i.e., $\ell^\infty = N(L) \oplus Z$ with a closed subspace $Z \subseteq \ell^\infty$; in addition, $L \in L(\ell^\infty)$ is onto. Thus the claim follows from Theorem A.3 applied to $G(\phi^*, \lambda^*) = 0$.

(b) can be shown as in Theorem 2.11(c). Here, the dichotomy roughness for one-sided time is due to [Kal94, p. 45, Satz 3.2.1]. □

Next we establish that (uniform) stability properties of ϕ^* persist:

Corollary 2.15. *There exists a $\rho > 0$ such that for every $\lambda \in B_\rho(\lambda^*)$ a solution $\phi(\lambda)$ of $(\Delta)_\lambda$ as in Theorem 2.11 or 2.14 satisfies:*

(a) *With ϕ^* also the perturbed solution $\phi(\lambda)$ of $(\Delta)_\lambda$ is exponentially stable,*
(b) *if ϕ^* is a complete solution and there exists a decomposition $\Sigma_{\mathbb{Z}}(\phi^*, \lambda^*) = \sigma_- \dot\cup \sigma_+$ with $\sup \sigma_+ < 1 < \inf \sigma_-$, then the perturbed solution $\phi(\lambda)$ of $(\Delta)_\lambda$ is unstable.*

Proof. Suppose $\mathbb{I} = \mathbb{Z}$ or $\mathbb{I} = \mathbb{Z}^+_\kappa$, depending if we deal with Theorem 2.11 or 2.14.

(a) Let Φ_{λ^*} denote the transition operator of (2.6). Since $\phi^* \in \ell(\Omega)$ is an exponentially stable solution of $(\Delta)_{\lambda^*}$, we know that 0 is an exponentially stable equilibrium for the equation of perturbed motion

$$x_{k+1} = f_k(x_k + \phi^*_k, \lambda^*) - f_k(\phi^*_k, \lambda^*).$$

So, the converse of the theorem on stability by first approximation due to Győri and Pituk (see [GP01, Theorem 4] and note that their proof remains valid for Banach spaces X instead of $\mathbb{R}^d$ as state space) implies the existence of constants $K_0 \geq 1$, $\alpha_0 \in (0, 1)$ such that $|\Phi_{\lambda^*}(k, \kappa)| \leq K_0\alpha_0^{k-\kappa}$ for $\kappa \leq k$. Hence, ϕ^* is hyperbolic with associated splitting projector $Q_k^* \equiv I$. We apply Theorem 2.11 or 2.14 in order to establish the existence of a $\rho > 0$ and of hyperbolic solutions $\phi(\lambda)$ of $(\Delta)_\lambda$ for $\lambda \in B_\rho(\lambda^*)$. Choosing $\rho > 0$ sufficiently small, we know that the splitting projectors Q_k^* and $Q_k(\lambda)$ associated with the EDs of the variational equations for $(\Delta)_{\lambda^*}$ and $(\Delta)_\lambda$ along ϕ^* and $\phi(\lambda)$, resp., are linearly conjugated. This guarantees that there exist constants $K \geq 1$, $\alpha \in (0,1)$ such that

$$|\Phi_\lambda(k,\kappa)| \leq K\alpha^{k-\kappa} \quad \text{for all } \kappa \leq k$$

holds for the corresponding transition operator Φ_λ of the variational equation for $(\Delta)_\lambda$ along $\phi(\lambda)$. The theorem on stability by first approximation (also for this classical case we refer to [GP01, Theorem 4]) implies that the zero solution of

$$x_{k+1} = f_k(x_k + \phi(\lambda)_k, \lambda) - f_k(\phi(\lambda)_k, \lambda) \tag{2.11}$$

is exponentially stable, i.e., $\phi(\lambda)$ is an exponentially stable solution of $(\Delta)_\lambda$.

(b) Due to our assumption, the dichotomy spectrum $\Sigma_{\mathbb{Z}}(\phi^*, \lambda^*)$ allows a decomposition $\sigma_+ \dot\cup \sigma_-$ as assumed. Referring to [Pöt09, Corollary 3] this decomposition persists for λ near λ^*. The resulting spectral gap implies an unstable fiber bundle for the zero solution of equation (2.11) (cf. [PR05, Theorem 3.2(b)]). Hence, $\phi(\lambda)$ is unstable. □

We close this section with a short proof of a nonautonomous stable manifold theorem. Thereto, let $\mathbb{I}$ be a discrete interval unbounded above and $\phi^* = (\phi_k^*)_{k\in\mathbb{I}}$ be a bounded solution of $(\Delta)_{\lambda^*}$, $\lambda^* \in \Lambda$, in Ω. Then the *stable set* of ϕ^* is defined to be

$$S^+(\lambda) := \left\{(\kappa,\xi) \in \mathbb{I} \times \Omega : \varphi(k;\kappa,\xi,\lambda) - \phi_k^* \xrightarrow[k\to\infty]{} 0\right\} \quad \text{for all } \lambda \in \Lambda$$

and we can describe its local structure as follows:

Corollary 2.16 (stable manifold theorem)**.** *If $\phi^* \in \ell^\infty(\Omega)$ is a permanent solution of $(\Delta)_{\lambda^*}$ on $\mathbb{I}$ satisfying* (2.10)*, then there exist $\varepsilon, \rho > 0$ and a unique C^m-function $s_\kappa^+ : B_\rho(0,\lambda^*) \subseteq R(P_\kappa) \times \Lambda \to N(P_\kappa)$ such that the fibers of $S^+(\lambda)$ fulfill*

$$S^+(\lambda)_\kappa \cap B_\varepsilon(\phi_\kappa^*) = \left\{\phi_\kappa^* + \xi + s_\kappa^+(\xi,\lambda) \in \Omega : \xi \in B_\rho(0) \subseteq R(P_\kappa)\right\}$$

for all $\lambda \in B_\rho(\lambda^*)$, with the fibers $S^+(\lambda)_\kappa := \{x \in \Omega : (\kappa, x) \in S^+(\lambda)\}$.

Proof. We consider the difference equation of perturbed motion

$$x_{k+1} = f_k(x_k + \phi_k^*, \lambda) - f_k(\phi_k^*, \lambda) \tag{2.12}$$

which admits the trivial solution on $\mathbb{I}$ for all $\lambda \in \Lambda$. Moreover, it satisfies (H_0)–(H_1) and if we choose $\kappa \in \mathbb{I}$, then Theorem 2.14 applies to (2.12) with $\ell = \ell_0$. This implies

$$S^+(\lambda)_\kappa \cap B_\varepsilon(\phi_\kappa^*) = \{\phi_\kappa^* + \psi(\xi,\lambda)_\kappa \in \Omega : \xi \in B_\rho(0) \subseteq R(P_\kappa)\}$$

and a closer look at Theorem A.3 yields that solutions of (2.12) decaying to 0 in forward time, start in the points $\psi(\xi,\lambda)_\kappa = \xi + \phi(\xi,\lambda)_\kappa$ with a C^m-function $\phi(\cdot)_\kappa$ having values in $N(P_\kappa)$. Then the assertion follows with $s_\kappa^+(\xi,\lambda) := \phi(\xi,\lambda)_\kappa$. □

In a more geometric language, the latter Corollary 2.16 states that the stable set $S^+(\lambda)$ is locally graph of a smooth function over the stable bundle. Analogously, one can describe the unstable set consisting of initial pairs for backward solutions converging to a complete solution ϕ^* as $k \to -\infty$; near ϕ^* it is a graph over the unstable bundle.

3. Functional differential equations

A theory parallel to Section 2 can be established for miscellaneous nonautonomous evolutionary differential equations. Among them, we focus on functional differential equations of retarded type, which include ODEs. However, an analogous treatment for classical solutions of various evolutionary partial differential equations seems possible. Since the corresponding theory is similar to Section 2, we keep our explanations more compact and concentrate on the more involved space setting.

Let $\Omega \subseteq \mathbb{R}^d$ denote a nonempty open set and $|\cdot|$ stands for a norm on $\mathbb{R}^d$. Given arbitrary $r \geq 0$, $t_0 \in \mathbb{R}$, for a convenient notation we also introduce the closed intervals

$$I_r := \begin{cases} \mathbb{R}, & r > 0, \\ [t_0, \infty), & r = 0. \end{cases}$$

The following spaces are central for our overall approach:

$$C^1(I_r, \Omega) := \{\phi : I_r \to \Omega |\, \phi \text{ is continuously differentiable}\},$$
$$BC(I_r, \Omega) := \left\{\phi : I_r \to \Omega |\, \phi \text{ is continuous and } \sup_{t \in I_r} |\phi(t)| < \infty\right\},$$
$$BC^1(I_r, \Omega) := \left\{\phi \in BC(I_r, \Omega) \cap C^1(I_r, \Omega) |\, \dot{\phi} \in BC(I_r, \mathbb{R}^d)\right\},$$

and in case $0 \in \Omega$ also

$$BC_0(I_r, \Omega) := \left\{\phi : I_r \to \Omega |\, \phi \text{ is continuous and } \lim \phi(t) = 0\right\}, \tag{3.1}$$
$$BC_0^1(I_r, \Omega) := \left\{\phi \in BC_0(I_r, \Omega) \cap C^1(I_r, \Omega) |\, \dot{\phi} \in BC_0(I_r, \mathbb{R}^d)\right\};$$

in (3.1) the limit $t \to \infty$ is meant for $r = 0$, and the two-sided limit $t \to \pm\infty$ for $r > 0$.

In the following, it is convenient to abbreviate $C^1 := C^1(I_r, \mathbb{R}^d)$ and we proceed accordingly with the other function spaces defined above. Then the sets BC, BC_0 and BC^1, BC_0^1 are Banach spaces equipped with the respective norms

$$\|\phi\|_0 := \sup_{t \in \mathbb{R}} |\phi(t)|, \qquad \|\phi\|_1 := \max\left\{\|\phi\|_0, \|\dot{\phi}\|_0\right\};$$

one has the continuous embeddings $BC_0 \hookrightarrow BC$ and $BC_0^1 \hookrightarrow BC^1 \hookrightarrow BC$.

From now on, we suppose $r \geq 0$ and $\Omega \subseteq \mathbb{R}^d$ is a nonempty open convex subset. For brevity we introduce the open set $C_r(\Omega) := C([-r, 0], \Omega)$ and as usual, the function $\phi_t \in C_r(\Omega)$ is defined by $\phi_t(s) := \phi(t+s)$ for a given continuous $\mathbb{R}^d$-valued function ϕ. The norm on $C_r := C_r(\mathbb{R}^d)$ is $|\phi|_r := \sup_{-r \leq s \leq 0} |\phi(s)|$.

With given right-hand side $f : \mathbb{R} \times C_r(\Omega) \times \Lambda \to \mathbb{R}^d$, we consider a nonautonomous parameter-dependent retarded functional differential equation (FDE for short)

$$\boxed{\dot{u}(t) = f(t, u_t, \lambda).} \tag{F}_\lambda$$

For initial times $t_0 \in \mathbb{R}$ and a fixed parameter $\lambda \in \Lambda$, we say $\phi : [t_0 - r, t_0 + R) \to \mathbb{R}^d$ is a *solution* of $(F)_\lambda$, provided $\phi_{t_0} \in C_r(\Omega)$ and the integral equation

$$\phi(t) = \phi(t_0) + \int_{t_0}^t f(s, \phi_s, \lambda)\, ds \quad \text{for all } t \in [t_0, t_0 + R)$$

holds with an interval length $R = R(t_0, \lambda) > 0$. Under our hypothesis stated below, it is shown in [HVL93, pp. 38ff, Chapter 2] that for each initial function $\theta \in C_r(\Omega)$ there exists a maximal $R > 0$ and a unique solution ϕ to $(F)_\lambda$ satisfying $\phi_{t_0} = \theta$; we denote it as *general solution* $\varphi(\cdot; t_0, \theta, \lambda)$. Furthermore, a *complete solution* of $(F)_\lambda$ is a globally defined C^1-function $\phi : \mathbb{R} \to \mathbb{R}^d$ with $\phi_t \in C_r(\Omega)$ satisfying the *solution identity* $\dot{\phi}(t) \equiv f(t, \phi_t, \lambda)$ on $\mathbb{R}$. In the situation $\phi \in BC(\mathbb{R}, \mathbb{R}^d)$ we speak of a *bounded complete solution* and for $\phi \in BC_0$ of a *homoclinic solution*. Finally, a solution of $(F)_\lambda$ defined on a maximal interval $I \subseteq \mathbb{R}$ is said to be a *permanent solution*, if $\inf_{t\in I} \operatorname{dist}(\phi(t), \Omega) > 0$.

3.1. **Substitution operators.** A variety of our results holds simultaneously for bounded complete, as well as for homoclinic solutions. Thus, it is convenient to introduce the symbol $\mathcal{C}$, which stands either for BC or for BC_0. However, once $\mathcal{C}$ is assigned, it remains fixed in each result.

In our present setting, the shift operator from the discrete case has to be replaced by a differential operator. We waive the proof of the following elementary

Lemma 3.1. *The operator $S : \mathcal{C}^1(I_r, \Omega) \to \mathcal{C}$, $(S\phi)(t) := \dot{\phi}(t)$ is linear and bounded.*

Proposition 3.2. *For every $t \in I_r$ the operator $E_t : \mathcal{C}(I_r, \mathbb{R}^d) \to \mathcal{C}(I_r, C_r)$, $E_t\phi := \phi_t$ is well-defined, linear with $|E_t| \le 1$ and satisfies $E_t\mathcal{C}(I_r, \Omega) \subseteq \mathcal{C}(I_r, C_r(\Omega))$.*

Proof. Let $t \in I_r$ be given. For $\mathcal{C} = BC$ we choose $\phi \in BC(I_r, \Omega)$. Thanks to

$$|E_t\phi|_r = |\phi_t|_r = \sup_{s\in[-r,0]} |\phi(t+s)| \le \|\phi\|_0 \quad \text{for all } t \in I_r$$

the linear function $E_t\phi : I_r \to C_r(\Omega)$ is bounded. Due to [HVL93, p. 40, Lemma 2.1], $E_t\phi$ is also continuous. In case $\mathcal{C} = BC_0$ we know that for each $\varepsilon > 0$ there exists an $T = T(\varepsilon) > 0$ such that $|\phi(t)| < \varepsilon/2$ for all $t \in I_r$ with $|t| \ge T$. This implies

$$|\phi(t+s)| < \varepsilon/2 \quad \text{for all } |t| \ge T + r,\ s \in [-r, 0]$$

and passing to the least upper bound over $s \in [-r, 0]$ yields $|\phi_t|_r < \varepsilon$ for $|t| \ge T + r$. We therefore also have $E_t\phi \in BC_0(I_r, C_r(\Omega))$. □

As indicated above (see [HVL93, p. 40, Lemma 2.1]), under the subsequent hypotheses one can guarantee existence and forward uniqueness of solutions for $(F)_\lambda$:

Hypothesis. *Let $m \in \mathbb{N}$, $r \ge 0$, suppose $f : \mathbb{R} \times C_r(\Omega) \times \Lambda \to \mathbb{R}^d$ is continuous and the partial derivatives $D^j_{(2,3)}f$, $0 \le j \le m$, exist, are continuous and satisfy:*

(H_0') *For all bounded $B \subseteq C_r(\Omega)$ one has*

$$\sup_{t\in\mathbb{R}} \sup_{\phi\in B} \left| D^j_{(2,3)} f(t, \phi, \lambda) \right| < \infty \quad \textit{for all } \lambda \in \Lambda$$

(well-definedness) *and for all $\lambda_0 \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|\phi - \psi|_r < \delta \quad \Rightarrow \quad \sup_{t\in\mathbb{R}} \left| D^j_{(2,3)} f(t, \phi, \lambda) - D^j_{(2,3)} f(t, \psi, \lambda_0) \right| < \varepsilon$$

for all $\phi, \psi \in C_r(\Omega)$ and $\lambda \in B_\delta(\lambda_0)$ (uniform continuity).

(H_1') *We have $0 \in \Omega$ and $\lim_{t\to\pm\infty} f(t, 0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

Having these assumptions satisfied, we formally introduce *substitution operators*

$$F(\phi, \lambda)(t) := f(t, E_t\phi, \lambda), \qquad F^j(\phi, \lambda)(t) := D^j_{(2,3)} f(t, E_t\phi, \lambda),$$
$$F^\upsilon(\phi, \lambda)(t) := D_2^{\upsilon_1} D_3^{\upsilon_2} f(t, E_t\phi, \lambda)$$

for all $t \in I_r$, with $0 \le j \le m$ and pairs $\upsilon = (\upsilon_1, \upsilon_2) \in \mathbb{N}_0^2$ such that $\upsilon_1 + \upsilon_2 \le m$.

Lemma 3.3. *Under (H_0') the operators $F^j : BC(I_r, \Omega) \times \Lambda \to BC\big(I_r, L_j(C_r \times Y, \mathbb{R}^d)\big)$, $F^\upsilon : BC(I_r, \Omega) \times \Lambda \to BC(I_r, L_{\upsilon_1}(C_r, L_{\upsilon_2}(Y, \mathbb{R}^d))$ are well-defined and continuous.*

Proof. Suppose $\lambda \in \Lambda$ and $\phi \in BC(I_r, \Omega)$ are given. Thanks to Proposition 3.2 and (H_0'), we deduce that the function $t \mapsto D^j_{(2,3)} f(t, \phi_t, \lambda)$ is bounded, continuous and additionally F^j is well-defined. To deduce the continuity of F^j, choose $\lambda_0 \in \Lambda$, $\phi^* \in BC(I_r, \Omega)$ and $\varepsilon > 0$. Given $\delta > 0$ and $\phi \in B_\delta(\phi^*)$ arbitrarily, we observe

$$|\phi_t - \phi_t^*|_r = \sup_{s \in [-r,0]} |\phi(t+s) - \phi^*(t+s)| \leq \|\phi - \phi^*\|_0 < \delta \quad \text{for all } t \in I_r$$

and by (H_0') we can choose $\delta > 0$ so that $\left|D^j_{(2,3)} f(t, \phi_t, \lambda) - D^j_{(2,3)} f(t, \phi_t^*, \lambda_0)\right| < \varepsilon$ is satisfied for all $t \in I_r$ and $\lambda \in B_\delta(\lambda_0)$. This implies the assertion. □

Before stating the next result, we point out that $BC(I_r, \Omega)$ is not necessarily open, whereas $BC_0(I_r, \Omega)$ is an open set.

Proposition 3.4. *Under (H_0') the operator $F : BC(I_r, \Omega) \times \Lambda \to BC$ is well-defined and m-times continuously differentiable on $BC(I_r, \Omega)^\circ \times \Lambda$ with partial derivatives*

$$D^\upsilon F(\phi, \lambda) = F^\upsilon(\phi, \lambda) \quad \textit{for all } \phi \in BC(I_r, \Omega)^\circ,\ \lambda \in \Lambda.$$

If (H_0')–(H_1') are satisfied, then the same holds for $F : BC_0^1(I_r, \Omega) \times \Lambda \to BC_0$.

Proof. With Lemma 3.3 at hand, the claim follows using analogous arguments as in the proof of Proposition 2.3. In particular, for $\phi \in BC_0(I_r, \Omega)$ one obtains $\lim_{t \to \pm\infty} \phi_t = 0$ in C_r from Proposition 3.2 and therefore (H_1') implies $F(\phi, \lambda) \in BC_0$. □

Corollary 3.5. *Under (H_0') the operator $G : BC^1(I_r, \Omega) \times \Lambda \to BC$,*

$$G(\phi, \lambda) := S\phi - F(\phi, \lambda)$$

is well-defined and m-times continuously differentiable on $BC^1(I_r, \Omega)^\circ \times \Lambda$. If (H_0')–(H_1') are satisfied, then the same holds for $G : BC_0^1(I_r, \Omega) \times \Lambda \to BC_0$.

Proof. Thanks to the continuous embedding $BC^1 \hookrightarrow BC$, Proposition 3.4 yields that also $F : BC^1(I_r, \Omega) \times \Lambda \to BC$ is well-defined and m-times continuously differentiable. With Lemma 3.1 this implies our claim. Under (H_0')–(H_1') the same holds with the corresponding spaces BC_0 and BC_0^1. □

After these preparations, we arrive at a counterpart to Theorem 2.4 for FDEs:

Theorem 3.6. *For $\lambda \in \Lambda$ the following holds under (H_0'):*

(a) *If $\phi \in BC(I_r, \Omega)$ is a solution of $(F)_\lambda$, then $\phi \in BC^1(I_r, \Omega)$ and*

$$G(\phi, \lambda) = 0; \tag{3.2}$$

conversely, if $\phi \in C^1(I_r, \Omega) \cap BC$ solves (3.2)*, then $\phi \in BC^1(I_r, \Omega)$ and ϕ is a bounded solution of $(F)_\lambda$.*

(b) *Under additionally (H_1'), if $\phi \in BC_0(I_r, \Omega)$ solves $(F)_\lambda$, then $\phi \in BC_0^1(I_r, \Omega)$ and* (3.2) *holds; conversely, if $\phi \in C^1(I_r, \Omega) \cap BC_0$ solves* (3.2)*, then $\phi \in BC_0^1(I_r, \Omega)$ and ϕ is a bounded solution of $(F)_\lambda$.*

Proof. (a) Referring to Proposition 3.2, the function $\phi_\cdot : I_r \to C_r(\Omega)$ is bounded and (H_0') yields that $\dot\phi$ is bounded, since $\dot\phi(t) \equiv f(t, \phi_t, \lambda)$ on I_r. This solution identity is obviously equivalent to (3.2) and we obtain $\phi \in BC^1(I_r, \Omega)$. The converse direction follows from the same arguments.

(b) can be shown analogously. □

3.2. **Linear functional differential equations.** Let $I \subseteq \mathbb{R}$ be an interval. In order to study hyperbolic solutions of $(F)_\lambda$ we have to introduce some further terminology for linear FDEs (cf. [HVL93, pp. 167ff, Chapter 6]). Given a continuous mapping $A : I \to L(C_r, \mathbb{R}^d)$ they are of the form

$$(LF) \qquad \boxed{\dot{u}(t) = A(t)u_t.}$$

Under the boundedness assumption $b := \sup_{t \in I} |A(t)| < \infty$, we deduce from [HVL93, p. 170, Theorem 1.2] that the general solution of (LF), denoted by $\varphi_A(\cdot; s, \theta) : [s, \infty) \cap I \to \mathbb{R}^d$, exists and we define the *transition operator* $\Phi(t, s) \in L(C_r)$ of (LF) by

$$\Phi(t, s)\theta := \varphi_A(\cdot; s, \theta)_t \quad \text{for all } s \le t,\ s, t \in I.$$

Using [HVL93, p. 172, Corollary 1.1] we get the estimate $|\Phi(t, s)|_r \le e^{b(t-s)}$ for $s \le t$ and in the terminology of [CL99], $(\Phi(t, s))_{s \le t}$ defines a strongly continuous, exponentially bounded evolution family on C_r. In addition, the operators $\Phi(t, s) \in L(C_r)$ are compact for $t - s \ge r$ (see [HVL93, p. 91, Corollary 6.2]).

We say (LF) or the associated transition operator Φ admits an *exponential dichotomy* (ED for short) on I, if there exists a projection-valued mapping $P : I \to L(C_r)$ and real numbers $\alpha > 0$, $K \ge 1$ so that

$$\Phi(t, s)P(s) = P(t)\Phi(t, s) \quad \text{for all } s \le t,$$

the restriction $\Phi(t, s)|_{N(P(s))}$ is an isomorphism onto $N(P(t))$ for $s \le t$, and

$$|\Phi(t, s)P(s)|_r \le Ke^{-\alpha(t-s)}, \qquad |\Phi(s, t)[I - P(t)]|_r \le Ke^{\alpha(s-t)} \quad \text{for all } s \le t.$$

Due to the compactness of $\Phi(t, s)$, $t - s \ge r$, we can suppose the unstable fibers $N(P(s))$, $s \in I$, to be finite-dimensional (cf. [Hen81, p. 226]). In this framework, the *dichotomy spectrum* of Φ or (LF) is given by

$$\Sigma_I(A) := \{\gamma \in \mathbb{R} : \Phi_\gamma \text{ has no ED on } I\}$$

with a scaled transition operator $\Phi_\gamma(t, s) := e^{\gamma(s-t)}\Phi(t, s)$ for all $s \le t$.

As counterpart to the difference operator (2.4) in the present framework of linear FDEs, we introduce the differential operator

$$(3.3) \qquad L : \mathcal{C}^1 \to \mathcal{C}, \qquad (L\phi)(t) := (S\phi)(t) - A(t)\phi_t \quad \text{for all } t \in I,$$

which is well-defined thanks to Lemma 3.1 and the boundedness of A.

Proposition 3.7. *If a linear FDE (LF) admits an ED on $\mathbb{R}$, then $L \in GL(\mathcal{C}^1, \mathcal{C})$.*

Proof. First we suppose $\mathcal{C} = BC$ and referring to the admissibility result [CL99, p. 108, Theorem 4.28], we know that for each $\bar{\psi} \in BC(\mathbb{R}, C_r)$ there exists a unique function (a mild solution) $\bar{\phi} \in BC(\mathbb{R}, C_r)$ solving the integral equation

$$\bar{\phi}(t) = \Phi(t, s)\bar{\phi}(s) + \int_s^t \Phi(t, \tau)\bar{\psi}(\tau)\, d\tau \quad \text{for all } s \le t$$

in the space C_r. This, in turn, guarantees that for each inhomogeneity $\psi \in BC$ there exists a unique solution $\bar{\phi} \in BC(\mathbb{R}, C_r)$ to $\bar{\phi}(t) = \varphi_A(\cdot; s, \bar{\phi}(s))_t + \int_s^t \Phi(t, \tau)\psi_\tau\, d\tau$ for all $s \le t$. Therefore, $\phi := \bar{\phi}(t)(0)$ is the unique bounded complete solution of

$$(3.4) \qquad \dot{u}(t) = A(t)u_t + \psi(t)$$

and for every inhomogeneity $\psi \in BC$ there exists a unique $\phi \in BC$ such that $L\phi = \psi$. Since ϕ is a solution of (3.4), one deduces $\phi \in BC^1$ and the claim follows.

In the remaining case $\mathcal{C} = BC_0$, one proceeds as above with the crucial admissibility property from [CL99, p. 112, Theorem 4.33(a)]. □

3.3. **Poincaré continuation for functional differential equations.** Let $I \subseteq \mathbb{R}$ be an interval and $\phi^* : I \to \mathbb{R}^d$ denote a bounded solutions of the FDE $(F)_{\lambda^*}$. Given a parameter $\lambda^* \in \Lambda$, we consider the *variational equation*

$$\dot{u}(t) = D_2 f(t, \phi_t^*, \lambda^*) u_t. \tag{3.5}$$

The solution ϕ^* of $(F)_{\lambda^*}$ is said to be *hyperbolic*, if (3.5) admits an ED on I. We define the *dichotomy spectrum* associated to ϕ^* by

$$\Sigma_I(\phi^*, \lambda^*) := \Sigma_I(A) \quad \text{with } A(t) := D_2 f(t, \phi_t^*, \lambda^*)$$

and ϕ^* is hyperbolic, if and only if $0 \notin \Sigma_I(\phi^*, \lambda^*)$ holds.

Theorem 3.8 (hyperbolic solutions on $\mathbb{R}$)**.** *Let $\lambda^* \in \Lambda$ and suppose (H_0') holds. If $\mathcal{C} = BC$ and $\phi^* \in \mathcal{C}(\mathbb{R}, \Omega)$ is a complete permanent solution of $(F)_{\lambda^*}$ such that*

$$0 \notin \Sigma_{\mathbb{R}}(\phi^*, \lambda^*),$$

then there exist $\rho, \varepsilon > 0$ and a C^m-function $\phi : B_\rho(\lambda^*) \to B_\varepsilon(\phi^*) \subseteq \mathcal{C}^1(\mathbb{R}, \Omega)$ with:

(a) $\phi(\lambda^*) = \phi^*$,
(b) *$\phi(\lambda)$ is the unique bounded complete solution of equation $(F)_\lambda$ in $B_\varepsilon(\phi^*) \times B_\rho(\lambda^*)$,*
(c) *$\phi(\lambda)$ is hyperbolic.*

If (H_0')–(H_1') are satisfied, then the same holds with $\mathcal{C} = BC_0$.

Proof. Let $\lambda^* \in \Lambda$ be fixed. We suppose $\mathcal{C} = BC$ and from Theorem 3.6(a) we know $\phi^* \in BC^1(\mathbb{R}, \Omega)$ and that $G(\phi^*, \lambda^*) = 0$ holds. Thanks to Proposition 3.4 it is

$$D_1 G(\phi^*, \lambda^*)\psi = S\psi - F^{(1,0)}(\phi^*, \lambda^*)\psi. = L\psi \quad \text{for all } \psi \in BC^1,$$

with a linear operator $(L\psi)(t) := S\psi(t) - D_2 f(t, \phi_t^*, \lambda^*)\psi_t$ (cf. (3.3)). Hence, Proposition 3.7 guarantees $D_1 G(\phi^*, \lambda^*) \in GL(BC^1, BC)$ and the claims (a), (b) follow from Theorem A.1. Finally, assertion (c) can be shown as in Theorem 2.11 using the corresponding dichotomy roughness for evolution families (cf. [CL99, p. 156, Theorem 5.24]).

In case of the function space $\mathcal{C} = BC_0$ and $\phi^* \in BC_0(\mathbb{R}, \Omega)$ the same arguments remain true and Theorem 3.6(b) yields the assertion under (H_1'). □

As in the case of difference equations, one can compute a Taylor approximation of the function $\phi(\lambda)$ from Theorem 3.8. The corresponding coefficients are obtained by recursively solving linearly inhomogeneous delay equations. Nevertheless, in order to illustrate this, we retreat to an ODE model.

Example 3.1. Let us consider the following planar ODE

$$\begin{cases} \dot{x} = s_1 - \frac{s_2 y}{b_1 + y} - \mu x - kxy + r_0 x \\ \dot{y} = \frac{gy}{b_2 + y} - cxy \end{cases} \tag{3.6}$$

with real parameters $b_1, b_2, c, g, k, s_1, s_2, \mu > 0$ and $r_0 \in (0, \mu)$. We are interested in the stability properties of the equilibrium $(x^*, y^*) := \left(\frac{s_1}{\mu - r_0}, 0\right)$. First of all, the linearization of (3.6) in (x^*, y^*) has the eigenvalues $r_0 - \mu$, $\frac{g}{b_2} + \frac{cs_1}{r_0 - \mu}$. Therefore, (x^*, y^*) is a saddle for $\frac{g}{b_2} > \frac{cs_1}{\mu - r_0}$, asymptotically stable for $\frac{g}{b_2} < \frac{cs_1}{\mu - r_0}$, and in any case a hyperbolic equilibrium for $\frac{g}{b_2} \neq \frac{cs_1}{\mu - r_0}$. Both kinds of hyperbolicity persist under small perturbations in the whole set of parameters occurring in (3.6). In the nongeneric case $\frac{g}{b_2} = \frac{cs_1}{\mu - r_0}$ however, stability properties can be determined using a center manifold analysis (cf. [Wig90, pp. 193ff]) — we leave this aspect to the interested reader.

As nonautonomous counterpart to (3.6), [KW98] investigate the equation

$$\begin{cases} \dot{x} = s_1 - \frac{s_2 y}{b_1 + y} - \mu x - kxy + r_\lambda(t)x \\ \dot{y} = \frac{gy}{b_1 + y} - cxy \end{cases} \tag{3.7}$$

to model the effects of an interleukin immunotherapy for HIV. In (3.6) and (3.7), x and y are the respective concentrations of uninfected T cells and the HIV population; we refer to [KW98, p. 74] for a biological interpretation of further involved parameters. Specifically in equation (3.7), it is assumed that the enhancement of the immune system through interleukin results in an increase in the T cells proportional to the population of these cells at a time-dependent rate r_λ. Here, we suppose the function $r_\lambda : \mathbb{R} \to \mathbb{R}$ is of the form $r_\lambda(t) := r_0 + \lambda r(t)$, where $r : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and bounded. Thus, (3.7) degenerates into (3.6) for $\lambda = 0$. It is obvious that $\phi_\lambda := (\phi_\lambda^1, 0) : \mathbb{R} \to \mathbb{R}^2$ solves the system (3.7), where the first component $\phi_\lambda^1 : \mathbb{R} \to \mathbb{R}$ is the unique complete bounded solution to the linearly inhomogeneous scalar equation $\dot{x} = s_1 - \mu x + r_\lambda(t)x$, i.e.,

$$\phi_\lambda^1(t) = s_1 \int_{-\infty}^t \exp\left(\int_s^t r_\lambda(\sigma) - \mu \, d\sigma\right) ds \quad \text{for all } t \in \mathbb{R}.$$

We conclude that the complete solution $\phi(\lambda) : \mathbb{R} \to \mathbb{R}^2$ postulated in Theorem 3.8 is given by the pair $(\phi_\lambda^1, 0)$, i.e., the equilibrium $(x^*, y^*) = (\frac{s_1}{\mu - r_0}, 0)$ for (3.6) persists as bounded solution $(\phi_\lambda^1, 0)$ to (3.7) under time-dependent perturbations. Moreover, it is possible to obtain a Taylor approximation

$$\phi_\lambda^1(t) = \frac{s_1}{\mu - r_0} + \sum_{n=1}^m \frac{\psi_n(t)}{n!} \lambda^n + R_m(t, \lambda)$$

of arbitrary order $m \in \mathbb{N}$, where R_m denotes the remainder. The Taylor coefficients $\psi_n(t)$, $n = 1, \ldots, m$, can be computed recursively from

$$\psi_n(t) := n \int_{-\infty}^t e^{(r_0 - \mu)(t-s)} \psi_{n-1}(s) r(s) \, ds \quad \text{for all } t \in \mathbb{R},$$

where $\psi_0(t) :\equiv \frac{s_1}{\mu - r_0}$.

In Figure 1—4 we have depicted the 5ths order Taylor approximations of the complete solutions $\phi_\lambda^1 : \mathbb{R} \to \mathbb{R}$ for different functions r. We have chosen parameter values $s_1 = 1$, $r_0 = 1$ and $\mu = 2$, so that $(x^*, y^*) = (1, 0)$.

3.4. **Stable manifolds for ordinary differential equations.** From now on we deal with ordinary differential equations, i.e., FDEs with vanishing delay $r = 0$. Under this assumption, we can identify the two sets $C_r(\Omega)$ and Ω. Thus, the variational equation (3.5) becomes $\dot{u} = D_2 f(t, \phi^*(t), \lambda^*)u$, while the crucial functional differential equations $(F)_\lambda$ and (LF) simplify to the ODEs

$$(O)_\lambda \qquad \dot{u} = f(t, u, \lambda),$$

$$(LD) \qquad \dot{u} = A(t)u,$$

respectively. In conclusion, our above results remain applicable. In particular, as in the case of difference equations (cf. Corollary 2.13), Taylor approximations of perturbed solutions can be obtained by successively solving linearly inhomogeneous ODEs. Beyond that, stable manifold results are based on the technical

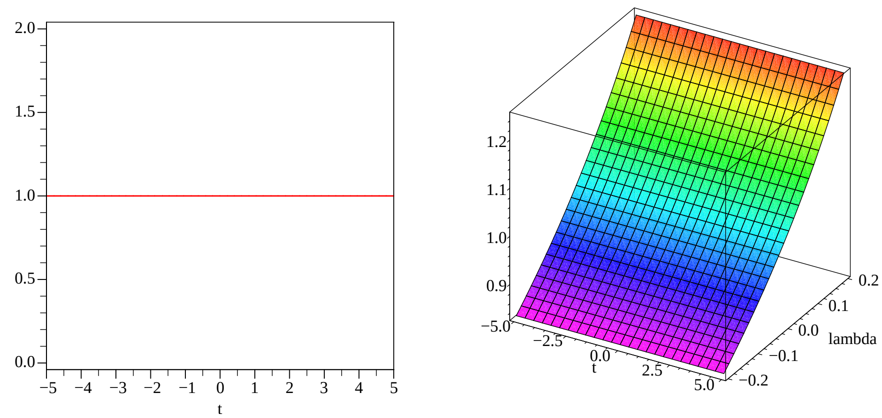

FIGURE 1. **Autonomous case:** Graph of the constant perturbation $r(t) = 1$ (left) and the resulting constant solutions $\phi^1_\lambda : \mathbb{R} \to \mathbb{R}$ (right).

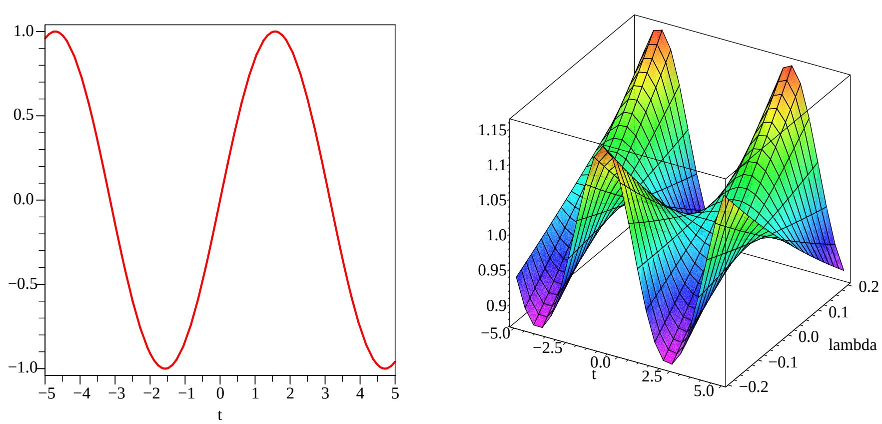

FIGURE 2. **Periodic case:** Graph of the 2π-periodic perturbation $r(t) = \sin(t)$ (left) and the resulting periodic solutions $\phi^1_\lambda : \mathbb{R} \to \mathbb{R}$ (right).

Proposition 3.9. *Let $t_0 \in \mathbb{R}$. If a linear ODE (LD) admits an ED on $[t_0, \infty)$, then L has a complemented kernel $N(L) \subseteq \mathcal{C}$ and satisfies*

$$N(L) = \left\{\Phi(\cdot, t_0)\xi \in \mathcal{C}^1 : \xi \in R(P(t_0))\right\}, \qquad R(L) = \mathcal{C}.$$

Proof. This follows as in Proposition 2.7 using [Cop78, p. 22, Proposition 3]. □

Theorem 3.10 (hyperbolic solutions on semiaxes)**.** *Let $\lambda^* \in \Lambda$, $t_0 \in \mathbb{R}$ and suppose (H_0') holds. If $\mathcal{C} = BC$ and $\phi^* \in \mathcal{C}([t_0, \infty), \Omega)$ is a permanent solution of $(O)_{\lambda^*}$ on the semiaxis $[t_0, \infty)$ with*

$$0 \notin \Sigma_{[t_0,\infty)}(\phi^*, \lambda^*) \quad \text{and associated invariant projector } P, \tag{3.8}$$

then there exist $\delta, \rho > 0$ and a unique C^m-function $\psi : B_\rho(0, \lambda^*) \subseteq R(P(t_0)) \times \Lambda \to B_\delta(\phi^*) \subseteq \mathcal{C}^1([t_0, \infty), \Omega)$ such that one has for all $(\xi, \lambda) \in B_\rho(0, \lambda^*)$:

(a) $\psi(0, \lambda^*) = \phi^*$,
(b) *$\psi(\xi, \lambda)$ is a hyperbolic solution of equation $(O)_\lambda$.*

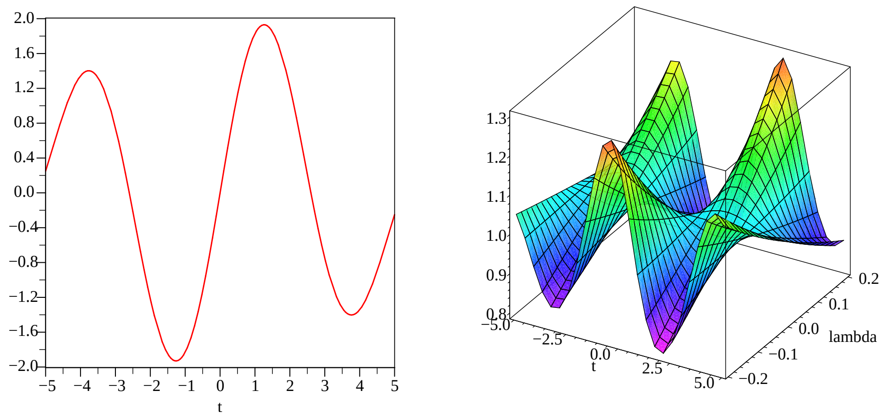

FIGURE 3. **Quasiperiodic case:** Graph of the quasiperiodic perturbation $r(t) = \sin(t) + \sin(\sqrt{2}t)$ (left) and the resulting complete solutions $\phi_\lambda^1 : \mathbb{R} \to \mathbb{R}$ (right).

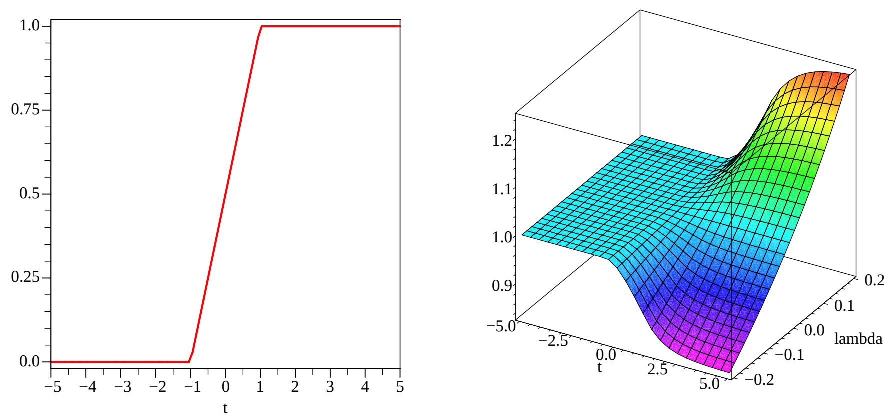

FIGURE 4. **Asymptotically autonomous case:** Graph of the perturbation $r(t) = \frac{1}{4}(|t+1| - |t-1|)$ (left) and the resulting complete solutions $\phi_\lambda^1 : \mathbb{R} \to \mathbb{R}$ (right).

If (H_0')–(H_1') *are satisfied, then the same holds with* $\mathcal{C} = BC_0$.

Proof. Using Proposition 3.9 one proceeds as in the proof of Theorem 2.14. □

Similarly to the case of nonautonomous difference equations in Section 2, given a hyperbolic bounded solution $\phi^* : [t_0, \infty) \to \Omega$ of $(O)_{\lambda^*}$, we define its *stable set*

$$S^+(\lambda) := \left\{ (t_0, \xi) \in \mathbb{R} \times \Omega : \varphi(t; t_0, \xi, \lambda) - \phi^*(t) \xrightarrow[t \to \infty]{} 0 \right\} \quad \text{for all } \lambda \in \Lambda,$$

whose local structure allows a description as in Corollary 2.16:

Corollary 3.11 (stable manifolds)**.** *If* $\phi^* \in BC([t_0, \infty), \Omega)$ *is a permanent solution of* $(O)_{\lambda^*}$ *on* $[t_0, \infty)$ *satisfying* (3.8)*, then there exist* $\varepsilon, \rho > 0$ *and a unique* C^m*-function*

$s^+_{t_0} : B_\rho(0, \lambda^*) \subseteq R(P(t_0)) \times \Lambda \to N(P(t_0))$ *such that the fibers of* $S^+(\lambda)$ *fulfill*

$$S^+(\lambda)_{t_0} \cap B_\varepsilon(\phi^*_{t_0}) = \left\{\phi^*(t_0) + \xi + s^+_{t_0}(\xi, \lambda) \in \Omega : \xi \in B_\rho(0) \subseteq R(P(t_0))\right\}$$

for all $\lambda \in B_\rho(\lambda^*)$*, with the fibers* $S^+(\lambda)_{t_0} := \{x \in \Omega : (t_0, x) \in S^+(\lambda)\}$.

Proof. Using Theorem 3.10 and Theorem A.3, the proof is analogous to the corresponding discrete result in Corollary 2.16. □

4. One-step discretizations of ordinary differential equations

Our results from Section 3 also apply in the delay-free case $r = 0$ of ODEs. We retreat to this situation and furthermore restrict to autonomous equations

$$\boxed{\dot{u} = g(u)} \tag{4.1}$$

with right-hand sides $g : \mathbb{R}^d \to \mathbb{R}^d$, which for simplicity are assumed to be defined on the whole space. Thus, bounded solutions to (4.1) are permanent.

Using the perturbation results from Section 2 we give an ad hoc approach on the behavior of hyperbolic solutions and stable manifolds under variable-stepsize one-step discretization. Here, it makes our explanations less technical, if we impose conditions directly on the flow generated by (4.1), instead of on the right-hand side g. This brings us to the following standing

Hypothesis. *Let* $m \in \mathbb{N}$, $H_0 > 0$, $g \in C^m(\mathbb{R}^d, \mathbb{R}^d)$, $\Xi \in C^m(\mathbb{R}^d \times (0, H_0], \mathbb{R}^d)$ *and suppose the flow* φ *generated by* (4.1) *satisfies for all* $0 \le j \le m$ *that*

- $\varphi(\cdot, \xi)$ *exists on* $[0, H_0]$ *for all* $\xi \in \mathbb{R}^d$,
$$\sup_{(t,\xi)\in B} \left|D^j\varphi(t, \xi)\right| < \infty \quad \text{for all bounded } B \subseteq [0, H_0] \times \mathbb{R}^d$$
and for every $\varepsilon > 0$ *there exists a* $\delta > 0$ *with*
$$|\xi - \eta| < \delta \quad \Rightarrow \quad \sup_{t\in[0,H_0]} \left|D^j\varphi(t, \xi) - D^j\varphi(t, \eta)\right| < \varepsilon \quad \text{for all } \xi, \eta \in \mathbb{R}^d.$$
- *For all bounded* $B \subseteq \mathbb{R}^d \times (0, H_0]$ *one has* $\sup_{(\xi,t)\in B} \left|D^j\Xi(\xi, t)\right| < \infty$ *and for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with*
$$|\xi - \eta| < \delta \quad \Rightarrow \quad \sup_{t\in(0,H_0]} \left|D^j\Xi(\xi, t) - D^j\Xi(\eta, t)\right| < \varepsilon \quad \text{for all } \xi, \eta \in \mathbb{R}^d.$$
- *The* local discretization error $E(x, h) := \frac{\varphi(h,x)-x}{h} - \Xi(x, h)$ *satisfies*
$$\lim_{h\searrow 0} E(x, h) = 0 \quad \text{for all } x \in \mathbb{R}^d.$$

Let $\mathbb{I}$ be a discrete interval unbounded above with $0 \in \mathbb{I}$. In order to understand the behavior of (4.1) under numerical discretization schemes with varying step-sizes, we introduce the space $\ell^\infty_\Delta := \{(\tau_k)_{k\in\mathbb{I}} : (\tau_{k+1} - \tau_k)_{k\in\mathbb{I}} \in \ell^\infty\}$ of real sequences. It becomes a Banach space w.r.t. the norm

$$|\tau|_\Delta := \max\Big\{|\tau_0|, \sup_{k\in\mathbb{I}} |\tau_{k+1} - \tau_k|\Big\}.$$

We fix a real number $H \in (0, H_0]$ and define the open convex subsets (whose elements are called *time meshes*) as H-balls in ℓ^∞_Δ, namely

$$T_H := \{(\tau_k)_{k\in\mathbb{I}} : |\tau|_\Delta < H\} \subseteq \ell^\infty_\Delta.$$

Now consider a nonautonomous difference equation $(\Delta)_\lambda$, with specific right-hand side

$$f_k(x,\lambda) = \varphi(\tau_{k+1}-\tau_k, x) + \tfrac{\theta}{H}(\tau_{k+1}-\tau_k)E(x,\tau_{k+1}-\tau_k) \tag{4.2}$$

depending on a parameter $\lambda = ((\tau_k)_{k\in\mathbb{I}}, \theta) \in \ell^\infty_\Delta \times \mathbb{R}$.

The idea behind the right-hand side (4.2) is to provide a homotopy between the continuous flow φ of (4.1) evaluated at discrete times τ_k ($\theta = 0$), and its numerical approximation in terms of the one-step method Ξ (where $\theta = H$) yielding the recursion

$$x_{k+1} = x_k + (\tau_{k+1}-\tau_k)\Xi(x_k, \tau_{k+1}-\tau_k). \tag{4.3}$$

For instance, in case of the Euler method one has $\Xi(x,h) = g(x)$. We now define the open and convex parameter space $\Lambda := T_H \times (-2H, 2H)$.

Lemma 4.1. *The map $f_k : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ from* (4.2) *is of class C^m and satisfies (H_0).*

Proof. It is easy to see that $\Delta_k : \ell^\infty_\Delta \to \mathbb{R}$, $\Delta_k\tau := \tau_{k+1}-\tau_k$ is a linear bounded operator with norm $|\Delta_k| \le 1$ for all $k \in \mathbb{I}$. Let $\tau \in \ell^\infty_\Delta$, $\theta \in \mathbb{R}$ and set $\lambda := (\tau,\theta)$. By the mean value theorem (cf. [Lan93, p. 341, Theorem 4.2]), the mapping (4.2) can be written as

$$f_k(x,\lambda) = \varphi(\Delta_k\tau, x) + \tfrac{\theta}{H}\Delta_k\tau\left(\int_0^1 g(\varphi(s\Delta_k\tau, x))\,ds - \Xi(x,\Delta_k\tau)\right) \quad \text{for all } k\in\mathbb{I}.$$

As consequence of chain and product rule (cf. [Lan93, pp. 336–337]), $f_k : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ is m-times continuously differentiable. Furthermore, referring to our standing hypothesis on the flow φ and the one-step scheme Ξ, f_k satisfies (H_0). □

Lemma 4.2. *Let I be a real interval unbounded above, $\tau \in T_H$ with*

$$\{\tau_k\}_{k\in\mathbb{I}} \subseteq I, \qquad \inf_{k\in\mathbb{I}}(\tau_{k+1}-\tau_k) > 0 \tag{4.4}$$

and set $\lambda^* := (\tau, 0) \in \Lambda$. If $\mathcal{C} = BC$ and $\psi^* \in \mathcal{C}(I,\mathbb{R}^d)$ is a hyperbolic solution of (4.1), *then $\phi^*_k := \psi^*(\tau_k)$ is a hyperbolic solution of $(\Delta)_{\lambda^*}$ in ℓ^∞, i.e., the variational equation*

$$x_{k+1} = D_1 f_k(\psi^*(\tau_k), \lambda^*)x_k \tag{4.5}$$

admits an ED on $\mathbb{I}$. In case $g(0) = 0$ and $\mathcal{C} = BC_0$ the same holds with $\phi^* \in \ell_0$.

Proof. Above all, we set $h := \inf_{k\in\mathbb{I}}(\tau_{k+1}-\tau_k) > 0$. Then hyperbolicity of the solution $\psi^* : I \to \mathbb{R}^d$ means that the variational equation

$$\dot{x} = Dg(\psi^*(t))x \tag{4.6}$$

corresponding to (4.1), whose transition operator is denoted by $\Psi(t,s)$, has an ED on I, say with data K, α and projector P. From (4.2) we deduce

$$D_1 f_k(\psi^*(\tau_k), \lambda^*) = D_2\varphi(\tau_{k+1}-\tau_k, \psi^*(\tau_k)) = \Psi(\tau_{k+1}, \tau_k) \quad \text{for all } k \in \mathbb{I}$$

and abbreviating the invertible transition operator of the variational equation (4.5) by Φ, we get $\Phi(k,l) = \Psi(\tau_k, \tau_l)$ for $k, l \in \mathbb{I}$. This yields a commutativity relation (2.3) and

$$\|\Phi(k,l)P(\tau_l)\| = \|\Psi(\tau_k,\tau_l)P(\tau_l)\| \le Ke^{-\alpha(\tau_k-\tau_l)} \le K(e^{-\alpha H})^{k-l},$$
$$\|\Phi(l,k)[I-P(\tau_k)]\| = \|\Psi(\tau_l,\tau_k)[I-P(\tau_k)]\| \le Ke^{\alpha(\tau_l-\tau_k)} \le K(e^{\alpha h})^{k-l}$$

for all $l \le k$. Thus, the ED of Ψ carries over to the difference equation (4.5). □

Next we show that hyperbolic solutions to (4.1) persist under the large class of discretization schemes considered in (4.3). Thereto, we make use of the class $\mathcal{K}_r$ from (A.1).

Theorem 4.3 (discretized hyperbolic solutions on $\mathbb{Z}$)**.** *Let $I = \mathbb{R}$, $\mathbb{I} = \mathbb{Z}$, $\mathcal{C} = BC$, $\ell = \ell^\infty$ and suppose $\psi^* \in \mathcal{C}$ is a hyperbolic complete solution of* (4.1). *For each sufficiently small $r > 0$ one finds functions $H, \rho_0 \in \mathcal{K}_r$ such that for $\tau^* \in T_{H(r)}$ satisfying* (4.4) *there exists a C^m-mapping $\hat{\phi} : B_{\rho_0(r)}(\tau^*) \subseteq T_{H(r)} \to \ell(\mathbb{R}^d)$ with*

(a) $\hat{\phi}(\tau)$ *is a complete solution of the one-step scheme* (4.3) *in $\ell(\mathbb{R}^d)$ and*

$$\left|\hat{\phi}(\tau)_k - \psi^*(\tau_k^*)\right| < r \quad \textit{for all } k \in \mathbb{Z},$$

(b) $\hat{\phi}(\tau)$ *is hyperbolic for all time meshes $\tau \in B_{\rho_0(r)}(\tau^*)$.*

In case $g(0) = 0$ and $\mathcal{C} = BC_0$ the same holds with $\ell = \ell_0$.

Proof. (I) Let $h > 0$ be arbitrary, choose $\tau^* \in T_h$ satisfying (4.4) and define $\lambda^* := (\tau^*, 0)$. From Lemma 4.2 we deduce that the sequence $\phi_k^* := \psi^*(\tau_k^*)$, $k \in \mathbb{Z}$, is a hyperbolic complete solution of $(\Delta)_{\lambda^*}$, where the right-hand side f_k is defined by (4.2) and depends on $\lambda = (\tau, \theta)$. Moreover, Lemma 4.1 guarantees that f_k satisfies (H_0). Thus, Theorem 2.11 is applicable yielding the existence of reals $\rho, \varepsilon > 0$ and a unique C^m-function $\phi : B_\rho(\lambda^*) \to B_\varepsilon(\phi^*) \subseteq \ell(\mathbb{R}^d)$ with $\phi(\lambda^*) = \phi^*$ and

- $\phi(\lambda)$ is the unique bounded complete solution of equation $(\Delta)_\lambda$,
- $\phi(\lambda)$ is hyperbolic for all $\lambda \in B_\rho(\lambda^*)$.

Here, due to $\frac{\tau_{k+1} - \tau_k}{h} \leq 1$ the right-hand side of (4.2) does not blow up in the limit $h \searrow 0$ and we see that $\rho > 0$ does not depend on $h > 0$.

(II) Now we choose $r \in (0, \varepsilon)$ and a function $\rho_0 \in \mathcal{K}_r$ with values in $(0, \rho)$ as claimed in Remark A.1. From step (I) we know that ρ_0 does not depend on h and as a result we find a function $H \in \mathcal{K}_r$ with values in $(0, \rho_0(r))$. Given $\tau \in B_{\rho_0(r)}(\tau^*)$ this yields the inclusion $(\tau, H(r)) \in B_{\rho_0(r)}(\lambda^*)$ and the mapping $\hat{\phi}(\tau) := \phi(\tau, H(r))$ fulfills our claims. □

Theorem 4.4 (discretized stable manifolds)**.** *Let $t_0 \in \mathbb{R}$, $\mathcal{C} = BC$, $\ell = \ell^\infty$ and suppose*

(i) $\psi^* \in \mathcal{C}([t_0, \infty), \mathbb{R}^d)$ *is a hyperbolic solution of* (4.1) *with projector P,*
(ii) $\psi : B_{\rho_0}(0) \subseteq R(P(t_0)) \to \mathcal{C}([t_0, \infty), \mathbb{R}^d)$ *is the C^m-function from Theorem 3.10 whose images are hyperbolic solutions of* (4.1) *forming the stable set of ψ^*.*

For each sufficiently small $r > 0$ one finds functions $H, \rho \in \mathcal{K}_r$ such that for $\tau^* \in T_{H(r)}$ satisfying (4.4) *there exists a C^m-function $\phi : B_{\rho(r)}(0, \tau^*) \subseteq R(P(t_0)) \times T_{H(r)} \to \ell$, $\rho \leq \rho_0$, such that the following holds:*

(a) $\phi(\xi, \tau)$ *is a solution of the one-step scheme* (4.3) *in ℓ and*

$$|\phi(\xi, \tau)_k - \psi(\xi)(\tau_k^*)| < r \quad \textit{for all } k \in \mathbb{I}, \tag{4.7}$$

(b) $\phi(\xi, \tau)$ *is a hyperbolic for all $(\xi, \tau) \in B_{\rho(r)}(0, \tau^*)$.*

In case $g(0) = 0$ and $\mathcal{C} = BC_0$ the same holds with $\ell = \ell_0$.

Remark 4.1. One sees as in Corollary 2.16 that the stable set of a one-step scheme (4.3) is locally graph of a function given by $\hat{s}_k^+(\xi, \tau) := \phi(\xi, \tau)_k$. This set is close to the stable set of a solution to the ODE (4.1) in the sense that the estimate (4.7) holds.

Proof. Replace Theorem 2.11 by Theorem 2.14 in the proof of Theorem 4.3. □

APPENDIX A. IMPLICIT FUNCTION THEOREMS

We supplement this paper with two versions of the implicit function theorem, among them a quantitative one.

Suppose X, Y, Z are Banach spaces, let $\Omega \subseteq X \times Y$ be nonempty open and $G : \Omega \to Z$ is assumed to be a C^m-mapping, $m \geq 1$.

Theorem A.1 (implicit function theorem)**.** *If a pair $(x_0, y_0) \in \Omega$ fulfills $G(x_0, y_0) = 0$ and moreover $D_1G(x_0, y_0) \in GL(X, Z)$ holds, then there exist $\rho, \varepsilon > 0$ and a unique C^m-mapping $\phi : B_\rho(y_0) \to B_\varepsilon(x_0)$ with $\phi(y_0) = x_0$ and $G(\phi(\eta), \eta) \equiv 0$ on $B_\rho(y_0)$.*

Remark A.1. In order to illuminate a further aspect of Theorem A.1 we introduce the class

$$\mathcal{K}_r := \left\{ f : (0, r] \to (0, \infty) \Big| \lim_{s \searrow 0} f(s) = 0 \right\}. \tag{A.1}$$

A closer look to the proof of Theorem A.1 shows that for every $r \in (0, \varepsilon)$ there exist a function $\rho_0 \in \mathcal{K}_r$ with values in $(0, \rho)$ such that $\phi(B_{\rho_0(r)}(y_0)) \subseteq B_r(x_0)$ holds.

Proof. See, for instance, [Zei93, pp. 150–151, Theorem 4.B]. □

Corollary A.2 (size of ρ, ε)**.** *If there exist functions $\omega_1 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, $\omega_2 : \mathbb{R} \to [0, \infty)$ nondecreasing in each argument so that*

$$|D_1G(x, y) - D_1G(x_0, y_0)| \leq \omega_1(|x - x_0|, |y - y_0|),$$
$$|G(x, y_0)| \leq \omega_2(|x - x_0|) \quad \textit{for all } x \in B_\rho(x_0),\ y \in B_\varepsilon(y_0),$$

then the reals $\rho, \varepsilon > 0$ from Theorem A.1 satisfy

$$\left| D_1G(x_0, y_0)^{-1} \right| \omega_1(\rho, \varepsilon) \leq \omega < 1, \qquad \left| D_1G(x_0, y_0)^{-1} \right| \omega_2(\rho) \leq \varepsilon(1 - \omega).$$

Proof. See [Hol70]. □

Theorem A.3 (surjective implicit function theorem)**.** *If a pair $(x_0, y_0) \in \Omega$ fulfills $G(x_0, y_0) = 0$ and $D_1G(x_0, y_0) \in L(X, Z)$ is onto with complemented kernel and projection operator $P \in L(X)$ onto $N(D_1G(x_0, y_0))$, then there exist $\rho > 0$, neighborhoods $U \subseteq R(P)$, $V \subseteq N(P)$ of zero and a unique C^m-mapping $\phi : U \times B_\rho(y_0) \to V$ satisfying*

$$G(x_0 + \xi + \phi(\xi, \eta), \eta) \equiv 0 \quad \textit{on } U \times B_\rho(y_0).$$

Proof. See [Zei93, p. 177, Theorem 4.H]. □

REFERENCES

[AG90] E.L. Allgower and K. Georg, *Numerical continuation methods. An introduction*, Springer Series in Computational Mathematics 13, Springer, Berlin etc., 1990.

[AHO98] A.I. Alonso, J. Hong, and R. Obaya, *Exponential dichotomy and trichotomy for difference equations*, Computers & Mathematics with Applications **38** (1998), 41–49.

[Ama90] H. Amann, *Ordinary differential equations: An introduction to nonlinear analysis*, Studies in Mathematics 13, Walter De Gruyter, Berlin-New York, 1990.

[AVM96] B. Aulbach and N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, Journal of Difference Equations and Applications **2** (1996), 251–262.

[AS01] B. Aulbach and S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*. Journal of Difference Equations and Applications **7** (2001), no. 6, pp. 895–913.

[Bas00] A.G. Baskakov, *Invertibility and the Fredholm property of difference operators*, Mathematical Notes **67** (2000), no. 6, 690–698.

[BAG91] A. Ben-Artzi and I. Gohberg, *Dichotomy, discrete Bohl exponents, and spectrum of block weighted shifts*, Integral Equations and Operator Theory **14** (1991), no. 5, 613–677.

[BS08] A. Berger and S. Siegmund, *Uniformly attracting solutions of nonautonomous differential equations*, Nonlinear Analysis (TMA) **68** (2008), no. 12, 3789–3811.

[BK97] W.-J. Beyn and J.-M. Kleinkauf, *The numerical computation of homoclinic orbits for maps*, SIAM Journal of Numerical Analysis **34** (1997), no. 3, 1209–1236.

[BM03] Z. Bishnani and R.S. Mackay, *Safety criteria for aperiodically forced systems*, Dynamical Systems **18** (2003), no. 2, 107–129.

[Boi01] O. Boichuk, *Solutions of linear and nonlinear difference equations bounded on the whole line*, Nonlinear Oscillations **4** (2001), no. 1, 16–27.

[CLRS06] A.N. Carvalho, J.A. Langa, J.C. Robinson and A. Suárez, *Characterization of non-autonomous attractors*, Cadernos de Matemática **07** (2006), 277–302.

[Chi06] C. Chicone, *Ordinary differential equations with applications*, 2nd ed., Texts in Applied Mathematics 34, Springer, Berlin etc., 2006.

[CL99] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs 70, AMS, Providence RI, 1999.

[Cop78] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Math. 629, Springer, Berlin, 1978.

[GP01] I. Győri and M. Pituk, *The converse of the theorem on stability by the first approximation for difference equations*, Nonlinear Analysis (TMA) **47** (2001), 4635–4640.

[Hag04] A. Hagen, *Hyperbolic trajectories of time discretizations*, Nonlinear Analysis (TMA) **59** (2004), no. 1–2, 121–132.

[HVL93] J.K. Hale and S.M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences 99, Springer, Berlin etc., 1993.

[HW04] J.K. Hale and M. Weedermann, *On perturbations of delay-differential equations with periodic orbits*, Journal of Differential Equations **197** (2004), no. 2, 219–246.

[Hen81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. 840, Springer, Berlin etc., 1981.

[HS74] M.W. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra*, Academic Press, Boston, 1974.

[Hol70] J.M. Holtzmann, *Explicit ε and δ for the implicit function theorem*, SIAM Review **12** (1970), no. 2, 284–286.

[Hül08] T. Hüls, *Homoclinic trajectories of non-autonomous maps*, Preprint, Universität Bielefeld, 2008.

[Ioo79] G. Iooss, *Bifurcation of maps and applications*, Mathematics Studies 36, North-Holland, Amsterdam, 1979.

[JSW03] N. Ju, D. Small and S. Wiggins, *Existence and computation of hyperbolic trajectories of aperiodically time dependent vector fields and their approximations*, International Journal of Bifurcation and Chaos **13** (2003), no. 6, 1449–1457.

[Kal94] J. Kalkbrenner, *Exponentielle Dichotomie und chaotische Dynamik nichtinvertierbarer Differenzengleichungen*, Ph.D. thesis, Universität Augsburg, Germany, 1994.

[Kie04] H. Kielhöfer, *Bifurcation theory: An introduction with applications to PDEs*, Applied Mathematical Sciences 156, Springer, New York, 2004.

[KW98] D. Kirschner and G. Webb. *Immunotherapy of HIV-1 infection*, Journal of Biological Systems **6** (1998, no. 1, 71–83.

[KOGV07] B. Krauskopf, H.M. Osinga and J. Galán-Vioque, *Numerical continuation methods for dynamical systems. Path following and boundary value problems*, Springer, Berlin etc., 2007.

[Lan93] S. Lang, *Real and functional analysis*, Graduate Texts in Mathematics 142, Springer, Berlin, 1993.

[Per97] P. Perfetti, *An infinite-dimensional extension of a Poincaré result concerning the continuation of periodic orbits*, Discrete and Continuous Dynamical Systems **3** (1997), no. 3, 401–418.

[Pöt09] C. Pötzsche, *A note on the dichotomy spectrum*, Journal of Difference Equations and Applications **to appear** (2009).

[PR05] C. Pötzsche and M. Rasmussen, *Taylor approximation of invariant fiber bundles*, Nonlinear Analysis (TMA) **60** (2005), no. 7, 1303–1330.

[SY02] G.R. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences 143, Springer, Berlin etc., 2002.

[Wig90] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Texts in Applied Mathematics 2, Springer, Berlin etc., 1990.

[Zei93] E. Zeidler, *Nonlinear functional analysis and its applications I (Fixed-points theorems)*, Springer, Berlin etc., 1993.

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