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NONAUTONOMOUS BIFURCATION OF BOUNDED SOLUTIONS II: A SHOVEL-BIFURCATION PATTERN

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Abstract. This paper continues our work on local bifurcations for nonautonomous difference and ordinary differential equations. Here, it is our premise that constant or periodic solutions are replaced by bounded entire solutions as bifurcating objects in order to encounter right-hand sides with an arbitrary time dependence.

We introduce a bifurcation pattern caused by a dominant spectral interval (of the dichotomy spectrum) crossing the stability boundary. As a result, differing from the classical autonomous (or periodic) situation, the change of stability appears in two steps from uniformly asymptotically stable to asymptotically stable and finally to unstable. During the asymptotically stable regime, a whole family of bounded entire solutions occurs (a so-called "shovel"). Our basic tools are exponential trichotomies and a quantitative version of the surjective implicit function theorem yielding the existence of strongly center manifolds.

1. **Motivation and introduction.** Bifurcation phenomena in dynamical systems typically go hand in hand with an exchange of stability induced by the fact that eigenvalues (or Floquet multipliers) of a variational equation along an equilibrium (or a periodic orbit) cross the stability boundary as parameters vary. Depending whether continuous or discrete systems are considered, this stability boundary is the imaginary axis resp. the unit circle in the complex plane, and stability properties change from being asymptotically stable to being unstable or conversely. Moreover, in nondegenerate cases, i.e. under a transversality condition, the crossing happens instantly and at the bifurcation value itself, the stability behavior depends on nonlinear effects and can be investigated using well-established techniques like center manifold reduction.

In case of nonautonomous difference or differential equations, the generic bifurcation phenomena slightly differ. Here, it is crucial to have a suitable hyperbolicity and spectral notion for the linear parts available, which appropriately replaces the discrete set of eigenvalues (or Floquet multipliers). It turned out that a natural concept for this purpose are exponential dichotomies, trichotomies and the associated dichotomy spectrum. Indeed, for nonautonomous equations, eigenvalues have to be

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replaced by spectral intervals of the dichotomy spectrum, also known as dynamical or Sacker-Sell spectrum (see [35]). More detailed, linearizing an equation along a heteroclinic solution, typically yields a dichotomy spectrum consisting of compact intervals with positive lengths. Thus, a nonautonomous bifurcation can consist of two transitions, namely a spectral interval entering, staying on, and leaving the stability boundary. As we will see, the corresponding loss (or gain) of stability happens in two steps as well.

We tackle nonautonomous bifurcation problems from an abstract functional-analytical perspective. A given nonautonomous difference or differential equation is formulated as operator equation in the space of bounded sequences resp. functions. Thus, a bifurcation is understood as a change in the number of bounded entire solutions when parameters vary. For the resulting operator equations it is essential to have a Fredholm theory for the linearization L in a (possible) bifurcation point available. From the perspective of an abstract branching theory, the operator L, in turn, offers three possibilities of interest:

(a) L is invertible
(b) L is noninvertible with Fredholm index 0
(c) L is onto with nontrivial kernel.

These functional-analytical conditions allow a dynamical interpretation, since Fredholm properties of L are closely connected to the notion of an exponential dichotomy (for this, see [24, 10]). Geometrically the latter notion means that the extended state space of a linear equation allows a hyperbolic splitting into a *stable bundle* of vector spaces containing forward solutions decaying to zero, and a complementary *unstable bundle* of limit zero backward solutions — both bundles are described using complementary projectors.

Above all, hyperbolicity in form of an exponential dichotomy on the whole time axis is a sufficiently robust property to prevent bifurcations (cf. [29, Props. 2.8, 3.7]). This corresponds to the above case (a) of an invertible operator L. Therefore, one approach to describe nonhyperbolicity and to obtain sufficient conditions for bifurcations, is to assume exponential dichotomies on both semiaxes. For the purpose of a suitable Fredholm theory capturing the above cases (b) and (c), on both time-axes the corresponding projectors onto the stable resp. unstable bundle have to fit together in an appropriate way. Regarding our case (b), the resulting 0-index Fredholm theory enabled us to tackle bifurcation problems via Lyapunov-Schmidt reduction yielding nonautonomous bifurcation scenarios of fold, transcritical and pitchfork type (among others, cf. [29]). Such an approach had the disadvantage that, beyond being applicable to at least 2-dimensional systems only, merely unstable solutions came into question as bifurcation points. This is somehow unsatisfying from the perspective of understanding bifurcations as a change in stability.

To address the remaining case (c), in this paper another form of nonhyperbolicity is investigated, namely an exponential trichotomy (cf. [15, 26, 16]). This concept weakens a dichotomy in the sense that there exists an additional *strongly center bundle* containing the entire solutions which decay exponentially in both time directions. In terms of the dichotomy spectrum, this means that a spectral interval exists on the stability boundary. We are interested in the situation when, under varying parameters, spectral intervals enter and cross this boundary, i.e. when we have a transition from an exponentially dichotomic into a trichotomic situation or conversely. In a functional-analytical language this implies a jump in the Fredholm index, which ensures that a unique entire bounded solution bifurcates into a whole family of bounded solutions. While the corresponding Fredholm operator L remains onto, its kernel becomes nontrivial. In order to describe this situation, our basic tool is a quantitative version of the surjective implicit function theorem (cf. Thm. 2.1 and Cor. 2.2). It guarantees the existence of a center-like fiber bundle (for differential equations we speak of a manifold) consisting of exponentially decaying entire solutions. In the bifurcation diagram the occurrence of such a strongly center fiber bundle (resp. manifold), i.e. of a family of bounded solutions, yields a shovel-shaped pattern — we therefore speak of a *shovel bifurcation* (cf. Fig. 4 or 5). Admittedly, the (dis)appearance of a family of bounded entire solutions is a quite crude description, but it features two characteristic prerequisites of a bifurcation phenomenon: A change in the number of bounded solutions, as well as a change in stability properties under varying parameters.

Our discussion divides into two parts, one devoted to difference equations (Section 3) and one presenting the analogous theory for ODEs (Section 4). We begin formulating our standing assumptions and the key idea behind our overall approach: Interpret difference or differential equations as operator equations in ambient function spaces. After that, the necessary linear theory is developed in terms of a suitable exponential trichotomy notion, different dichotomy spectra and the corresponding Fredholm theory. We relate the dichotomy spectra to classical stability notions, prove the existence of strongly center bundles (resp. manifolds) and proceed to the central topic of the paper — sufficient conditions are given that whole families of bounded entire solutions bifurcate in a super- or subcritical fashion.

We close this introduction with the remark that our earlier paper [29] contains a brief overview of various approaches to describe bifurcation and transition patterns for nonautonomous equations. For this reason, we restrict our discussion of the related literature to a supplementary two-step bifurcation pattern for stochastic equations observed by L. Arnold and his coworkers (cf. [3]): Given a family of stochastic ODEs in the plane $\mathbb{R}^2$ depending on a parameter $\lambda \in \mathbb{R}$, where the origin 0 is an equilibrium of each member of this family. Suppose that 0 looses its asymptotic stability for a particular parameter value $\lambda = \lambda_1$ (interpret these statements in the almost everywhere sense with respect to the measure μ on the path space). Then various examples feature the following behavior: For $\lambda > \lambda_1$ near λ_1 there is an attracting invariant random measure on $\mathbb{R}^2$, whose topological support is essentially a two-point set. The first step of the two-step pattern consists of the transfer of the stability to this random invariant measure. Second, there is a value $\lambda_2 > \lambda_1$ such that for $\lambda > \lambda_2$ close to λ_2, then there is a random attracting invariant (topological) circle, which contains two attractor-repeller pairs of points. The two-step bifurcation process is completed when λ crosses λ_2. We refer to the paper [20] discussing the above phenomenon, in particular, for deterministic but nonautonomous ODEs.

2. **Preliminaries.**

As usual, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ are the positive integers and a *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$; sometimes it is convenient to introduce the shifted interval $\mathbb{I}' := \{k \in \mathbb{I} :\, k+1 \in \mathbb{I}\}$. Given an integer $\kappa \in \mathbb{Z}$ we define the discrete intervals

$$\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} :\, \kappa \le k\}\,, \qquad \mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} :\, \kappa \ge k\}\,.$$

Real Banach spaces are denoted by X, Y, Z and equipped with norm $|\cdot|$; $B_\varepsilon(x)$ is the open ball with radius $\varepsilon > 0$ and center x. The space of bounded linear operators from X to Y is $L(X,Y)$, $L(X) := L(X,X)$ and for the corresponding toplinear isomorphisms we write $GL(X,Y)$, $GL(X) := GL(X,X)$. Given $T \in L(X,Y)$, we write $R(T) := TX$ for the *range* and $N(T) := T^{-1}(0)$ for the *kernel*.

Suppose $\Omega \subseteq X \times Y$ is a nonempty open convex set and the mapping $G : \Omega \to Z$ is of class C^m, $m \in \mathbb{N}$. We begin with a quantitative version of the surjective implicit function theorem. It combines results of [37, p. 177, Thm. 4.H] and [18].

Theorem 2.1 (surjective implicit function theorem)**.** *Suppose that $(x^*, y^*) \in \Omega$ fulfills $G(x^*, y^*) = 0$ and $D_1G(x^*, y^*) \in L(X,Z)$ is onto with complemented kernel. Provided $P \in L(X)$ denotes the projection onto $N(D_1G(x^*, y^*))$, then there exist reals $\varepsilon, \rho > 0$ and a unique C^m-mapping $\phi : B_\rho(0, y^*) \subseteq R(P) \times Y \to B_\varepsilon(0) \subseteq N(P)$ with*

$$G(x^* + \xi + \phi(\xi, \eta), \eta) \equiv 0 \quad \text{on } B_\rho(0, y^*)$$

and $\phi(0, y^*) = 0$. Moreover, for the derivatives of ϕ one has

$$D_i\phi(0, y^*) = [D_1G(x^*, y^*)(I - P)]^{-1} D_iG(x^*, y^*) \quad \text{for all } i = 1, 2. \tag{2.1}$$

Proof. Due to our assumption we have the splitting $X = X_1 \oplus N\big(D_1(G(x^*, y^*))\big)$ with closed subspaces $X_1 = N(P)$ and $N\big(D_1(G(x^*, y^*))\big) = R(P)$. Having this at our disposal, we define a C^m-mapping $F(x_1, x_2, y) := G(x^* + x_1 + x_2, y)$ depending on variables $x_1 \in N(P)$, $x_2 \in R(P)$ from a suitable open neighborhood of 0. This yields the relation

$$D_1F(x_1, x_2, y) = D_1G(x^* + x_1 + x_2, y)[I - P]$$

and $D_1F(0, 0, y^*) = D_1G(x^*, y^*) \in GL(X_1, Z)$. By the implicit function theorem (see, for instance, [37, pp. 150–151, Thm. 4.B]) there exist $\rho, \varepsilon > 0$, neighborhoods $B_\rho(0) \subseteq R(P)$, $B_\varepsilon(0) \subseteq N(P)$ and a unique C^m-mapping $\phi : B_\rho(0, y^*) \to B_\varepsilon(0)$ satisfying the identity

$$0 \equiv F(\xi, \phi(\xi, \eta), \eta) \equiv G(x^* + \xi + \phi(\xi, \eta), \eta) \quad \text{on } B_\rho(0, y^*).$$

Finally, if we differentiate the latter identity, then (2.1) follows. □

Corollary 2.2. *If there exist functions $\omega_1 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, $\omega_2 : \mathbb{R} \to [0, \infty)$, nondecreasing in each argument, so that*

$$\begin{aligned} |[D_1G(x, y) - D_1G(x^*, y^*)]\,[I - P]| &\le \omega_1(|x - x^*|, |y - y^*|), \\ |G(x, y^*)| &\le \omega_2(|x - x^*|) \end{aligned} \tag{2.2}$$

for all $x \in B_\rho(x^*)$, $y \in B_\varepsilon(y^*)$, then the reals $\rho, \varepsilon > 0$ from Thm. 2.1 can be determined using the conditions

$$\left|[D_1G(x^*, y^*)(I - P)]^{-1}\right| \omega_1(\rho, \varepsilon) \le \omega < 1,$$
$$\left|[D_1G(x^*, y^*)(I - P)]^{-1}\right| \omega_2(\rho) \le \varepsilon(1 - \omega).$$

Proof. The assertion on the size of $\rho, \varepsilon > 0$ results, if we apply the quantitative result [18] in the above implicit function theorem argument. □

3. **Difference equations.** A central aspect of our overall approach is to rephrase difference equations as operator equations in suitable sequence spaces (cf. [29, 30]). Throughout, for a nonempty open convex subset $\Omega \subseteq X$ we denote the set of bounded sequences $\phi = (\phi_k)_{k \in \mathbb{Z}}$ with $\phi_k \in \Omega$, $k \in \mathbb{Z}$, by $\ell^\infty(\Omega)$ and in case $0 \in \Omega$ we write $\ell_0(\Omega)$ for the space of such sequences converging to 0 in both time directions.

We briefly write $\ell^\infty := \ell^\infty(X)$, $\ell_0 := \ell_0(X)$ or simply ℓ for one of these two spaces, which both are Banach spaces equipped with the natural norm

$$\|\phi\| := \sup_{k\in\mathbb{I}} |\phi_k| .$$

Since Ω is convex, also the sets $\ell^\infty(\Omega)$ and $\ell_0(\Omega)$ inherit this property. Nevertheless, in general $\ell^\infty(\Omega)$ needs not to be open, whereas $\ell_0(\Omega)$ is.

Let the *parameter space* $\Lambda \subseteq Y$ be a nonempty open convex set. We consider functions $f_k : \Omega\times\Lambda \to X$, $k \in \mathbb{Z}$, which serve as right-hand sides of nonautonomous parameter-dependent difference equations

$$\boxed{x_{k+1} = f_k(x_k, \lambda).} \tag{$(\Delta)_\lambda$}$$

For a fixed parameter $\lambda \in \Lambda$, a *solution* of the difference equation $(\Delta)_\lambda$ is a sequence $\phi = (\phi_k)_{k\in\mathbb{I}}$ with $\phi_k \in \Omega$ for $k \in \mathbb{I}$, satisfying the recursion $(\Delta)_\lambda$ on a discrete interval $\mathbb{I}'$. In order to emphasize the dependence on λ, we sometimes write $\phi(\lambda)$. Particularly, a *complete* or *entire solution* is a solution defined on the whole integer axis $\mathbb{Z}$ and a *persistent solution* satisfies $\inf_{k\in\mathbb{I}} \operatorname{dist}(\phi_k, \Omega) > 0$. For given times $\kappa \in \mathbb{Z}$ and states $\xi \in \Omega$, the solution ϕ of $(\Delta)_\lambda$ satisfying $\phi_\kappa = \xi$ is denoted as *general solution* $\varphi_\lambda(\cdot; \kappa, \xi)$.

The following assumptions hold for C^m-smooth right-hand sides f_k of $(\Delta)_\lambda$, whose derivatives map bounded into bounded sets uniformly in time.

Hypothesis. *Let $m \in \mathbb{N}$ and suppose each $f_k : \Omega\times\Lambda \to X$, $k \in \mathbb{Z}$, is a C^m-function such that the following holds for $0 \le j \le m$:*

(H_0) *For all bounded $B \subseteq \Omega$ one has*

$$\sup_{k\in\mathbb{Z}} \sup_{x\in B} \left|D^j f_k(x, \lambda)\right| < \infty \quad \text{for all } \lambda \in \Lambda$$

(well-definedness) *and for all $\lambda^* \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|x - y| < \delta \quad\Rightarrow\quad \sup_{k\in\mathbb{Z}} \left|D^j f_k(x, \lambda) - D^j f_k(y, \lambda)\right| < \varepsilon \quad \text{for all } x, y \in \Omega \tag{3.1}$$

and $\lambda \in B_\delta(\lambda^*)$ (uniform continuity).

(H_1) *We have $0 \in \Omega$ and $\lim_{k\to\pm\infty} f_k(0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

With this at hand, we arrive at the crucial

Theorem 3.1. *For every parameter $\lambda \in \Lambda$ a sequence ϕ in Ω is a solution of the difference equation $(\Delta)_\lambda$, if and only if ϕ solves the nonlinear equation*

$$G(\phi, \lambda) = 0 \tag{3.2}$$

with $G(\phi, \lambda)_k = \phi_{k+1} - f_k(\phi_k, \lambda)$. For the mapping G one has under (H_0):

(a) $G : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ is well-defined and of class C^m on $\ell^\infty(\Omega)^\circ \times \Lambda$.
(b) If (H_1) holds, then $G : \ell_0(\Omega) \times \Lambda \to \ell_0$ is well-defined and of class C^m.

Proof. See [30, Lemma 2.3, Prop. 2.4 and Thm. 2.5]. □

3.1. **Linear difference equations.** In this subsection suppose $\mathbb{I}$ is a discrete interval. For a given operator sequence $A_k \in L(X)$, $k \in \mathbb{Z}$, linear difference equations are of the form

$$\boxed{x_{k+1} = A_k x_k} \tag{$(L\Delta)$}$$

with associated *transition operator* $\Phi(k,l) \in L(X)$, $k,l \in \mathbb{Z}$, defined by

$$\Phi(k,l) := \begin{cases} I & \text{for } k = l, \\ A_{k-1} \cdots A_l & \text{for } k > l; \end{cases}$$

if every A_k is invertible, we additionally set $\Phi(k,l) := A_k^{-1} \cdots A_{l-1}^{-1}$ for $k < l$.

A dichotomy (or trichotomy) means that the *extended state space* $\mathbb{Z} \times X$ of ($L\Delta$) splits into invariant vector bundles consisting of solutions with a specific asymptotic behavior; these vector bundles are described using projectors. More precisely, we say a sequence of projections $P_k \in L(X)$, $k \in \mathbb{I}$, is an *invariant projector* for ($L\Delta$), provided

$$A_k P_k = P_{k+1} A_k \quad \text{for all } k \in \mathbb{I}'.$$

We speak of a *regular projector* for ($L\Delta$), if the restriction $A_k : R(P_k) \to R(P_{k+1})$ is an isomorphism. Thus, the restricted transition operator $\Phi(k,l) : R(P_l) \to R(P_k)$, $k \leq l$, exists.

The following terminology generalizes earlier considerations from [26, 16] to the noninvertible situation: A linear nonautonomous difference equation ($L\Delta$) is said to have an *exponential trichotomy* (ET for short) on $\mathbb{I}$, if the following holds:

(i) There exist invariant projectors P_k, Q_k satisfying $P_k Q_k = Q_k P_k = 0$ and moreover the projectors $Q_k, I - P_k - Q_k$, $k \in \mathbb{I}$, are regular,
(ii) there exist reals $K \geq 1$, $\alpha \in (0,1)$ and a $\kappa \in \mathbb{I}$ such that

$$\begin{aligned} |\Phi(k,l)P_l| &\leq K\alpha^{k-l} \quad \text{for all } l \leq k, \\ |\Phi(k,l)Q_l| &\leq K\alpha^{|k-l|} \quad \text{for all } \kappa \leq l \leq k \text{ or } k \leq l \leq \kappa, \\ |\Phi(k,l)[I - P_l - Q_l]| &\leq K\alpha^{l-k} \quad \text{for all } k \leq l. \end{aligned} \tag{3.3}$$

To provide some dynamic insight, an ET means that the extended state space $\mathbb{I} \times X$ of equation ($L\Delta$) splits into three invariant vector bundles, namely

- the *stable bundle* consisting of exponentially decaying forward solutions on $\mathbb{Z}_\kappa^+$ and given by the ranges $R(P_k) \oplus R(Q_k)$ (if $\mathbb{I}$ is unbounded above),
- the *unstable bundle* consisting of solutions which exist in backward time on $\mathbb{Z}_\kappa^-$ and are exponentially decaying, given by the kernel $N(P_k)$ (if $\mathbb{I}$ is unbounded below),
- the *strongly center bundle* consisting of solutions existing in backward time and exponentially decaying both in forward and backward time, given by $R(Q_k)$ (if $\mathbb{I} = \mathbb{Z}$). Therefore, Q_k is called *central projector*.

In absence of a strongly center bundle, i.e. for $Q_k \equiv 0$, we speak of an *exponential dichotomy* (ED for short, cf. [17, p. 229, Def. 7.6.4]) on $\mathbb{I}$. Note that our trichotomy notion is stronger than the *Sacker-Sell trichotomy* (i.e. the discrete counterpart to [36, pp. 197ff]), where the above estimates hold with condition (3.3) weakened to

$$|\Phi(k,l)Q_l| \leq K \quad \text{for all } k,l \in \mathbb{I};$$

this means the *center bundle consists* of bounded solutions existing on the whole axis $\mathbb{Z}$, as opposed to exponentially decaying ones in the strongly center bundle.

Remark 3.1. (1) Linear difference equations ($L\Delta$) with bounded growth yield examples of EDs on $\mathbb{Z}$ with trivial projectors $P_k \equiv 0$ or $P_k \equiv I$. More detailed, one says ($L\Delta$) has *bounded forward growth* resp. *bounded backward growth*, provided there exist reals $K_0 \geq 1$, $\omega_-, \omega_+ > 0$ such that

$$|\Phi(k,l)| \leq K_0 \omega_+^{k-l} \quad \text{for all } k \geq l, \qquad |\Phi(k,l)| \leq K_0 \omega_-^{k-l} \quad \text{for all } l \geq k, \tag{3.4}$$

where in the latter case we suppose the invertibility assumption

$$A_k \in GL(X). \tag{3.5}$$

The condition $\sup_{k\in\mathbb{Z}} |A_k| < \infty$ is necessary and sufficient for bounded forward growth; bounded backward growth holds when $\sup_{k\in\mathbb{Z}} \left|A_k^{-1}\right| < \infty$. Thus, finite-dimensional equations possessing bounded forward growth and fulfilling

$$\inf_{k\in\mathbb{Z}} \det A_k > 0$$

also have bounded backward growth (see [12, p. 74, Lemma 1]).

(2) A θ-periodic, $\theta \in \mathbb{N}$, linear equation with *monodromy operator*

$$M := \Phi(\kappa+\theta, \kappa)$$

admits an ED on $\mathbb{Z}$, provided the spectrum of M does not intersect the closed concentric annulus $\left\{z \in \mathbb{C} : |z| \in \left[\alpha^\theta, \alpha^{-\theta}\right]\right\}$ (cf., for instance, [31, Prop. 2.2] for the finite-dimensional case). If semisimple eigenvalues of M exist on the complex unit circle, then $(L\Delta)$ has a Sacker-Sell trichotomy on $\mathbb{Z}$. Thus, this notion in a natural extension of classical autonomous nonhyperbolic behavior.

Differing from the commonly used Sacker-Sell trichotomy, the above trichotomy concept is tailor-made to investigate linear equations with EDs on both semiaxes. Indeed, if a linear equation $(L\Delta)$ admits an ET on $\mathbb{Z}$, then it has an ED on $\mathbb{Z}_\kappa^+$ with projector $P_k + Q_k$, as well as an ED on $\mathbb{Z}_\kappa^-$ with P_k (see also [16, Lemma 2]). For the converse we have

Lemma 3.2. *Let $\underline{\kappa}, \overline{\kappa} \in \mathbb{Z}$ with $\underline{\kappa} \le \overline{\kappa}$ and suppose (3.5) holds. If a linear difference equation $(L\Delta)$ has an ED on $\mathbb{Z}_{\overline{\kappa}}^+$ (with projector P_k^+) and on $\mathbb{Z}_{\underline{\kappa}}^-$ (with projector P_k^-), then it admits an ET on $\mathbb{Z}$, provided one of the following conditions holds:*

- *(i) Every solution of $(L\Delta)$ is the sum of a solution bounded on $\mathbb{Z}_{\overline{\kappa}}^+$ and a solution bounded on $\mathbb{Z}_{\underline{\kappa}}^-$,*
- *(ii) one has the relation*

$$P_{\underline{\kappa}}^- = P_{\underline{\kappa}}^- \Phi(\underline{\kappa}, \overline{\kappa}) P_{\overline{\kappa}}^+ \Phi(\overline{\kappa}, \underline{\kappa}) = \Phi(\underline{\kappa}, \overline{\kappa}) P_{\overline{\kappa}}^+ \Phi(\overline{\kappa}, \underline{\kappa}) P_{\underline{\kappa}}^-, \tag{3.6}$$

where the projectors associated to the ET read as $P_k := \Phi(k, \underline{\kappa}) P_{\underline{\kappa}}^- \Phi(\underline{\kappa}, k)$ and $Q_k := \Phi(k, \overline{\kappa}) P_{\overline{\kappa}}^+ \Phi(\overline{\kappa}, k) - \Phi(k, \underline{\kappa}) P_{\underline{\kappa}}^- \Phi(\underline{\kappa}, k)$.

Remark 3.2. (1) If $(L\Delta)$ has an almost-periodic coefficient operator A_k, an ED on a semiaxis $\mathbb{Z}_\kappa^+$ or $\mathbb{Z}_\kappa^-$ yields an ED on $\mathbb{Z}$ (cf. [2, Prop. 3.2] or [25]). Hence, for almost-periodic equations the notions of ED and ET on $\mathbb{Z}$ coincide, i.e. the strongly center bundle is trivial.

(2) In case $\kappa := \underline{\kappa} = \overline{\kappa}$ the relation (3.6) simplifies to

$$P_\kappa^- = P_\kappa^- P_\kappa^+ = P_\kappa^+ P_\kappa^- \tag{3.7}$$

and has a couple of consequences. Above all, (3.6) is equivalent to $N(P_\kappa^+) \subseteq N(P_\kappa^-)$ and $R(P_\kappa^-) \subseteq R(P_\kappa^+)$. In addition, this implies

$$R(P_\kappa^+(I - P_\kappa^-)) = R(P_\kappa^+) \cap N(P_\kappa^-), \qquad R(P_\kappa^+) + N(P_\kappa^-) = X$$

and the relation between $R(P_\kappa^+)$ and $N(P_\kappa^-)$ will be crucial for our further investigations. In this sense, condition (3.6) is complementary to the assumptions imposed previously in [29].

Proof. We extend the projector P_k^+ from $\mathbb{Z}_{\overline{\kappa}}^+$ to the discrete interval $\mathbb{Z}_{\underline{\kappa}}^+$ as follows

$$P_k^+ := \Phi(k, \overline{\kappa}) P_{\overline{\kappa}}^+ \Phi(\overline{\kappa}, k) \quad \text{for all } \underline{\kappa} \leq k \leq \overline{\kappa}$$

and verify that P_k^+ serves as invariant projector of an ED for $(L\Delta)$ on the larger interval $\mathbb{Z}_{\underline{\kappa}}^+$. Hence, the linear difference equation $(L\Delta)$ admits an ED both on $\mathbb{Z}_{\underline{\kappa}}^-$ and on $\mathbb{Z}_{\underline{\kappa}}^+$. Now, under assumption (i) the claim follows from [26, Lemma 4], and assumption (ii) yields the assertion using [16, Lemma 2]. In particular, (3.6) implies that Q_k is a projector □

Our upcoming results allow an elegant formulation extending the classical autonomous situation using the *dichotomy spectrum*, which in finite dimensions has been introduced in [9] (for one-sided time) and in [5] (for two-sided time, see also [4] for the noninvertible case). Thereto, for given reals $\gamma > 0$ we consider the scaled difference equation

$$\boxed{x_{k+1} = \gamma^{-1} A_k x_k} \qquad (L)_\gamma$$

and we define the

dichotomy spectrum	$\Sigma(A) := \{\gamma > 0 : (L)_\gamma \text{ has no ED on } \mathbb{Z}\}$,
forward dichotomy spectrum	$\Sigma^+(A) := \{\gamma > 0 : (L)_\gamma \text{ has no ED on } \mathbb{Z}_\kappa^+\}$,
backward dichotomy spectrum	$\Sigma^-(A) := \{\gamma > 0 : (L)_\gamma \text{ has no ED on } \mathbb{Z}_\kappa^-\}$

of equation $(L\Delta)$. It is not hard to see that $\Sigma^+(A), \Sigma^-(A) \subseteq \Sigma(A)$ and under (3.5) the sets $\Sigma^+(A), \Sigma^-(A)$ are independent of $\kappa \in \mathbb{Z}$. Furthermore, the dichotomy spectra $\Sigma^+(A), \Sigma^-(A)$ are invariant under compact perturbations, i.e.

$$\Sigma^\pm(A) = \Sigma^\pm(A + B)$$

for sequences of compact operators $B_k \in L(X)$ satisfying $\lim_{k\to\pm\infty} B_k = 0$ (see [9, Thm. 2.3]).

For a linear equation $(L\Delta)$ with bounded forward growth (cf. (3.4)) one has $\Sigma(A) \subseteq (0, \omega_+]$ and additional bounded backward growth yields $\Sigma(A) \subseteq [\omega_-, \omega_+]$. In case $d := \dim X < \infty$ it was shown in [9, 4] that $\Sigma(A)$ is the disjoint union of $n \leq d$ nonempty *spectral intervals* $\sigma_1, \dots, \sigma_n \subseteq (0, \infty)$, i.e.

$$\Sigma(A) = \bigcup_{i=1}^n \sigma_i, \qquad \sup \sigma_i < \inf \sigma_{i+1} \quad \text{for all } 1 \leq i < n.$$

Here, σ_n is called the *dominant spectral interval* of $(L\Delta)$.

Under bounded forward growth $\sigma_2, \dots, \sigma_n$ are compact and bounded backward growth yields the compactness of σ_1. We follow [4] to provide a further dynamical interpretation, suppose $\mathbb{I} = \mathbb{Z}$ and pick reals $0 < b_1 < a_2 < \dots < b_{n-1} < a_n$ such that

$$\Sigma(A) \cap \bigcup_{i=1}^{n-1} (b_i, a_{i+1}) = \emptyset.$$

If possible, choose $\gamma_0 \in (0, \infty) \setminus \Sigma(A)$ with $(0, \gamma_0) \subseteq (0, \infty) \setminus \Sigma(A)$ and otherwise set

$$\mathcal{S}_{\gamma_0}^- := \mathbb{Z} \times X, \qquad \mathcal{S}_{\gamma_0}^+ := \mathbb{Z} \times \{0\}.$$

We choose $\gamma_n \in (0, \infty) \setminus \Sigma(A)$ with $(\gamma_n, \infty) \subseteq (0, \infty) \setminus \Sigma(A)$ and otherwise define

$$\mathcal{S}_{\gamma_n}^- := \mathbb{Z} \times \{0\}, \qquad \mathcal{S}_{\gamma_n}^+ := \mathbb{Z} \times X.$$

For reals $\gamma > 0$ we introduce nonautonomous sets

$$\mathcal{S}_\gamma^\pm := \left\{(\kappa, \xi) \in \mathbb{Z} \times X : \sup_{k \in \mathbb{Z}_\kappa^\pm} |\Phi(k,\kappa)\xi| \, \gamma^{\kappa - k} < \infty \right\}$$

and finally choose $\gamma_i \in (b_i, a_{i+1})$ for $1 \le i < n$ in order to define *spectral bundles*

$$\mathcal{W}_i := \mathcal{S}_{\gamma_{i-1}}^- \cap \mathcal{S}_{\gamma_i}^+ \quad \text{for all } 1 \le i \le n.$$

They are invariant vector bundles for $(L\Delta)$, independent of γ_i, and satisfy

$$\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_n = \mathbb{Z} \times X. \tag{3.8}$$

The (constant) dimension of $\mathcal{W}_i$ is called *multiplicity* of the spectral interval σ_i. The Whitney sum (3.8) generalizes the autonomous situation, where the state space X is the decomposition of the generalized eigenspaces corresponding to the spectrum of $A_k \equiv A$.

In the *hyperbolic* case

$$1 \notin \Sigma(A) \tag{3.9}$$

we can choose growth rates $\gamma_+, \gamma_- \in (0,\infty) \setminus \Sigma(A)$ satisfying $\gamma_+ < 1 < \gamma_-$ and the two sets $\mathcal{S}_{\gamma_+}^+$, $\mathcal{S}_{\gamma_-}^-$ are denoted as *stable* resp. *unstable bundle* of $(L\Delta)$. In the *nonhyperbolic* situation $1 \in \Sigma(A)$, i.e. $1 \in \sigma_i$ for some $1 \le i < n$, we say that $\mathcal{S}_{\gamma_i}^+$ is the *center-stable*, $\mathcal{S}_{\gamma_{i-1}}^-$ the *center-unstable* and $\mathcal{W}_i$ the *center bundle* of $(L\Delta)$.

An ET is a specific form of nonhyperbolicity:

Proposition 3.3. *In case $(L\Delta)$ has an ET on $\mathbb{Z}$ with $Q_k \not\equiv 0$, then $[\alpha, \alpha^{-1}] \subseteq \Sigma(A)$ and the center fiber bundle is given by $\{(\kappa,\xi) \in \mathbb{Z} \times X : \xi \in R(Q_\kappa)\}$.*

Proof. Let $\kappa \in \mathbb{Z}$, $\gamma > 0$. The transition operator of $(L)_\gamma$ is $\Phi_\gamma(k,l) := \gamma^{l-k}\Phi(k,l)$. Since $(L\Delta)$ admits an ET, we obtain

$$|\Phi_\gamma(k,l)Q_l| \overset{(3.3)}{\le} K \max\left\{\alpha\gamma, \tfrac{\alpha}{\gamma}\right\}^{|k-l|} \quad \text{for all } \kappa \le l \le k \text{ or } k \le l \le \kappa,$$

and therefore $(L)_\gamma$ has nontrivial solutions bounded on $\mathbb{Z}$, provided $\alpha \le \gamma \le \alpha^{-1}$. Hence, $(L)_\gamma$ has no ED on $\mathbb{Z}$ (cf. [17, p. 230, Thm. 7.6.5]) and the first claim follows. The assertion on the center bundle is clear. □

In general the dichotomy spectra can be computed only numerically (cf. [19]). Nonetheless, we can illustrate the above by combining results of [7] and [8, Section 4] to deduce the following finite-dimensional examples.

Example 3.1 (scalar equations). For scalar equations, the dichotomy spectrum is related to *Bohl exponents*. More precisely, let $a_k \in \mathbb{R} \setminus \{0\}$ satisfy

$$\sup_{k \in \mathbb{Z}} \left\{|a_k|, \left|a_k^{-1}\right|\right\} < \infty.$$

For a scalar equation $(L\Delta)$ with $A_k = a_k$ one has $\Sigma(a) = [\beta_-(\mathbb{Z}), \beta_+(\mathbb{Z})]$,

$$\Sigma^+(a) = \left[\beta_-(\mathbb{Z}_\kappa^+), \beta_+(\mathbb{Z}_\kappa^+)\right], \qquad \Sigma^-(a) = \left[\beta_-(\mathbb{Z}_\kappa^-), \beta_+(\mathbb{Z}_\kappa^-)\right] \quad \text{for } \kappa \in \mathbb{Z},$$

where the *lower* resp. *upper Bohl exponent* is given by

$$\beta_-(\mathbb{I}) = \sup\left\{\gamma > 0 : \sup_{k \le l,\, k,l \in \mathbb{I}} \gamma^{k-l} |\Phi(k,l)| < \infty\right\},$$

$$\beta_+(\mathbb{I}) = \inf\left\{\gamma > 0 : \sup_{l \le k,\, k,l \in \mathbb{I}} \gamma^{k-l} |\Phi(k,l)| < \infty\right\}.$$

In particular, for asymptotically autonomous scalar difference equations, where $\alpha_+, \alpha_- \in \mathbb{R} \setminus \{0\}$ and $\kappa \in \mathbb{Z}$ such that — with $a_k = \alpha_-$ for $k < \kappa$ and $a_k = \alpha_+$ for $k \geq \kappa$ — one immediately deduces $\Sigma(a) = [\min\{|\alpha_-|, |\alpha_+|\}, \max\{|\alpha_-|, |\alpha_+|\}]$,

$$\Sigma^+(a) = \{|\alpha_+|\}, \qquad \Sigma^-(a) = \{|\alpha_-|\}.$$

Example 3.2 (autonomous systems). For autonomous difference equations $(L\Delta)$ with coefficient matrix $A_k \equiv A \in L(\mathbb{R}^d)$ and $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, $n \leq d$, one has

$$\Sigma(A) = \Sigma^+(A) = \Sigma^-(A) = \bigcup_{i=1}^{n} \{|\lambda_i|\}.$$

Example 3.3 (periodic systems). Let $\theta \in \mathbb{N}$. For θ-periodic difference equations $(L\Delta)$ with monodromy matrix M and $\sigma(M) = \{\lambda_1, \dots, \lambda_n\}$, $n \leq d$, one gets

$$\Sigma(A) = \Sigma^+(A) = \Sigma^-(A) = \bigcup_{i=1}^{n} \left\{ \sqrt[\theta]{|\lambda_i|} \right\}.$$

Example 3.4. Suppose that $A_-, A_+ \in GL(\mathbb{R}^d)$ and $\kappa \in \mathbb{Z}$. For a real $\rho > 0$ we denote by $N(A_-, \rho)$ (resp. $R(A_+, \rho)$) the kernel (resp. range) of the Riesz projection corresponding to $\{z \in \mathbb{C} : |z| \leq \rho\}$. For a difference equations $(L\Delta)$ with $A_k = A_-$ for $k < \kappa$ and $A_k = A_+$ for $k \geq \kappa$ we suppose $\sigma(A_-) \cup \sigma(A_+) = \{\lambda_1, \dots, \lambda_{2d}\}$, where the complex numbers λ_i are ordered according to

$$|\lambda_1| = \ldots = |\lambda_{n_1}| < |\lambda_{n_1+1}| = \ldots = |\lambda_{n_k}| < |\lambda_{n_k+1}| = \ldots = \left|\lambda_{n_{k+1}}\right|,$$

i.e. the indices $n_1 < \ldots < n_k$ indicate one of the $k < 2d$ jumps in the moduli of the elements of $\sigma(A_-) \cup \sigma(A_+)$, and we set $n_{k+1} := 2d$. Moreover, choose $i_1 < \ldots < i_{l-1}$ in the set $\{1, \dots, k\}$ such that $N(A_-, \left|\lambda_{n_{i_m}}\right|) \oplus R(A_+, \left|\lambda_{n_{i_m}}\right|) = \mathbb{R}^d$ holds for $0 \leq m < l$. Then one obtains $l \leq d+1$ and, with $i_0 = 0$, $i_l = k+1$, the dichotomy spectra for $(L\Delta)$ read as

$$\Sigma(A) = \bigcup_{m=0}^{l-1} \left[\left|\lambda_{n_{i_m}+1}\right|, \left|\lambda_{n_{i_{m+1}}}\right| \right],$$

$$\Sigma^+(A) = \{|\lambda| \in \mathbb{R} : \lambda \in \sigma(A_+)\},$$

$$\Sigma^-(A) = \{|\lambda| \in \mathbb{R} : \lambda \in \sigma(A_-)\}.$$

Before discussing two further examples, some functional-analytical results are due. As previously in [29] we study Fredholm properties of the difference operator

$$L : \ell \to \ell, \qquad (L\phi)_k := \phi_{k+1} - A_k \phi_k \quad \text{for all } k \in \mathbb{Z}, \tag{3.10}$$

which is easily seen to be well-defined and continuous under bounded forward growth of equation $(L\Delta)$ — an assumption we impose throughout this remaining subsection.

Proposition 3.4. *Let $\kappa \in \mathbb{Z}$. If a linear difference equation $(L\Delta)$ has an ET on the full axis $\mathbb{Z}$, then*

(a) $L : \ell \to \ell$ is semi-Fredholm with

$$N(L) = \{\Phi(\cdot, \kappa)\xi \in \ell : \xi \in R(Q_\kappa)\}, \qquad R(L) = \ell,$$

(b) $N(L)$ is complemented and the projector $P \in L(\ell)$ with $R(P) = N(L)$ is given by $(P\phi)_k := Q_k \phi_k$

and, provided $\dim R(Q_\kappa) < \infty$, it is Fredholm with index $\dim R(Q_\kappa) \geq 0$. In case of an ED on $\mathbb{Z}$ one has $L \in GL(\ell)$.

Remark 3.3. Referring to the estimate (3.3), the kernel $N(L)$ consists of solutions to $(L\Delta)$ in the strongly center bundle. Indeed, we have the dynamical characterization

$$N(L) = \left\{ \Phi(\cdot,\kappa)\xi \in \ell : \sup_{k\in\mathbb{Z}} |\Phi(k,\kappa)\xi|\, \alpha^{-|k-\kappa|} < \infty \right\} \subseteq \ell_0.$$

Proof of Prop. 3.4. For the case of an ED we refer to [17, p. 230, Thm. 7.6.5] (where $\ell = \ell^\infty$) and to [7, Cor. 3] (where $\ell = \ell_0$), while the assertion for an ET can be shown along the lines of [26, Prop. 1]. Moreover, since Q_k is an invariant projector for $(L\Delta)$, the linear mapping $P : \ell \to \ell$ defined above is a bounded projector onto $N(L)$ and therefore the kernel $N(L)$ is complemented. □

Corollary 3.5. *The restriction $L|_{N(P)}$ is invertible and the inverse $L|_{N(P)}^{-1} : \ell \to N(P)$ fulfills $\|L|_{N(P)}^{-1}\| \le \frac{K}{1-\alpha}$.*

Proof. For every given sequence $\psi \in \ell$ it has been shown in [26, Proof of Prop. 1] or [16, Proof of Thm. 4] that $\phi := \sum_{n\in\mathbb{Z}} G(\cdot, n+1)\psi_n$ is a solution to the linearly inhomogeneous equation

$$x_{k+1} = A_k x_k + \psi_k \tag{3.11}$$

in the space ℓ, where Green's function reads as

$$G(k,l) := \begin{cases} \Phi(k,l)P_l, & l \le \kappa \le k,\ l \le k \le \kappa, \\ -\Phi(k,l)Q_l, & k < l \le \kappa, \\ \Phi(k,l)[P_l + Q_l], & \kappa < l \le k \\ -\Phi(k,l)[I - P_l - Q_l], & \kappa \le k < l,\ k \le \kappa < l. \end{cases}$$

This means we have $L\phi = \psi$ and $L \in L(\ell)$ is onto. Due to the relation

$$\phi = LP\phi + L[I-P]\phi = L[I-P]\phi$$

we see that $[I-P]\phi$ is the unique solution of (3.11) in the space $N(P)$. Due to the relations $P_k G(k,l) = 0$ for $k < l$ and $|G(k,l)| \le K\alpha^{k-l}$ for $l \le k$ we deduce

$$|((I-P)\phi)_k| \le \sum_{n<k} |G(k,n+1)|\,|\psi_n| \le K \sum_{n=-\infty}^{k-1} \alpha^{k-n-1} \|\psi\| = \frac{K}{1-\alpha}\|\psi\|$$

for all $k \in \mathbb{Z}$ and consequently one has $\left\| L|_{N(P)}^{-1} \right\| \le \frac{K}{1-\alpha}$. □

Corollary 3.6. *If $R(L) = \ell$ and there is an $\alpha \in (0,1)$ such that $[\alpha, \alpha^{-1}] \subseteq \Sigma(A)$, then $(L\Delta)$ admits an ET on $\mathbb{Z}$ with nonzero central projector $Q_k \not\equiv 0$.*

Proof. By assumption, we know that for each inhomogeneity $\psi \in \ell$ there exists a solution $\phi \in \ell$ of (3.11) and thanks to [26, Prop. 1] or [16, Proof of Thm. 4] we know that $(L\Delta)$ has an ET on $\mathbb{Z}$. Since there exists a spectral interval $\sigma^* \supseteq [\alpha, \alpha^{-1}]$, we define Q_k to be the projector onto the fibers of the corresponding spectral bundle, which is nontrivial. □

Proposition 3.7 (nodal operator)**.** *Let $\underline{\kappa}, \overline{\kappa} \in \mathbb{Z}$ with $\underline{\kappa} < \overline{\kappa}$. Suppose a linear difference equation $(L\Delta)$ admits an ED on $\mathbb{Z}_{\overline{\kappa}}^+$ (with projector P_k^+) and on $\mathbb{Z}_{\underline{\kappa}}^-$ (with projector P_k^-). Then $L : \ell \to \ell$ is Fredholm, if and only if the* nodal operator

$$\Xi(\overline{\kappa}, \underline{\kappa}) := (I - P_{\overline{\kappa}}^+)\Phi(\overline{\kappa}, \underline{\kappa})(I - P_{\underline{\kappa}}^-) : N(P_{\underline{\kappa}}^-) \to N(P_{\overline{\kappa}}^+)$$

is Fredholm. Both operators have the same Fredholm index, which for finite-dimensional kernels $N(P_{\underline{\kappa}}^-), N(P_{\overline{\kappa}}^+)$ is given by $\dim N(P_{\underline{\kappa}}^-) - \dim N(P_{\overline{\kappa}}^+)$.

Proof. See [10, Thm. 8] and [37, p. 367, Example 8.15] for the index. □

Corollary 3.8. $N(L) = \left\{\Phi(\cdot,\kappa)\xi \in \ell : \xi \in N((I - P^+_{\kappa+1})A_\kappa(I - P^-_\kappa))\right\}$ *for* $\kappa \in \mathbb{Z}$.

Proof. We refer to [10, Thm. 4]. □

The following two prototype examples illustrate the above concepts:

Example 3.5. Let $\alpha_-, \alpha_+ \in \mathbb{R} \setminus \{0\}$ be given and suppose $X = \mathbb{R}$. As in Ex. 3.1 we define a piecewise constant coefficient matrix for $(L\Delta)$, resp. its transition matrix by

$$A_k := \begin{cases} \alpha_-, & k < 0, \\ \alpha_+, & k \geq 0, \end{cases} \qquad \Phi(k,l) := \begin{cases} \alpha_+^{k-l}, & k \geq l \geq 0, \\ \alpha_+^k \alpha_-^{-l}, & k \geq 0 > l, \\ \alpha_-^{k-l}, & 0 > k \geq l; \end{cases}$$

and due to the invertibility of A_k one sets $\Phi(k,l) := \Phi(l,k)^{-1}$ for $k < l$. We discuss several cases in order to describe dichotomy and Fredholm properties of $(L\Delta)$. First of all, the dichotomy spectra are given in Ex. 3.1; in particular $\Sigma^+(A) = \{|\alpha_+|\}$.

Moreover, due to Prop. 3.7 the operator $L : \ell \to \ell$ is Fredholm for $|\alpha_\pm| \neq 1$ and using Lemma 3.2 we arrive at:

(a) $|\alpha_+| < 1$: ED on $\mathbb{Z}_0^+$ with $P_k^+ \equiv 1$ and $\Sigma^+(A) = \{|\alpha_+|\} \subseteq (0,1)$
- (a_1) $|\alpha_-| < 1$: ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv 1$, $\Sigma(A) \subseteq (0,1)$, L is invertible, $(L\Delta)$ has an ED on $\mathbb{Z}$ with $P_k \equiv 1$ and 0 is uniformly asymptotically stable
- (a_2) $|\alpha_-| = 1$: $\Sigma(A) = [|\alpha_+|, 1]$ and 0 is uniformly asymptotically stable
- (a_3) $|\alpha_-| > 1$: ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv 0$, $1 \in \Sigma(A) = [|\alpha_+|, |\alpha_-|]$, L is Fredholm with index 1, $(L\Delta)$ has an ET on $\mathbb{Z}$ with $Q_k \equiv 1$ and 0 is asymptotically stable

See Fig. 1 to illustrate the loss of stability in the transition from (a_1) to (a_3).

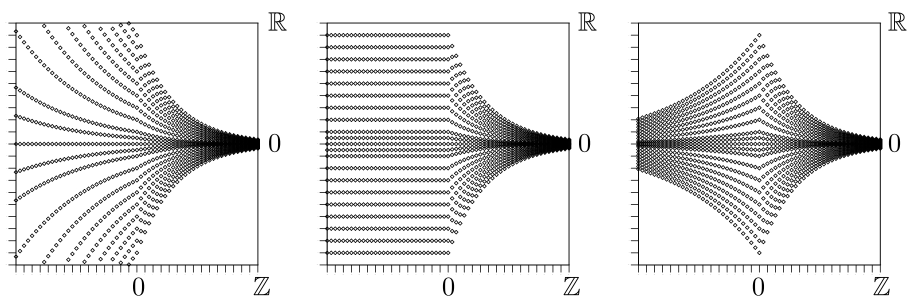

FIGURE 1. Example 3.5(a): Solutions in the asymptotically stable case $|\alpha_+| < 1$ with $|\alpha_-| < 1$ (left), $|\alpha_-| = 1$ (middle) and $|\alpha_-| > 1$ (right)

(b) $|\alpha_+| = 1$: No ED on $\mathbb{Z}_0^+$.
- (b_1) $|\alpha_-| < 1$: ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv 1$, $\Sigma(A) = [|\alpha_-|, 1]$ and 0 is uniformly stable
- (b_2) $|\alpha_-| = 1$: $\Sigma(A) = \{1\}$, $(L\Delta)$ has a Sacker-Sell trichotomy on $\mathbb{Z}$ and 0 is uniformly stable
- (b_3) $|\alpha_-| > 1$: $\Sigma(A) = [1, |\alpha_-|]$ and 0 is stable, but not uniformly stable (Fig. 2).

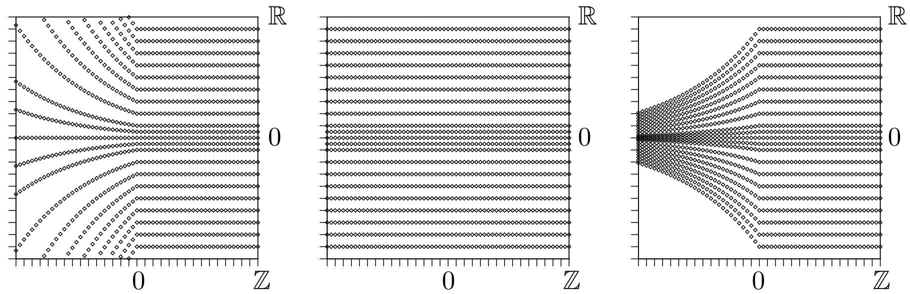

FIGURE 2. Example 3.5(b): Solutions in the stable case $|\alpha_+| = 1$ with $|\alpha_-| < 1$ (left), $|\alpha_-| = 1$ (middle) and $|\alpha_-| > 1$ (right)

(c) $|\alpha_+| > 1$: ED on $\mathbb{Z}_0^+$ with $P_k^+ \equiv 0$
- (c_1) $|\alpha_-| < 1$: ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv 1$, $1 \in \Sigma(A) = [|\alpha_-|, |\alpha_+|]$, L is Fredholm with index -1 and 0 is unstable
- (c_2) $|\alpha_-| = 1$: $\Sigma(A) = [1, |\alpha_+|]$ and 0 is unstable
- (c_3) $|\alpha_-| > 1$: ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv 1$, $\Sigma(A) \subseteq (1, \infty)$, $(L\Delta)$ has an ED on $\mathbb{Z}$ with $P_k \equiv 0$, i.e. 0 is uniformly unstable (see Fig. 3).

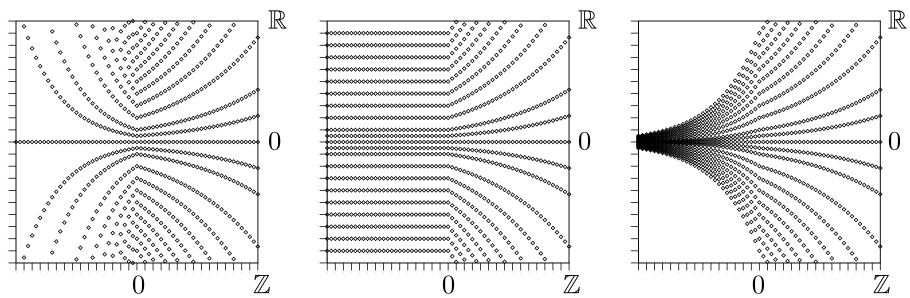

FIGURE 3. Example 3.5(c): Solutions in the unstable case $|\alpha_+| > 1$ with $|\alpha_-| < 1$ (left), $|\alpha_-| = 1$ (middle) and $|\alpha_-| > 1$ (right)

Of particular interest is the nonhyperbolic situation $1 \in \Sigma(A)$. Here, by virtue of the cases (a_3) and (c_1) it is easy to construct examples showing that the dichotomy spectrum $\Sigma(A)$ is insufficient for stability assertions, whereas $\Sigma^+(A)$ is.

Example 3.6. Let $\gamma_-, \beta_-, \gamma_+, \beta_+ \in \mathbb{R} \setminus \{0\}$ be given and suppose $X = \mathbb{R}^2$. We define a piecewise constant coefficient matrix for $(L\Delta)$ by

$$A_k := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix}, \qquad b_k := \begin{cases} \beta_-, & k < 0, \\ \beta_+, & k \geq 0, \end{cases} \qquad c_k := \begin{cases} \gamma_-, & k < 0, \\ \gamma_+, & k \geq 0 \end{cases}$$

and arrive at the transition matrix

$$\Phi(k, l) := \begin{cases} \operatorname{diag}(\beta_+^{k-l}, \gamma_+^{k-l}), & k \geq l \geq 0, \\ \operatorname{diag}(\beta_+^k \beta_-^{-l}, \gamma_+^k \gamma_-^{-l}), & k \geq 0 > l, \\ \operatorname{diag}(\beta_-^{k-l}, \gamma_-^{k-l}), & 0 > k \geq l; \end{cases}$$

due to the invertibility of A_k one sets $\Phi(k,l) := \Phi(l,k)^{-1}$ for $k < l$. In our present situation the dichotomy spectra read as

$$\Sigma(A) = \Sigma(b) \cup \Sigma(c), \qquad \Sigma^+(A) = \{|\beta_+|, |\gamma_+|\},$$

where $\Sigma(b), \Sigma(c)$ can be computed as in Ex. 3.1. We distinguish several generic cases in order to describe the dichotomy and Fredholm properties of $(L\Delta)$. In each case, $(L\Delta)$ admits an ED on $\mathbb{Z}_0^+$ and $\mathbb{Z}_{-1}^-$ with constant projectors P_k^- resp. P_k^+; it is easy to see that the ED on $\mathbb{Z}_{-1}^-$ extends to $\mathbb{Z}_0^-$. Referring to Prop. 3.7 the operator $L : \ell \to \ell$ is Fredholm for $|\beta_\pm|, |\gamma_\pm| \neq 1$ and using Lemma 3.2(ii) we arrive at:

(a) $|\beta_+|, |\gamma_+| < 1$: $P_k^+ \equiv I$ and $\Sigma^+(A) \subseteq (0,1)$
  (a_1) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$,
  $$\Sigma(A) \subseteq (0,1),$$
  L is invertible, $(L\Delta)$ has an ED on $\mathbb{Z}$ and 0 is uniformly asymptotically stable
  (a_2) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$,
  $$1 \in \Sigma(A) = \Sigma(b) \cup [|\gamma_+|, |\gamma_-|],$$
  L has 1-dimensional kernel, index 1, $(L\Delta)$ admits an ET on $\mathbb{Z}$ and 0 is asymptotically stable
  (a_3) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,
  $$1 \in \Sigma(A) = [|\beta_+|, |\beta_-|] \cup \Sigma(c),$$
  L has 1-dimensional kernel, index 1, $(L\Delta)$ admits an ET on $\mathbb{Z}$ and and 0 is asymptotically stable
  (a_4) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$,
  $$1 \in \Sigma(A) = [\min\{|\beta_+|, |\gamma_+|\}, \max\{|\beta_-|, |\gamma_-|\}],$$
  L has 2-dimensional kernel, index 2, $(L\Delta)$ admits an ET on $\mathbb{Z}$ and 0 is asymptotically stable

(b) $|\beta_+| < 1 < |\gamma_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}_0^+$ with $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and 0 is unstable
  (b_1) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$,
  $$1 \in \Sigma(A) = \Sigma(b) \cup [|\gamma_-|, |\gamma_+|],$$
  L has 0-dimensional kernel and index -1
  (b_2) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$,
  $$1 \notin \Sigma(A) = \Sigma(b) \dot{\cup} \Sigma(c),$$
  L is invertible, $(L\Delta)$ has an ED on $\mathbb{Z}$ and 0 has an unstable manifold
  (b_3) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,
  $$1 \in \Sigma(A) = [\min\{|\beta_+|, |\gamma_-|\}, \max\{|\beta_-|, |\gamma_+|\}],$$
  L has 1-dimensional kernel and index 0
  (b_4) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$,
  $$1 \in \Sigma(A) = [|\beta_+|, |\beta_-|] \cup \Sigma(c),$$
  L has 1-dimensional kernel, index 1 and $(L\Delta)$ admits an ET on $\mathbb{Z}$

(c) $|\gamma_+| < 1 < |\beta_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}_0^+$ with $P_k^+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and 0 is unstable

(c_1) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$,

$$1 \in \Sigma(A) = [|\beta_-|, |\beta_+|] \cup \Sigma(c),$$

L has 0-dimensional kernel and index -1

(c_2) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$,

$$1 \in \Sigma(A) = [\min\{|\beta_-|, |\gamma_+|\}, \max\{|\beta_+|, |\gamma_-|\}],$$

L has 1-dimensional kernel and index 0

(c_3) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,

$$1 \notin \Sigma(b) \dot{\cup} \Sigma(c),$$

L is invertible and $(L\Delta)$ has an ED on $\mathbb{Z}$

(c_4) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$,

$$1 \in \Sigma(A) = \Sigma(b) \cup [|\gamma_+|, |\gamma_-|],$$

L has 1-dimensional kernel, index 1 and $(L\Delta)$ admits an ET on $\mathbb{Z}$

(d) $1 < |\beta_+|, |\gamma_+|$: $(L\Delta)$ admits an ED on $\mathbb{Z}_0^+$ with $P_k^+ = 0$ and 0 is unstable

(d_1) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$,

$$1 \in \Sigma(A) = [\min\{|\beta_-|, |\beta_+|\}, \max\{|\gamma_-|, |\gamma_+|\}],$$

L has 0-dimensional kernel and index -2

(d_2) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$,

$$1 \in \Sigma(A) = [|\beta_-|, |\beta_+|] \cup \Sigma(c),$$

L has 0-dimensional kernel and index -1

(d_3) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,

$$1 \in \Sigma(A) = \Sigma(b) \cup [|\gamma_-|, |\gamma_+|],$$

L has 0-dimensional kernel and index -1

(d_4) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$,

$$\Sigma(A) \subseteq (1, \infty),$$

L is invertible and $(L\Delta)$ has an ED on $\mathbb{Z}$, i.e. 0 is uniformly unstable

3.2. **Stability, spectra and strongly center fiber bundles.** Let us suppose that, for a given parameter $\lambda \in \Lambda$, our nonautonomous difference equation $(\Delta)_\lambda$ has a fixed entire reference solution $\phi(\lambda) \in \ell^\infty(\Omega)$. We consider the *variational equation*

$$\boxed{x_{k+1} = D_1 f_k(\phi(\lambda)_k, \lambda) x_k,} \qquad (V)_\lambda$$

denote its transition operator by $\Phi_\lambda(k, l) \in L(X)$ and its dichotomy spectra by $\Sigma(\lambda)$ and $\Sigma^+(\lambda), \Sigma^-(\lambda)$. If $(V)_\lambda$ admits an ET on $\mathbb{Z}$, we say the solution $\phi(\lambda)$ is *weakly hyperbolic* and for a *hyperbolic* solution $\phi(\lambda)$, equation $(V)_\lambda$ has an ED on the whole axis $\mathbb{Z}$.

While $\Sigma(\lambda)$ is a useful notion to characterize hyperbolicity (cf. [30]) and in particular uniform asymptotic stability, as seen in Ex. 3.5 it is not adequate to detect asymptotic stability: Thus, we need a finer insight into $\Sigma(\lambda)$ by virtue of the one-sided dichotomy spectra $\Sigma^+(\lambda)$ and $\Sigma^-(\lambda)$. Indeed, one has:

Proposition 3.9. *Let $\lambda \in \Lambda$. Under (H_0) the following holds:*

(a) If $\max \Sigma^+(\lambda) < 1$, then $\phi(\lambda)$ is asymptotically stable,

(b) if $\max \Sigma(\lambda) < 1$, then $\phi(\lambda)$ is uniformly asymptotically stable.

Proof. Let $\kappa \in \mathbb{Z}$. We neglect the dependence on the fixed parameter value $\lambda \in \Lambda$ in our notation and consider the *equation of perturbed motion*

$$x_{k+1} = D_1 f_k(\phi_k) x_k + r_k(x_k) \tag{3.12}$$

with general solution $\varphi(\cdot; \kappa, \xi)$ and $r_k(x) := f_k(x + \phi_k) - f_k(\phi_k) - D_1 f_k(\phi_k) x$ satisfying

$$\lim_{x \to 0} \tfrac{r_k(x)}{|x|} = 0 \quad \text{uniformly in } k \in \mathbb{Z}.$$

(a) By assumption $\max \Sigma^+ < 1$ we know that $(V)_\lambda$ admits an ED on $\mathbb{Z}_\kappa^+$ with invariant projector $P_k \equiv I$ for some $\kappa \in \mathbb{Z}$, i.e. $|\Phi(k,l)| \le K\alpha^{k-l}$ holds for all $\kappa \le l \le k$. Using the standard cut-off technique one can modify r_k outside a neighborhood $B_\rho(0) \subseteq X$, $\rho > 0$, such that (3.12) satisfies the global assumptions of [1, p. 256, Thm. 5.6.2], yielding that its trivial solution is exponentially asymptotically stable. This means that there exist a $\gamma \in (0,1)$ such that for any given $\varepsilon > 0$, there is a $\delta = \delta(\kappa) > 0$ with

$$|\varphi(l; \kappa, \xi)| < \delta \quad \Rightarrow \quad |\varphi(k; \kappa, \xi)| \le \varepsilon \gamma^{k-l} \quad \text{for all } \kappa \le l \le k$$

and in particular the trivial solution of (3.12) is attractive. By continuous dependence on the initial conditions, for every $\kappa_0 \le \kappa$ there exists a $\delta_0 > 0$ with $|\xi| < \delta_0$ guaranteeing $|\varphi(k; \kappa_0, \xi)| < \delta$ for all $\kappa_0 \le k \le \kappa$. This yields that the trivial solution of equation (3.12) is asymptotically stable and, in turn, ϕ is an asymptotically stable solution of $(\Delta)_\lambda$.

(b) In contrast to (a), one has the uniform estimate $|\Phi(k,l)| \le K\alpha^{k-l}$ for all $l \le k$, and $\delta > 0$ can be chosen independently of $\kappa \in \mathbb{Z}$. Therefore, the zero solution of (3.12) and consequently the solution ϕ of $(\Delta)_\lambda$ are uniformly asymptotically stable. □

We say that an entire solution $\phi(\lambda)$ of the difference equation $(\Delta)_\lambda$ admits an *unstable fiber bundle*, if for every $\kappa \in \mathbb{Z}$ and $\varepsilon > 0$ there exists a backward solution $\psi : \mathbb{Z}_\kappa^- \to \Omega$ with $\psi(\kappa) \in \dot{B}_\varepsilon(\phi(\lambda)_\kappa)$ satisfying the limit relation

$$\lim_{k \to -\infty} |\psi_k - \phi(\lambda)_k| = 0.$$

Proposition 3.10. *Let $\lambda \in \Lambda$. Under (H_0) suppose the dichotomy spectrum of $(V)_\lambda$ allows a splitting $\Sigma(\lambda) = \Sigma_- \dot{\cup} \sigma$, where σ and σ^+ are the dominant spectral intervals of $\Sigma(\lambda)$ and $\Sigma^+(\lambda)$ respectively. The following holds:*

(a) If $\min \sigma^+ > 1$, then $\phi(\lambda)$ is unstable,

(b) if $\Sigma_- \neq \emptyset$, $\max \Sigma_- < 1$ and $\min \sigma > 1$, then $\phi(\lambda)$ is unstable and admits an unstable fiber bundle.

Proof. We again suppress the dependence on $\lambda \in \Lambda$.

(a) The assumption $\min \sigma^+ > 1$ guarantees that $(V)_\lambda$ admits an ED on $\mathbb{Z}_\kappa^+$ for some $\kappa \in \mathbb{Z}$, with associated invariant projector $P_k \neq I$. Thus, as in the proof of Prop. 3.9 an appropriate modification of the nonlinearity r_k in (3.12) outside a neighborhood of 0 makes [1, p. 261, Cor. 5.6.9] applicable, yielding that the solution ϕ is unstable.

(b) The claim follows from [32, Thm. 3.2(b)] applied to (3.12). □

One easily constructs examples showing that the center bundle in a Sacker-Sell trichotomy does not survive under (nonlinear) perturbations as bundle consisting of bounded entire solutions. Our trichotomy notion is more robust and we obtain that the strongly center bundle persists as graph of a smooth mapping consisting of entire solutions decaying exponentially to 0 in both time directions.

Proposition 3.11 (strongly center fiber bundles)**.** *Let $\lambda^* \in \Lambda$, $\kappa \in \mathbb{Z}$ and suppose that (H_0) is fulfilled. If $\ell = \ell^\infty$ and $\phi^* \in \ell(\Omega)$ is an entire permanent and weakly hyperbolic solution of equation $(\Delta)_{\lambda^*}$, then there exist reals $\varepsilon, \rho > 0$ and a unique C^m-function $\psi : B_\rho(0, \lambda^*) \subseteq R(Q_\kappa) \times \Lambda \to B_\varepsilon(\phi^*) \subseteq \ell(\Omega)$ such that one has:*

(a) $\psi(0, \lambda^*) = \phi^*$,

(b) $\psi(\xi, \lambda)$ is an entire weakly hyperbolic solution of difference equation $(\Delta)_\lambda$

for all $(\xi, \lambda) \in B_\rho(0, \lambda^*)$. If (H_0)–(H_1) are satisfied, then the same holds with $\ell = \ell_0$.

Remark 3.4. Our goal is to employ the above results to the difference equation $(\Delta)_\lambda$ resp. its operator formulation (3.2), as well as the Fredholm theory for the linear operator L defined in (3.10) presented in Subsection 3.1 with $A_k = D_1 f_k(\phi_k^*, \lambda^*)$. Here, it is important to note that the variational equation $(V)_\lambda$ has bounded forward growth, since (H_0) yields

$$\sup_{k\in\mathbb{Z}} |D_1 f_k(\phi_k^*, \lambda^*)| < \infty \quad \text{for entire solutions } \phi^* \in \ell^\infty(\Omega).$$

Proof of Prop. 3.11. We give the proof only for $\ell^\infty(\Omega)$ and set $L := D_1 G(\phi^*, \lambda^*)$.

(a) Our assumption and Prop. 3.4 imply that $N(L) \cong R(Q_\kappa)$ is complemented, i.e. we have $\ell^\infty = X_1 \oplus N(L)$ with a closed subspace $X_1 \subseteq \ell^\infty$; in addition, the map $L \in L(\ell^\infty)$ is onto. Since Thm. 3.1(a) implies that $G : \ell^\infty(\Omega)^\circ \times \Lambda \to \ell^\infty$ is of class C^m, we can apply Thm. 2.1 to the equation $G(\phi, \lambda) = 0$ with $X = Z = \ell^\infty$. This yields a family of entire solutions to $(\Delta)_\lambda$ given by

$$\psi(\xi, \lambda) := \phi^* + \Phi(\cdot, \kappa)\xi + \phi(\xi, \lambda).$$

(b) follows from the ℓ^∞-roughness of ETs (cf. [26, Prop. 2]). □

Thanks to Cor. 2.2 we can obtain quantitative information on the domain and range of the function ψ, i.e. the ℓ^∞-norm of the bounded entire solutions.

Corollary 3.12. *If there exist functions $\omega_0 : \mathbb{R} \to [0, \infty)$, $\omega_1 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, nondecreasing in each argument, so that*

$$|f_k(x, \lambda) - f_k(\phi_k^*, \lambda^*)| \le \omega_0(|x - \phi_k^*|),$$
$$|D_1 f_k(x, \lambda) - D_1 f_k(\phi_k^*, \lambda^*)| \le \omega_1(|x - \phi_k^*|, |\lambda - \lambda^*|)$$

for all $k \in \mathbb{Z}$ and $x \in \Omega$, $\lambda \in \Lambda$, then the reals $\varepsilon, \rho > 0$ from Prop. 3.10 can be determined from the relations

$$\tfrac{K(1+K)}{1-\alpha}\omega_1(\rho, \varepsilon) \le \omega < 1, \qquad \tfrac{K}{1-\alpha}(\rho + \omega_0(\rho)) \le \varepsilon(1 - \omega).$$

Proof. We have to verify the conditions of Cor. 2.2. First of all, thanks to the above Cor. 3.5 we derive $\left|(D_1 G(\phi^*, \lambda^*)(I - P))^{-1}\right| \le \frac{K}{1-\alpha}$. Furthermore, due to the relations

$$G(\phi^*, \lambda^*)_k = 0, \qquad (D_1 G(\phi, \lambda)\psi)_k = \psi_{k+1} - D_1 f_k(\phi_k, \lambda)\psi_k \quad \text{for all } k \in \mathbb{Z}$$

and $\psi \in \ell$, it is not difficult to deduce the estimates (2.2) from our assumptions. □

3.3. **Jump and shovel bifurcations.** In the standard terminology from branching theory (cf., for example, [37]) already used in [29], an entire solution ϕ^* of $(\Delta)_{\lambda^*}$ *bifurcates* at a fixed parameter value $\lambda^* \in \Lambda$, if there exists a convergent sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with limit λ^* such that each difference equation $(\Delta)_{\lambda_n}$ has two distinct entire solutions $\phi^1(\lambda_n), \phi^2(\lambda_n) \in \ell$ satisfying the limit relation

$$\lim_{n\to\infty} \phi^1(\lambda_n) = \lim_{n\to\infty} \phi^2(\lambda_n) = \phi^* \quad \text{in } \ell.$$

For a parameter space $\Lambda \subseteq \mathbb{R}$, one speaks of a *subcritical* or a *supercritical bifurcation*, if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ can be chosen according to $\lambda_n < \lambda^*$ or $\lambda_n > \lambda^*$, respectively.

In [30, Prop. 2.8] we showed that, in order to establish bifurcation results, it is essential to consider nonhyperbolic solutions ϕ^* of $(\Delta)_\lambda$, i.e. solutions satisfying the inclusion $1 \in \Sigma(\lambda^*)$. Based on an index 0 Fredholm theory and a Lyapunov-Schmidt reduction, we deduced criteria for finitely many bounded entire solutions to bifurcate (see [29]). Now we are interested in an although nonhyperbolic but weakly hyperbolic situation, where an index 0 Fredholm approach is not applicable. In addition, the bifurcation scenario we are about to study now, differs from the results in [29], since they also apply to linear equations.

Hypothesis. *Let $\kappa \in \mathbb{Z}$ and $\Lambda \subseteq \mathbb{R}$.*

(H_2) *For all $\lambda \in \Lambda$ the equation $(\Delta)_\lambda$ has an entire permanent solution $\phi(\lambda) \in \ell(\Omega)$.*

Now suppose that for every $\lambda \neq \lambda^*$ the variational equation $(V)_\lambda$ along $\phi(\lambda)$ admits an ET on $\mathbb{Z}$ with projectors $P_k(\lambda), Q_k(\lambda)$ and $\dim R(Q_k(\lambda)) < \infty$. This ensures that $(V)_\lambda$ has an ED on $\mathbb{Z}_\kappa^+$ with projector $P_k^+(\lambda) = P_k(\lambda) + Q_k(\lambda)$, and an ED on $\mathbb{Z}_\kappa^-$ with projector $P_k^-(\lambda) = P_k(\lambda)$. Consequently, by Prop. 3.4 the operator $L_\lambda \in L(\ell)$ given by

$$(L_\lambda \psi)_k := \psi_{k+1} - D_1 f_k(\phi(\lambda)_k, \lambda)\psi_k$$

is onto with Fredholm index $\dim R(P_\kappa^+(\lambda)(I - P_\kappa^-(\lambda))) = \dim R(Q_\kappa(\lambda))$ — we are dealing with positive index bifurcations. Choosing a fixed $\lambda^* \in \Lambda$, we define the function $r : \Lambda \setminus \{\lambda^*\} \to \mathbb{N}_0$ by

$$r(\lambda) := \dim R\big(P_\kappa^+(\lambda)(I - P_\kappa^-(\lambda))\big) = \dim R(Q_\kappa(\lambda)).$$

Introducing the one-sided limits

$$r_+(\lambda^*) := \lim_{\lambda \searrow \lambda^*} r(\lambda), \qquad r_-(\lambda^*) := \lim_{\lambda \nearrow \lambda^*} r(\lambda),$$

we say r has a *jump* at λ^*, if the following difference is nonzero:

$$j(\lambda^*) := r_+(\lambda^*) - r_-(\lambda^*).$$

For instance, as the next result shows, an ET does not guarantee that a finite number of solutions bifurcates.

Proposition 3.13 (jump bifurcation)**.** *Suppose that hypotheses (H_0), (H_2) hold and for every $\lambda \neq \lambda^*$ the variational equation $(V)_\lambda$ admits an ET on $\mathbb{Z}$ with projectors $P_k(\lambda), Q_k(\lambda)$ and $\dim R(Q_k(\lambda)) < \infty$. In case $\ell = \ell^\infty$ and $\phi^* = \phi(\lambda^*) \in \ell(\Omega)$, there exist open convex neighborhoods $U \subseteq \ell(\Omega)$ of ϕ^* and $\Lambda_1 \subseteq \Lambda$ of λ^* such that one has for $\lambda \in \Lambda_1$:*

(a) Subcritical case*: If $-r_-(\lambda^*) = j(\lambda^*) < 0$, then the unique entire bounded solution of $(\Delta)_\lambda$ is $\phi(\lambda)$ for $\lambda > \lambda^*$, while $\phi(\lambda)$ is embedded into a $r_-(\lambda^*)$-parameter family $\psi(\xi, \lambda) \in \ell(\Omega)$ of entire solutions in U of $(\Delta)_\lambda$ for $\lambda < \lambda^*$.*

(b) Supercritical case*: If $r_+(\lambda^*) = j(\lambda^*) > 0$, then the unique entire bounded solution of $(\Delta)_\lambda$ is $\phi(\lambda)$ for $\lambda < \lambda^*$, while $\phi(\lambda)$ is embedded into a $r_+(\lambda^*)$-parameter family $\psi(\xi, \lambda) \in \ell(\Omega)$ of entire solutions in U of $(\Delta)_\lambda$ for $\lambda > \lambda^*$.*

(c) The properties of the mapping ψ are given in Prop. 3.11 and Cor. 3.12.

If (H_0)–(H_1) are satisfied, then the same holds with $\ell = \ell_0$.

Proof. Let $\lambda \in \Lambda \setminus \{\lambda^*\}$ and due to analogy we restrict to assertion (b). By Prop. 3.11 we see that a difference equation $(\Delta)_\lambda$ has a $r_+(\lambda)$-parameter family of bounded entire solutions $\psi : B_\rho(0, \lambda^*) \subseteq R(Q_\kappa(\lambda^*)) \times \Lambda \to \ell(\Omega)$ forming an $r(\lambda)$-dimensional manifold $C(\lambda) \subseteq \ell^\infty(\Omega)$. By assumption, the function r has a jump at λ^* and $\dim C(\lambda)$ increases from 0 by $j(\lambda^*)$, as λ grows trough λ^*. On the other hand, $r_-(\lambda) = 0$ and thus $(V)_\lambda$ admits an ED on $\mathbb{Z}$ for $\lambda < \lambda^*$. This implies that equation $(\Delta)_\lambda$ has a unique bounded entire solution for $\lambda < \lambda^*$. Finally, the same argument applies in case $\ell = \ell_0$. □

Corollary 3.14. *If $j(\lambda^*) \neq 0$, then $1 \in \Sigma(\lambda^*)$.*

Proof. Let $\kappa \in \mathbb{Z}$. Arguing indirectly, we suppose that $1 \notin \Sigma(\lambda^*)$, i.e. $(V)_{\lambda^*}$ has an ED on $\mathbb{Z}$. By the roughness theorem for EDs (see [17, p. 232, Thm. 7.6.7]) there exists a neighborhood Λ_1 of λ^* such that also $(V)_\lambda$ admits an ED on $\mathbb{Z}$ for all $\lambda \in \Lambda_1$ with projector $P_k(\lambda)$. In particular, $(V)_\lambda$ has dichotomies on both semiaxes $\mathbb{Z}_\kappa^+$ and $\mathbb{Z}_\kappa^-$ with the same projector $P_k(\lambda)$. This guarantees $P_\kappa(\lambda)(I - P_\kappa(\lambda)) = 0$ and therefore $r(\lambda) \equiv 0$ on Λ_1, which contradicts the assumption $j(\lambda^*) \neq 0$. □

Example 3.7. The quantity $r(\lambda) \in \mathbb{N}_0$ can also be computed in the parameter-free situations from Examples 3.5 (left) and 3.6 (right). One obtains the following tables

$r(\lambda)$	(a_i)	(c_i)
$i = 1$	0	0
$i = 3$	1	0

$r(\lambda)$	(a_i)	(b_i)	(c_i)	(d_i)
$i = 1$	0	0	0	0
$i = 2$	1	0	1	0
$i = 3$	1	1	0	0
$i = 4$	2	1	1	0

where $(a_i), \dots, (d_i)$ refers to the corresponding constellations for $\alpha_\pm, \beta_\pm, \gamma_\pm$.

Hypothesis. *Let $D_1 f_k(\phi_k^*, \lambda^*) \in GL(X)$ for all $\lambda \in \Lambda \subseteq \mathbb{R}$ and $k \in \mathbb{Z}$.*

(H_3) *Suppose the dichotomy spectra of $(V)_\lambda$ allow a splitting*

$$\Sigma(\lambda) = \Sigma_-(\lambda)\dot{\cup}\sigma(\lambda), \qquad \Sigma^\pm(\lambda) = \Sigma_-^\pm(\lambda)\dot{\cup}\sigma^\pm(\lambda) \quad \text{for all } \lambda \in \Lambda,$$

and $\sup_{\lambda \in \Lambda} \max \Sigma_-(\lambda) < 1$.

Remark 3.5. Let m be the multiplicity of $\sigma(\lambda)$. In case $\max \Sigma_-(\lambda) < 1$ the equation $(\Delta)_\lambda$ possesses a center fiber bundle (cf. [27, Thm. 2.4]) and the stability analysis for the permanent entire solution $\phi(\lambda) \in \ell$ reduces to an m-dimensional problem, where a corresponding nonautonomous reduction principle can be found in [27, Thm. 3.5]. In the remaining two results we neglect the situation $\max \sigma^+(\lambda) = 1$. Here, the behavior of ϕ^* is determined by the restriction of $(\Delta)_\lambda$ on a center fiber bundle and particularly on Taylor coefficients of nonlinear terms (cf. [32]). As opposed to this setting, in the following stability and bifurcation results are determined by the linear part alone.

In the autonomous (or periodic) situation one has a powerful perturbation theory for isolated eigenvalues available, yielding their differentiable dependence on the parameters (see, for instance, [21, Chapter 7]). Since the dichotomy spectrum depends only upper-semicontinuously on parameters (cf. [28, Cor. 4]), one cannot expect a similar behavior for the boundary points of spectral intervals and instead we have to assume certain monotonicity properties for them. In this context, given a function $f : \Lambda \to \mathbb{R}$, a convenient terminology is as follows: We briefly say $f(\lambda^*) = 1$ *increases* (*decreases*), if $f(\lambda^*) = 1$ and the function f is strictly increasing (decreasing) in a neighborhood of λ^*.

In addition, a typical case where the function r has a jump is when spectral intervals cross the stability boundary 1. In the following two theorems we restrict to the dominant spectral interval $\sigma(\lambda)$, since it yields the essential stability information.

Theorem 3.15 (shovel bifurcation I)**.** *Let $\ell = \ell^\infty$ and suppose (H_0), (H_2), (H_3) hold. If*

$$\max \sigma(\lambda^*) = 1$$

and the dominant spectral interval $\sigma^-(\lambda)$ has constant multiplicity $m < \infty$, then there exists a neighborhood $\Lambda_1 \subseteq \Lambda$ of λ^* such that:

(a) Subcritical case*: If $\max \sigma$ is decreasing at λ^*, then*
- *(a_1) for $\lambda < \lambda^*$ one has*
  - *$\bullet_1$ if $\max \sigma^+(\lambda^*) < 1$ or $\max \sigma^+(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is asymptotically stable, and if also $\min \sigma^-(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is embedded into an m-parameter family of bounded entire solutions to $(\Delta)_\lambda$ in $\ell(\Omega)$,*
  - *$\bullet_2$ if $\min \sigma^+(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is unstable,*
- *(a_2) for $\lambda = \lambda^*$ and $\max \sigma^+(\lambda^*) < 1$ the solution $\phi(\lambda)$ is asymptotically stable,*
- *(a_3) for $\lambda > \lambda^*$ the unique entire bounded solution of $(\Delta)_\lambda$ is $\phi(\lambda)$; it is uniformly asymptotically stable*

(b) Supercritical case*: If $\max \sigma$ is increasing at λ^*, then*
- *(b_1) for $\lambda < \lambda^*$ the unique entire bounded solution of $(\Delta)_\lambda$ is $\phi(\lambda)$; it is uniformly asymptotically stable,*
- *(b_2) for $\lambda = \lambda^*$ and $\max \sigma^+(\lambda^*) < 1$ the solution $\phi(\lambda)$ is asymptotically stable,*
- *(b_3) for $\lambda > \lambda^*$ one has*
  - *$\bullet_1$ if $\max \sigma^+(\lambda^*) < 1$ or $\max \sigma^+(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is asymptotically stable, and if also $\min \sigma^-(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is embedded into an m-parameter family of bounded entire solutions to $(\Delta)_\lambda$ in $\ell(\Omega)$,*
  - *$\bullet_2$ if $\min \sigma^+(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is unstable*

for all $\lambda \in \Lambda_1$. If (H_0)–(H_3) are satisfied, then the same holds with $\ell = \ell_0$.

We refer to Fig. 4 for a schematic illustration of the bifurcation patterns described in the previous Thm. 3.15. To explain our terminology, the set of solutions in ℓ for different values of the parameter λ resembles a shovel rather than e.g. a pitchfork. The shape of the shovel depends on the nonlinearity (see the discussion in Ex. 3.8). For linear difference equations, the bifurcating family of bounded solutions fills the whole half-plane left (subcritical case) resp. right (supercritical case) of the critical parameter λ^*.

Proof. Let $\kappa \in \mathbb{Z}$ and suppose that we have $\max \Sigma(\lambda^*) = \max \sigma(\lambda^*) = 1$.

(a) Let $\max \sigma : \Lambda \to \mathbb{R}$ be decreasing at λ^*.

(a_1) Thus, for $\lambda < \lambda^*$ one has $\max \sigma(\lambda) > 1$ and information on the stability of $\phi(\lambda)$ can be obtained from the forward spectrum: If $\max \sigma^+(\lambda^*) < 1$ then the upper-semicontinuity of $\Sigma^+(\cdot)$ (see [28, Cor. 4]) guarantees $\max \Sigma^+(\lambda) < 1$ in a

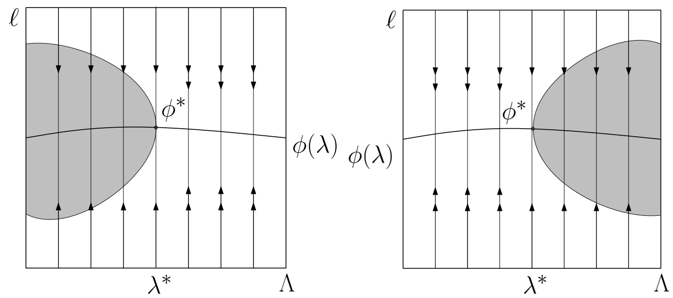

FIGURE 4. Bifurcation diagram for Thm. 3.15$\bullet_1$ with a subcritical shovel bifurcation (left) and a supercritical shovel bifurcation (right) of an entire solution ϕ^* (double arrows indicate uniform asymptotic stability).

neighborhood of λ^* and consequently $\phi(\lambda)$ is asymptotically stable by Prop. 3.9(a). In case $\max\sigma^+(\lambda^*)=1$ increases, it is $\max\Sigma^+(\lambda)<1$ in a left-sided neighborhood of λ^* and $\phi(\lambda)$ is asymptotically stable as above. Both situations feature an ED on $\mathbb{Z}_\kappa^+$ with identity projector $P_k^+(\lambda)\equiv I$. If we additionally have that $\min\sigma^-(\lambda^*)=1$ is decreasing, then $\min\sigma^-(\lambda)>1$. This ensures that $(V)_\lambda$ has an ED on $\mathbb{Z}_\kappa^-$ with projector $P_k^-(\lambda)$ and $\dim N(P_k^-(\lambda))=m$, where m is the multiplicity of σ^-. Consequently, the projectors $P_\kappa^-(\lambda),P_\kappa^+(\lambda)\equiv I$ fulfill (3.7) and $(V)_\lambda$ has an ET on $\mathbb{Z}$ by Lemma 3.2(b) and the solution $\phi(\lambda)$ is weakly hyperbolic. Moreover, the associated central projector satisfies $\dim R(Q_k(\lambda))=\dim N(P_k^-(\lambda))=m$. Thanks to Prop. 3.11 the solution $\phi(\lambda)$ is embedded into a m-parameter family of entire solutions in ℓ^∞. Finally, if $\max\sigma^+(\lambda^*)=1$ decreases, then $\min\sigma^+(\lambda^*)>1$ for $\lambda<\lambda^*$ and Prop. 3.10(a) guarantees that $\phi(\lambda)$ is unstable.
(a_2) The assertion follows from Prop. 3.9(a).
(a_3) For $\lambda>\lambda^*$ one has $\min\Sigma(\lambda)<1$ and consequently the stability claim follows from Prop. 3.9(b). In particular, $\phi(\lambda)$ is hyperbolic in the sense that $(V)_\lambda$ admits an ED on $\mathbb{Z}$. Hence, the uniqueness assertion on the solution $\phi(\lambda)$ is a consequence of [30, Thm. 2.11].

(b) The proof in the supercritical situation is dual to (a). □

Theorem 3.16 (shovel bifurcation II)**.** *Let $\ell=\ell^\infty$ and suppose (H_0), (H_2), (H_3) hold. If*

$$\min\sigma(\lambda^*)=1$$

and the dominant spectral interval $\sigma^-(\lambda)$ has constant multiplicity $m<\infty$, then there exists a neighborhood $\Lambda_1\subseteq\Lambda$ of λ^* such that:

(a) Subcritical case*: If $\min\sigma$ is increasing at λ^*, then*
  (a_1) for $\lambda<\lambda^*$ one has
    $\bullet_1$ *if $\max\sigma(\lambda^*)=1$ increases, then $\phi(\lambda)$ is uniformly asymptotically stable and the unique entire bounded solution of $(\Delta)_\lambda$,*
    $\bullet_2$ *if $\max\sigma^+(\lambda^*)=1$ increases, then $\phi(\lambda)$ is asymptotically stable, and if also $\min\sigma^-(\lambda^*)>1$ or $\min\sigma^-(\lambda^*)=1$ decreases, then $\phi(\lambda)$ is embedded into an m-parameter family of bounded entire solutions to $(\Delta)_\lambda$ in $\ell(\Omega)$,*
    $\bullet_3$ *if $\min\sigma^+(\lambda^*)>1$ or $\min\sigma^+(\lambda^*)=1$ decreases, then $\phi(\lambda)$ is unstable,*
  (a_2) for $\lambda=\lambda^*$ and $\min\sigma^+(\lambda^*)>1$ the solution $\phi(\lambda)$ is unstable,

(a_3) *for* $\lambda > \lambda^*$ *the solution* $\phi(\lambda)$ *is unstable and the unique entire bounded solution of* $(\Delta)_\lambda$*; in case* $\Sigma(\lambda) \neq \emptyset$ *it has an unstable fiber bundle*

(b) Supercritical case*: If* $\min \sigma$ *is decreasing at* λ^**, then*

(b_1) *for* $\lambda < \lambda^*$ *the unique entire bounded solution of* $(\Delta)_\lambda$ *is* $\phi(\lambda)$*; it is unstable; in case* $\Sigma(\lambda) \neq \emptyset$ *it has an unstable fiber bundle,*

(b_2) *for* $\lambda = \lambda^*$ *and* $\min \sigma^+(\lambda^*) > 1$ *the solution* $\phi(\lambda)$ *is unstable,*

(b_3) *for* $\lambda > \lambda^*$ *one has*

$\bullet_1$ *if* $\max \sigma(\lambda^*) = 1$ *decreases, then the unique entire bounded solution of* $(\Delta)_\lambda$ *is* $\phi(\lambda)$*; it is uniformly asymptotically stable,*

$\bullet_2$ *if* $\max \sigma^+(\lambda^*) = 1$*, then* $\phi(\lambda)$ *is asymptotically stable, and if also* $\min \sigma^-(\lambda^*) > 1$ *or* $\min \sigma^-(\lambda^*) = 1$ *increases, then* $\phi(\lambda)$ *is embedded into an* m*-parameter family of bounded entire solutions to* $(\Delta)_\lambda$ *in* $\ell(\Omega)$*,*

$\bullet_3$ *if* $\min \sigma^+(\lambda^*) > 1$ *or* $\min \sigma^+(\lambda^*) = 1$ *increases, then* $\phi(\lambda)$ *is unstable*

for all $\lambda \in \Lambda_1$*. If* (H_0)*–*(H_3) *are satisfied, then the same holds with* $\ell = \ell_0$*.*

See Fig. 5 for the bifurcation diagram illustrating Thm. 3.16.

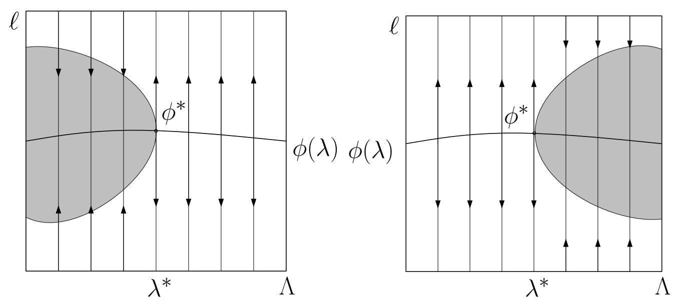

FIGURE 5. Bifurcation diagram for Thm. 3.16$\bullet_2$ with a subcritical shovel bifurcation (left) and a supercritical shovel bifurcation (right) of an entire solution ϕ^*

Remark 3.6 (size of the shovel). In Prop. 3.13 and Thms. 3.15, 3.16 we have established the existence of a strongly center fiber bundle using Prop. 3.11. Thus, an estimate on the ℓ^∞-norm of the bifurcating entire solutions in $\ell(\Omega)$ can be derived using Cor. 3.12.

Proof. Let $\kappa \in \mathbb{Z}$ and suppose the relation $\min \sigma(\lambda^*) = 1$.

(a) Let $\min \sigma$ be increasing at λ^*.

(a_1) For parameters $\lambda < \lambda^*$ one obtains $\min \sigma(\lambda) < 1$. If $\max \sigma(\lambda^*) = 1$ increases, then $\max \sigma(\lambda) = \max \Sigma(\lambda) < 1$ and the claim immediately follows from Prop. 3.9(b) and [30, Thm. 2.11], since $(V)_\lambda$ has an ED on $\mathbb{Z}$ with $P_k(\lambda) \equiv I$. If $\max \sigma^+(\lambda^*) = 1$ increases, then $\max \sigma^+(\lambda) = \max \Sigma^+(\lambda) < 1$ and $\phi(\lambda)$ is asymptotically stable by Prop. 3.9(a). In particular, we have an ED on $\mathbb{Z}_\kappa^+$ with $P_k^+(\lambda) \equiv I$. However, if $\min \sigma^-(\lambda^*) > 1$ or $\min \sigma^-(\lambda^*) = 1$ decreases, one deduces $\min \sigma^-(\lambda) > 1$ and $(V)_\lambda$ admits an ED on $\mathbb{Z}_\kappa^-$ with $\dim N(P_k^-(\lambda)) = m$. We conclude from Lemma 3.2(ii) that $(V)_\lambda$ has an ET on the whole axis $\mathbb{Z}$ and Prop. 3.11 implies that the weakly hyperbolic solution $\phi(\lambda) \in \ell$ is embedded into a m-parametric family of entire solutions in ℓ. Finally, provided $\min \sigma^+(\lambda^*) > 1$ or $\min \sigma^+(\lambda^*) = 1$ decreases, then $\min \sigma^+(\lambda^*) > 1$ and thus, the solution $\phi(\lambda)$ is unstable by Prop. 3.10(a).

(a_2) For $\min \sigma^+(\lambda^*) > 1$ the assertion follows from Prop. 3.10(a).

(a_3) For $\lambda > \lambda^*$ one has $\min \sigma(\lambda) > 1$. Hence, the solution $\phi(\lambda)$ is unstable by Prop. 3.10; in case $\Sigma^-(\lambda) \neq \emptyset$ one has an unstable fiber bundle.

(b) The situation is dual to the above case (a). □

We conclude this section with a simple nonlinear example illustrating the essential assertions of Prop. 3.13, as well as both Thm. 3.15 and 3.16.

Example 3.8 (shovel bifurcation). For given parameters $\delta \in (0,1)$ and a bifurcation parameter $\lambda > 0$, let us consider the scalar nonlinear difference equation

$$x_{k+1} = a_k(\lambda)x_k + x_k^3, \qquad a_k(\lambda) := \begin{cases} \lambda, & k \geq 0, \\ \lambda + \delta, & k < 0, \end{cases} \tag{3.13}$$

whose right-hand side $f_k(x,\lambda) = a_k(\lambda)x + x^3$ satisfies (H_0)–(H_2) with $\Omega = \mathbb{R}$ and the family of entire solutions $\phi(\lambda) \equiv 0$ for all λ. From Ex. 3.5 the linearization

$$x_{k+1} = a_k(\lambda)x_k \tag{3.14}$$

along $\phi(\lambda)$ has the dichotomy spectra

$$\Sigma(\lambda) = [\lambda, \lambda + \delta], \qquad \Sigma^-(\lambda) = \{\lambda + \delta\}, \qquad \Sigma^+(\lambda) = \{\lambda\}$$

and the corresponding dominant spectral intervals

$$\sigma(\lambda) = \{\lambda + \delta\}, \qquad \sigma^-(\lambda) = \{\lambda + \delta\}, \qquad \sigma^+(\lambda) = \{\lambda\}.$$

Obviously, the functions $\max \sigma$, $\min \sigma$ and $\max \sigma^\pm = \min \sigma^\pm$ are increasing. For the sake of a bifurcation analysis, the parameter values $\lambda^* = 1 - \delta$ and $\lambda^* = 1$ are of interest. Apart from these values, the linear part (3.14) admits an ET on $\mathbb{Z}$ with

$$P_k(\lambda) \equiv \begin{cases} 1, & \lambda < 1 - \delta, \\ 0, & \lambda > 1 - \delta, \end{cases} \qquad Q_k(\lambda) \equiv \begin{cases} 1, & \lambda \in (1-\delta, 1), \\ 0, & \lambda \notin [1-\delta, 1] \end{cases} \qquad \text{on } \mathbb{Z}$$

for all $\lambda \notin \{1 - \delta, 1\}$. This yields the bifurcation results:

- For $\lambda^* = 1 - \delta$ one has

$$r_-(\lambda^*) = 0, \qquad r_+(\lambda^*) = 1, \qquad j(\lambda^*) = 1$$

  and Prop. 3.13(b) implies a supercritical jump bifurcation. For $\lambda < 1 - \delta$ the unique entire bounded solution of (3.13) is 0 and for $\lambda \in (1 - \delta, 1)$ the zero solution embeds into a 1-parameter family of entire solutions in ℓ_0. By Thm. 3.15(b) the above jump bifurcation is actually a supercritical shovel bifurcation, where the uniformly asymptotically stable trivial solution of (3.13) becomes asymptotically stable, as λ passes through the value $1 - \delta$. The corresponding qualitative change in the solution portrait of (3.13) is depicted in Fig. 6 (left and middle).
- For $\lambda^* = 1$ one has

$$r_-(\lambda^*) = 1, \qquad r_+(\lambda^*) = 0, \qquad j(\lambda^*) = -1$$

  and Prop. 3.13(a) ensures a subcritical jump bifurcation; the 1-parameter family of entire solutions in ℓ_0 collapses to the zero solution as unique bounded solution, when λ passes through 1. We obtain a subcritical shovel bifurcation from Prop. 3.16(a), where the asymptotically stable zero solution becomes unstable. In Fig. 6 (middle, right) we have illustrated the related qualitative change in the solution portrait of (3.13). Note that Thm. 3.16 does not imply stability assertions of the trivial solution in case $\lambda^* = 1$. Here nevertheless, [32, Prop. 5.4] yields that 0 is unstable.

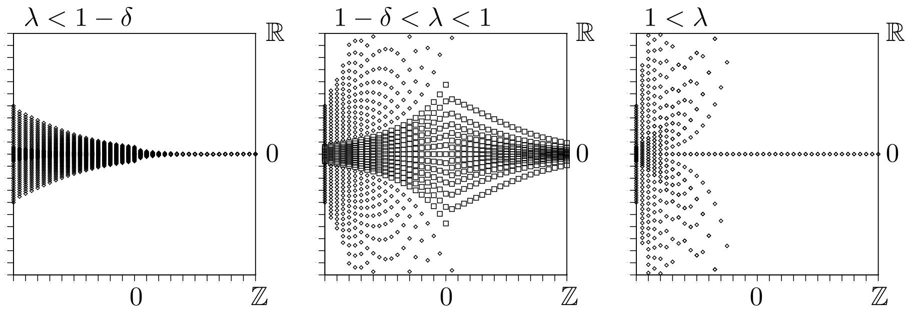

FIGURE 6. **Shovel bifurcation**: Solution portraits of the difference equation (3.13) from Ex. 3.8 with $\delta = \frac{1}{5}$ for different values of the parameter λ and $k = -20, \ldots, 20$. For $\lambda < \frac{4}{5}$ the trivial solution is the unique entire solution in ℓ_0 and uniformly asymptotically stable (left, $\lambda = \frac{7}{10}$). For $\lambda \in (\frac{4}{5}, 1)$ the trivial solution becomes asymptotically stable and is embedded into a 1-parameter family of solutions in ℓ_0 (middle, the points of the bounded solutions are marked as boxes $\Box$, $\lambda = 0.9$). Finally, for $1 < \lambda$ the zero solution is the unique bounded solution in ℓ_0, but unstable (right, $\lambda = \frac{11}{10}$).

Since the right-hand side $f_k(\cdot, \lambda) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, $\lambda \in \mathbb{R}$, of (3.13) is a C^∞-diffeomorphism, through every pair $(\kappa, \xi) \in \mathbb{Z} \times \mathbb{R}$ there exists a uniquely determined entire solution $\varphi_\lambda(\cdot; \kappa, \xi)$. For $k \geq 0$ and $\lambda \in [1-\delta, 1]$ the mappings $f_k(\cdot, \lambda)$ have the fixed points $0, \pm\sqrt{1-\lambda}$ and using graphical iterations one therefore sees that the entire solutions of (3.13) in ℓ_0 are exactly those with initial values satisfying

$$|\varphi_\lambda(0; \kappa, \xi)| \leq \sqrt{1-\lambda}.$$

In particular, for $\kappa = 0$ the pairs (λ, ξ) yielding entire solutions for (3.13) in ℓ_0 are given by $\left\{(\lambda, \xi) \in \mathbb{R}^2 : \xi = 0 \text{ or } \lambda \in [1-\delta, 1], |\xi| \leq \sqrt{1-\lambda}\right\}$.

Various other examples for transitions from an ED into an ET or backwards can be constructed by virtue of the linear parts discussed in Ex. 3.6.

4. **Differential equations.** In this section, we deal with finite-dimensional nonautonomous ordinary differential equations. Our investigations are parallel to the above case of discrete systems, although the dichotomy and Fredholm theory is simpler due to the existence of backward solutions. For this reason we prefer a more compact presentation.

As state space we consider an open convex nonempty subset $\Omega \subseteq \mathbb{R}^d$ and the continuously differentiable functions $\phi : \mathbb{R} \to \Omega$ are denoted by $C^1(\mathbb{R}, \Omega)$. The space of bounded continuous functions $BC(\mathbb{R}, \mathbb{R}^d)$ is equipped with the natural norm

$$\|\phi\| := \sup_{t \in \mathbb{R}} |\phi(t)|;$$

moreover, $BC^1(\mathbb{R}, \Omega)$ denotes the bounded continuously differentiable functions with bounded derivative, $BC_0(\mathbb{R}, \Omega)$ the functions in $BC(\mathbb{R}, \Omega)$ decaying to 0 in the limit $t \to \pm\infty$ and $BC_0^1(\mathbb{R}, \Omega)$ such functions $\phi \in BC^1(\mathbb{R}, \Omega)$ with $\phi, \dot{\phi} \in BC_0(\mathbb{R}, \Omega)$. It is convenient to abbreviate $BC := BC(\mathbb{R}, \mathbb{R}^d)$ and to proceed accordingly with the other spaces.

The nonempty *parameter space* $\Lambda \subseteq Y$ is assumed to be open and convex. We consider functions $f : \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ and nonautonomous parameter-dependent ordinary differential equations (ODE for short)

$$\boxed{\dot{x} = f(t, x, \lambda).} \tag{$(D)_\lambda$}$$

Under the assumptions given below, solutions $\phi : I \to \Omega$ of $(D)_\lambda$ are uniquely determined and exist on maximal open intervals $I \subseteq \mathbb{R}$. An *entire solution* to $(D)_\lambda$ is a solution existing on the whole axis $\mathbb{R}$, and a *permanent solution* fulfills

$$0 < \inf_{t \in I} \operatorname{dist}(\phi(t), \Omega).$$

Hypothesis. *Let $m \in \mathbb{N}$, suppose $f : \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ is continuous and $f(t, \cdot)$, $t \in \mathbb{R}$, is a C^m-function such that the following holds for $0 \le j \le m$:*

(H_0) *For all bounded $B \subseteq \Omega$ one has $\sup_{t \in \mathbb{R}} \sup_{x \in B} \left| D^j_{(2,3)} f(t, x, \lambda) \right| < \infty$ for all $\lambda \in \Lambda$* (well-definedness) *and for all $\lambda^* \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|x - y| < \delta \quad \Rightarrow \quad \sup_{t \in \mathbb{R}} \left| D^j_{(2,3)} f(t, x, \lambda) - D^j_{(2,3)} f(t, y, \lambda) \right| < \varepsilon$$

for all $x, y \in \Omega$ and $\lambda \in B_\delta(\lambda^*)$ (uniform continuity).

(H_1) *We have $0 \in \Omega$ and $\lim_{t \to \pm\infty} f(t, 0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

Under these assumptions the following *substitution operators*

$$F(\phi, \lambda)(t) := f(t, \phi(t), \lambda), \qquad F^\upsilon(\phi, \lambda)(t) := D_2^{\upsilon_1} D_3^{\upsilon_2} f(t, \phi(t), \lambda)$$

are defined for $t \in \mathbb{R}$, $0 \le j \le m$ and $\upsilon = (\upsilon_1, \upsilon_2) \in \mathbb{N}_0^2$ such that $\upsilon_1 + \upsilon_2 \le m$. We profit from our earlier preparations on such operators in [30] and simply quote:

Proposition 4.1. *Under (H_0) the operator $F : BC(\mathbb{R}, \Omega) \times \Lambda \to BC$ is well-defined and m-times continuously differentiable on $BC(\mathbb{R}, \Omega)^\circ \times \Lambda$ with partial derivatives*

$$D^\upsilon F(\phi, \lambda) = F^\upsilon(\phi, \lambda) \quad \text{for all } \phi \in BC(\mathbb{R}, \Omega)^\circ,\ \lambda \in \Lambda.$$

If (H_0)–(H_1) are satisfied, then the same holds for $F : BC_0(\mathbb{R}, \Omega) \times \Lambda \to BC_0$.

Proof. See [30, Lemma 3.3 and Prop. 3.4]. □

Corollary 4.2. *Under (H_0) the operator $G : BC^1(\mathbb{R}, \Omega) \times \Lambda \to BC$,*

$$G(\phi, \lambda) = \dot{\phi} - F(\phi, \lambda)$$

is well-defined and m-times continuously differentiable on $BC^1(\mathbb{R}, \Omega)^\circ \times \Lambda$. If (H_0)–(H_1) are satisfied, then the same holds for $G : BC^1_0(\mathbb{R}, \Omega) \times \Lambda \to BC_0$.

Proof. See [30, Cor. 3.5]. □

Finally, we can state that the bounded solutions to the ODE $(D)_\lambda$ and zeros of the abstract nonlinear operator G are related by

Theorem 4.3. *For $\lambda \in \Lambda$ the following holds under (H_0):*

(a) If $\phi \in BC(\mathbb{R}, \Omega)$ is an entire solution of $(D)_\lambda$, then $\phi \in BC^1(\mathbb{R}, \Omega)$ and

$$G(\phi, \lambda) = 0; \tag{4.1}$$

conversely, if $\phi \in C^1(\mathbb{R}, \Omega) \cap BC$ solves (4.1), then $\phi \in BC^1(\mathbb{R}, \Omega)$ and ϕ is an entire bounded solution of $(D)_\lambda$.

(b) *Under additionally* (H_1)*, if* $\phi \in BC_0(\mathbb{R}, \Omega)$ *is an entire solution of* $(D)_\lambda$*, then* $\phi \in BC_0^1(\mathbb{R}, \Omega)$ *and* (4.1) *holds; conversely, if* $\phi \in C^1(\mathbb{R}, \Omega) \cap BC_0$ *solves* (4.1)*, then* $\phi \in BC_0^1(\mathbb{R}, \Omega)$ *and* ϕ *is an entire bounded solution of* $(D)_\lambda$*.*

Proof. See [30, Thm. 3.6]. □

4.1. **Linear differential equations.** We consider a linear homogeneous differential equation

$$\boxed{\dot{x} = A(t)x} \tag{LD}$$

with a given continuous coefficient operator $A : \mathbb{R} \to L(\mathbb{R}^d)$. Its *transition operator* $\Phi(t, s) \in L(\mathbb{R}^d)$, $t, s \in \mathbb{R}$, is the unique solution of the matrix valued initial value problem $\dot{X} = A(t)X$, $X(s) = I$ in $L(\mathbb{R}^d)$.

Let us suppose that $\mathbb{I} \subseteq \mathbb{R}$ is a real interval. We describe invariant splittings of the *extended state space* $\mathbb{I} \times \mathbb{R}^d$ for (LD) using *invariant projectors*, i.e. projection-valued functions $P : \mathbb{I} \to L(\mathbb{R}^d)$ which commute with the transition operator by means of the relation $\Phi(t, s)P(s) = P(t)\Phi(t, s)$ for all $t, s \in \mathbb{I}$. The following terminology due to [15] defines (LD) to admit an *exponential trichotomy* (ET for short) on $\mathbb{I}$, provided:

(i) There exist invariant projectors $P, Q : \mathbb{I} \to L(\mathbb{R}^d)$ with $P(t)Q(t) \equiv Q(t)P(t) \equiv 0$ on $\mathbb{I}$,
(ii) there exist reals $K \geq 1$, $\alpha > 0$ and $\tau \in \mathbb{I}$ such that

$$\begin{aligned} |\Phi(t, s)P(s)| &\leq Ke^{-\alpha(t-s)} \quad \text{for all } s \leq t, \\ |\Phi(t, s)Q(s)| &\leq Ke^{-\alpha|t-s|} \quad \text{for all } \tau \leq s \leq t \text{ or } t \leq s \leq \tau, \\ |\Phi(t, s)[I - P(s) - Q(s)]| &\leq Ke^{\alpha(t-s)} \quad \text{for all } t \leq s. \end{aligned}$$

Similarly to the difference equations case, the ranges of P, Q form the stable and strongly center bundle for (LD), resp., while the kernels of $P + Q$ are the unstable bundle. In particular, for $Q = 0$, we say (LD) admits an *exponential dichotomy* (ED for short) on $\mathbb{I}$.

Remark 4.1. Let $\theta > 0$. A θ-periodic linear differential equation (LD) with *monodromy operator* $M := \Phi(\tau + \theta, \tau)$ admits an ED on $\mathbb{R}$, provided the spectrum $\sigma(M)$ does not intersect the complex unit circle (cf. [13, p. 203, Thm. 2.1]).

Lemma 4.4. *Let* $\overline{\tau}, \underline{\tau} \in \mathbb{R}$ *with* $\overline{\tau} \leq \underline{\tau}$*. If a linear differential equation* (LD) *has an ED both on* $[\underline{\tau}, \infty)$ *(with projector* P^+*) and on* $(-\infty, \overline{\tau}]$ *(with projector* P^-*), then it has an ET on* $\mathbb{R}$*, provided one of the following conditions holds:*

(i) Every solution of (LD) *is the sum of a solution bounded on* $[\underline{\tau}, \infty)$ *and a solution bounded on* $(-\infty, \overline{\tau}]$*,*
(ii) one has the relation

$$P^-(\overline{\tau}) = P^-(\overline{\tau})\Phi(\overline{\tau}, \underline{\tau})P^+(\underline{\tau})\Phi(\underline{\tau}, \overline{\tau}) = \Phi(\overline{\tau}, \underline{\tau})P^+(\underline{\tau})\Phi(\underline{\tau}, \overline{\tau})P^-(\overline{\tau}), \tag{4.2}$$

where the projectors associated to the ET read as $P := \Phi(\cdot, \overline{\tau})P^-(\overline{\tau})\Phi(\overline{\tau}, \cdot)$ *and* $Q := \Phi(\cdot, \underline{\tau})P^+(\underline{\tau})\Phi(\underline{\tau}, \cdot) - \Phi(\cdot, \overline{\tau})P^-(\overline{\tau})\Phi(\overline{\tau}, \cdot)$*.*

Remark 4.2. (1) The argument of Rem. 3.2(1) and [12, p. 70, Prop. 3] (for this see also [23]) yield that for almost-periodic differential equations (LD) one necessarily has $Q = 0$.

(2) In case $\tau := \overline{\tau} = \underline{\tau}$ we know that (4.2) is equivalent to $N(P^+(\tau)) \subseteq N(P^-(\tau))$ and $R(P^-(\tau)) \subseteq R(P^+(\tau))$; moreover one has

$$R(P^+(\tau)(I - P^-(\tau))) = R(P^+(\tau)) \cap N(P^-(\tau)), \quad R(P^+(\tau)) + N(P^-(\tau)) = \mathbb{R}^d.$$

Proof. As in the proof of Lemma 3.2 we establish EDs on the intervals $[\underline{\tau}, \infty)$ and $(-\infty, \underline{\tau}]$ and refer to [15, Lemma 1.2]. □

Let $\gamma \in \mathbb{R}$. As in [35, 34], a spectral notion for linear differential equations (LD) can be formulated using the shifted linear differential equation

$$\boxed{\dot{x} = [A(t) - \gamma I]\, x} \tag{$(LD)_\gamma$}$$

and now we are in a position to define the

dichotomy spectrum $\Sigma(A) := \{\gamma \in \mathbb{R} : (LD)_\gamma \text{ has no ED on } \mathbb{R}\}$,

forward dichotomy spectrum $\Sigma^+(A) := \{\gamma \in \mathbb{R} : (LD)_\gamma \text{ has no ED on } [\tau, \infty)\}$,

backward dichotomy spectrum $\Sigma^-(A) := \{\gamma \in \mathbb{R} : (LD)_\gamma \text{ has no ED on } (-\infty, \tau]\}$

of (LD) for some $\tau \in \mathbb{I}$. The dichotomy spectra $\Sigma^+(A), \Sigma^-(A)$ are independent of the instant $\tau \in \mathbb{R}$ and the inclusions $\Sigma^+(A), \Sigma^-(A) \subseteq \Sigma(A)$ hold true. However, for general coefficient matrices A only numerical methods enable us to approximate $\Sigma(A)$ or $\Sigma^\pm(\lambda)$ (see [14]). From a theoretical side the following is known:

The dichotomy spectrum $\Sigma(A)$ is a compact subset of $\mathbb{R}$, provided (LD) has *bounded growth* (cf. [34, Thm. 3.1]), i.e. there exist reals $K_0 \geq 1$, $\omega \geq 0$ such that

$$|\Phi(t,s)| \leq K_0 e^{\omega|t-s|} \quad \text{for all } s, t \in \mathbb{I}. \tag{4.3}$$

In this situation it has been shown in [34, Spectral Theorem] that $\Sigma(A)$ is the disjoint union of $n \leq d$ nonempty compact *spectral intervals* $\sigma_1, \ldots, \sigma_n \subseteq \mathbb{R}$, i.e.

$$\Sigma(A) = \bigcup_{i=1}^n \sigma_i, \qquad \sup \sigma_i < \inf \sigma_{i+1} \quad \text{for all } 1 \leq i < n$$

and we call σ_n the *dominant spectral interval.* Information on $\Sigma(A)$ yields a decomposition of the extended state space into invariant vector bundles. For this, suppose $\mathbb{I} = \mathbb{R}$ and choose reals $b_1 < a_2 < \ldots < b_{n-1} < a_n$ with $\Sigma(A) \cap \bigcup_{i=1}^{n-1}(b_i, a_{i+1}) = \emptyset$. If possible, pick $\gamma_0 \in \mathbb{R} \setminus \Sigma(A)$ with $(-\infty, \gamma_0) \subseteq \mathbb{R} \setminus \Sigma(A)$ and otherwise set

$$\mathcal{S}^-_{\gamma_0} := \mathbb{R} \times \mathbb{R}^d, \qquad \mathcal{S}^+_{\gamma_0} := \mathbb{R} \times \{0\}.$$

Moreover, choose $\gamma_n \in \mathbb{R} \setminus \Sigma(A)$ with $(\gamma_n, \infty) \subseteq \mathbb{R} \setminus \Sigma(A)$ and otherwise define

$$\mathcal{S}^-_{\gamma_n} := \mathbb{R} \times \{0\}, \qquad \mathcal{S}^+_{\gamma_n} := \mathbb{R} \times \mathbb{R}^d.$$

For growth rates $\gamma \in \mathbb{R}$ we introduce nonautonomous sets

$$\mathcal{S}^+_\gamma := \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \sup_{\tau \leq t} |\Phi(t,\tau)\xi|\, e^{\gamma(\tau - t)} < \infty\right\},$$

$$\mathcal{S}^-_\gamma := \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \sup_{t \leq \tau} |\Phi(t,\tau)\xi|\, e^{\gamma(\tau - t)} < \infty\right\}$$

and choose $\gamma_i \in (b_i, a_{i+1})$ for $1 \leq i < n$ to define *spectral manifolds* via the intersection $\mathcal{W}_i := \mathcal{S}^-_{\gamma_{i-1}} \cap \mathcal{S}^+_{\gamma_i}$, $1 \leq i \leq n$. They are linear integral manifolds for (LD), independent of γ_i, and also fulfill $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_n = \mathbb{R} \times \mathbb{R}^d$. We adopt our notation and terminology from the above discrete case, where the hyperbolicity condition (3.9) has to be replaced by $0 \notin \Sigma(A)$.

Proposition 4.5. *In case (LD) has an ET on $\mathbb{R}$ with $Q \neq 0$, then $[-\alpha, \alpha] \subseteq \Sigma(A)$.*

Proof. Proceed as in Prop. 3.3 and use [15, Thm. 5.2]. Here the transition operator of the shifted equation $(LD)_\gamma$ reads as $\Phi_\gamma(t,s) := e^{\gamma(s-t)}\Phi(t,s)$. □

Crucial for our approach are surjectivity properties of the differential operator

$$L : \mathcal{C}^1 \to \mathcal{C}, \qquad (L\phi)(t) := \dot{\phi}(t) - A(t)\phi(t) \quad \text{for all } t \in \mathbb{R}, \tag{4.4}$$

where $\mathcal{C}$ denotes one of the linear spaces BC or BC_0. A bounded coefficient function $A : \mathbb{R} \to L(\mathbb{R}^d)$ ensures that L is well-defined and continuous — an assumption we impose from now on.

Proposition 4.6. *Let $\tau \in \mathbb{R}$. If a linear equation (LD) has an ET on $\mathbb{R}$, then $L : \mathcal{C}^1 \to \mathcal{C}$ is Fredholm, has a complemented kernel with*

$$N(L) = \left\{\Phi(\cdot, \tau)\xi \in \mathcal{C}^1 : \xi \in R(Q(\tau))\right\}, \qquad R(L) = \mathcal{C}$$

and index $\dim R(Q(\tau))$. In case of an ED on $\mathbb{R}$ one has $L \in GL(\mathcal{C}^1, \mathcal{C})$.

Proof. This follows from the proof of [15, Thm. 5.2]. □

Corollary 4.7. *If $R(L) = \mathcal{C}$ and there exists a real $\alpha > 0$ such that $[-\alpha, \alpha] \subseteq \Sigma(A)$, then (LD) admits an ET on $\mathbb{R}$ with nonzero central projector $Q \neq 0$.*

Proof. Proceed dually to the proof of Cor. 3.6. □

Proposition 4.8. *Let $\tau \in \mathbb{R}$. If a linear differential equation (LD) admits an ED both on $[\tau, \infty)$ (with projector P^+) and on $(-\infty, \tau]$ (with projector P^-), then $L : \mathcal{C}^1 \to \mathcal{C}$ is Fredholm with index $\dim R(P^+(\tau)) - \dim R(P^-(\tau))$.*

Proof. See [22, Lemma 4.2]. □

4.2. **Stability, spectra and strongly center integral manifolds.** In this subsection, we keep a parameter $\lambda \in \Lambda$ fixed and suppose that ϕ_λ stands for a bounded entire solution of $(D)_\lambda$. The corresponding stability properties of ϕ_λ can be investigated using the dichotomy spectra of the *variational equation*

$$\boxed{\dot{x} = D_2 f(t, \phi_\lambda(t), \lambda)x,} \tag{$VD)_\lambda$}$$

whose transition operator is denoted by $\Phi_\lambda(t,s)$ and its dichotomy spectrum by $\Sigma(\lambda)$. We say the solution ϕ_λ is *weakly hyperbolic*, if $(VD)_\lambda$ has an ET on $\mathbb{R}$ and for a *hyperbolic* solution the equation $(VD)_\lambda$ is exponentially dichotomous on $\mathbb{R}$. A comprehensive stability theory for $(VD)_\lambda$ has been developed for instance in [11] and we obtain

Proposition 4.9. *Let $\lambda \in \Lambda$. Under (H_0) the following holds:*

(a) If $\max \Sigma^+(\lambda) < 0$, then ϕ_λ is asymptotically stable,
(b) if $\max \Sigma(\lambda) < 0$, then ϕ_λ is uniformly asymptotically stable.

Proof. We proceed as in the proof of Prop. 3.9 and use [11, p. 68, Thm. 8] to justify (a), while [11, p. 70, Thm. 9] is appropriate to show assertion (b). □

An entire solution $\phi : \mathbb{R} \to \Omega$ of $(D)_\lambda$ is said to admit an *unstable integral manifold*, if for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ there exists a backward solution $\psi : (-\infty, \tau] \to \Omega$ satisfying the relations $\psi(\tau) \in \dot{B}_\varepsilon(\phi(\tau))$ and $\lim_{t \to -\infty} |\psi(t) - \phi(t)| = 0$.

Proposition 4.10. *Let $\lambda \in \Lambda$. Under (H_0) suppose the dichotomy spectrum of $(VD)_\lambda$ allows a splitting $\Sigma(\lambda) = \Sigma_- \dot\cup \sigma$, where σ and σ^+ are the dominant spectral intervals of $\Sigma(\lambda)$ resp. $\Sigma^+(\lambda)$. Then the following holds:*

(a) If $\min \sigma^+ > 0$, then ϕ_λ is unstable,
(b) if $\Sigma_- \neq \emptyset$, $\max \Sigma_- < 0$ and $\min \sigma > 0$, then ϕ_λ is unstable and admits an unstable integral manifold.

Proof. Proceed analogously to the difference equations case considered in Prop. 3.10 using the result [11, p. 74, Thm. 10]. The existence of an unstable integral manifold can be deduced from [33, Thm. 3.2] applied to the equation of perturbed motion. □

The continuous counterpart to Prop. 3.11 reads as follows:

Proposition 4.11 (strongly center integral manifolds)**.** *Let $\lambda^* \in \Lambda$, $\tau \in \mathbb{R}$ and suppose that (H_0) is fulfilled. If $\mathcal{C} = BC$ and $\phi^* \in \mathcal{C}(\mathbb{R}, \Omega)$ is an entire permanent and weakly hyperbolic solution of equation $(D)_{\lambda^*}$, then there exists a $\rho > 0$ and a unique C^m-function $\psi : B_\rho(0, \lambda^*) \subseteq R(Q(\tau)) \times \Lambda \to \mathcal{C}(\mathbb{R}, \Omega)$ such that one has:*

(a) $\psi(0, \lambda^*) = \phi^*$,
(b) $\psi(\xi, \lambda)$ is an entire weakly hyperbolic solution of equation $(D)_\lambda$

for all $(\xi, \lambda) \in B_\rho(0, \lambda^*)$. If (H_0)–(H_1) are satisfied, then the same holds with $\mathcal{C} = BC_0$.

Proof. This can be shown as in Prop. 3.11. Here, the appropriate counterpart to Prop. 3.4 is Prop. 4.6, while Cor. 4.2 serves as substitute for Thm. 3.1(a). A combination of Thm. 2.1 with $X = \mathcal{C}^1$, $Z = \mathcal{C}$ and Thm. 4.3 yields our claim. □

4.3. **Jump and shovel bifurcations.** We adopt our bifurcation notion for bounded entire solutions ϕ^* of $(D)_{\lambda^*}$ from the previous explanations for difference equations given in Subsection 3.3 and suppose:

Hypothesis. *Let $\tau \in \mathbb{R}$ and $\Lambda \subseteq \mathbb{R}$.*

(H_2) *For all $\lambda \in \Lambda$ the differential equation $(D)_\lambda$ has an entire permanent solution $\phi_\lambda \in \mathcal{C}(\mathbb{R}, \Omega)$.*

Let us assume that the variational equation $(VD)_\lambda$ along ϕ_λ admits an ET on the whole axis $\mathbb{R}$ with associated projectors P_λ, Q_λ for all $\lambda \neq \lambda^*$. Hence, the assumptions of Prop. 4.8 are met with $P_\lambda^- = P_\lambda$ and $P_\lambda^+ = P_\lambda + Q_\lambda$. Therefore, we know that the operator

$$L_\lambda : \mathcal{C}^1 \to \mathcal{C}, \qquad (L_\lambda \psi)(t) := \dot\psi(t) - D_2 f(t, \phi_\lambda(t), \lambda)\psi(t)$$

is Fredholm. By Prop. 4.6 it is onto with Fredholm index

$$\dim R\big(P_\lambda^+(\tau)(I - P_\lambda^-(\tau))\big) = \dim R(Q_\lambda(\tau)).$$

As in the discrete case we are confronted with positive index bifurcations. We define $r : \Lambda \setminus \{\lambda^*\} \to \mathbb{N}_0$ by $r(\lambda) := \dim R\big(P_\lambda^+(\tau)(I - P_\lambda^-(\tau))\big) = \dim R(Q_\lambda(\tau))$ and say that the function r has a *jump* at λ^*, if

$$j(\lambda^*) := r_+(\lambda^*) - r_-(\lambda^*) \neq 0,$$

with the one-sided limits

$$r_+(\lambda^*) := \lim_{\lambda \searrow \lambda^*} r(\lambda), \qquad r_-(\lambda^*) := \lim_{\lambda \nearrow \lambda^*} r(\lambda).$$

Proposition 4.12 (jump bifurcation)**.** *Suppose that hypotheses* (H_0)*,* (H_2) *hold and for every* $\lambda \neq \lambda^*$ *the variational equation* $(VD)_\lambda$ *admits an ET on* $\mathbb{R}$ *with projectors* P_λ, Q_λ*. In case* $\mathcal{C} = BC$ *and* $\phi^* = \phi_{\lambda^*} \in \mathcal{C}(\mathbb{R}, \Omega)$*, there exist open convex neighborhoods* $U \subseteq \mathcal{C}(\mathbb{R}, \Omega)$ *of* ϕ^* *and* $\Lambda_1 \subseteq \Lambda$ *of* λ^* *such that the following holds for* $\lambda \in \Lambda_1$*:*

(a) Subcritical case*: If* $-r_-(\lambda^*) = j(\lambda^*) < 0$*, then the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ *for* $\lambda > \lambda^*$*, while* ϕ_λ *is embedded into a* $r_-(\lambda^*)$*-parameter family* $\psi(\xi, \lambda) \in \mathcal{C}(\mathbb{R}, \Omega)$ *of entire solutions in* U *of* $(D)_\lambda$ *for* $\lambda < \lambda^*$*.*

(b) Supercritical case*: If* $r_+(\lambda^*) = j(\lambda^*) > 0$*, then the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ *for* $\lambda < \lambda^*$*, while* ϕ_λ *is embedded into a* $r_+(\lambda^*)$*-parameter family* $\psi(\xi, \lambda) \in \mathcal{C}(\mathbb{R}, \Omega)$ *of entire solutions in* U *of* $(D)_\lambda$ *for* $\lambda > \lambda^*$*.*

(c) The properties of the mapping ψ *are given in Prop. 4.11.*

If (H_0)*–*(H_1) *are satisfied, then the same holds with* $\mathcal{C} = BC_0$*.*

Proof. The proof is mimics that of Prop. 3.13, with Prop. 3.11 replaced by the above Prop. 4.11. □

Corollary 4.13. *If* $j(\lambda^*) \neq 0$*, then* $0 \in \Sigma(\lambda^*)$*.*

Proof. Proceed analogously to Cor. 3.14. Concerning the corresponding roughness theorem for EDs we refer to [12, p. 42, Prop. 1]. □

Hypothesis. *Let* $\lambda \in \Lambda \subseteq \mathbb{R}$*.*

(H_3) *Suppose the dichotomy spectra of* $(VD)_\lambda$ *allow a splitting*

$$\Sigma(\lambda) = \Sigma_-(\lambda)\dot{\cup}\sigma(\lambda), \qquad \Sigma^\pm(\lambda) = \Sigma_-^\pm(\lambda)\dot{\cup}\sigma^\pm(\lambda) \quad \text{for all } \lambda \in \Lambda,$$

and $\sup_{\lambda\in\Lambda} \max \Sigma_-(\lambda) < 0$*.*

In the subsequent results stability properties are determined by the linear part $(VD)_\lambda$ alone. We neglect the situation $\max \sigma^+(\lambda^*) = 0$, where a nonautonomous center manifold reduction is required, which involves preparations from [6, 33].

Theorem 4.14 (shovel bifurcation I)**.** *Let* $\mathcal{C} = BC$ *and suppose* (H_0)*,* (H_2)*,* (H_3) *hold. If*

$$\max \sigma(\lambda^*) = 0$$

and the dominant spectral interval $\sigma^-(\lambda)$ *has constant multiplicity* $m < \infty$*, then there exists a neighborhood* $\Lambda_1 \subseteq \Lambda$ *of* λ^* *such that:*

(a) Subcritical case*: If* $\max \sigma$ *is decreasing at* λ^**, then*

- (a_1) *for* $\lambda < \lambda^*$ *one has*
  - $\bullet_1$ *if* $\max \sigma^+(\lambda^*) < 0$ *or* $\max \sigma^+(\lambda^*) = 0$ *increases, then* ϕ_λ *is asymptotically stable, and if also* $\min \sigma^-(\lambda^*) = 0$ *decreases, then* ϕ_λ *is embedded into an* m*-parameter family of bounded entire solutions to* $(D)_\lambda$ *in* $\mathcal{C}(\mathbb{R}, \Omega)$*,*
  - $\bullet_2$ *if* $\min \sigma^+(\lambda^*) = 0$ *decreases, then* ϕ_λ *is unstable,*
- (a_2) *for* $\lambda = \lambda^*$ *and* $\max \sigma^+(\lambda^*) < 0$ *the solution* ϕ_λ *is asymptotically stable,*
- (a_3) *for* $\lambda > \lambda^*$ *the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ*; it is uniformly asymptotically stable*

(b) Supercritical case*: If* $\max \sigma$ *is increasing at* λ^**, then*

- (b_1) *for* $\lambda < \lambda^*$ *the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ*; it is uniformly asymptotically stable,*
- (b_2) *for* $\lambda = \lambda^*$ *and* $\max \sigma^+(\lambda^*) < 0$ *the solution* ϕ_λ *is asymptotically stable,*
- (b_3) *for* $\lambda > \lambda^*$ *one has*

$\bullet_1$ *if* $\max \sigma^+(\lambda^*) < 0$ *or* $\max \sigma^+(\lambda^*) = 0$ *decreases, then* ϕ_λ *is asymptotically stable, and if also* $\min \sigma^-(\lambda^*) = 0$ *increases, then* ϕ_λ *is embedded into an* m*-parameter family of bounded entire solutions to* $(D)_\lambda$ *in* $\mathcal{C}(\mathbb{R}, \Omega)$*,*
$\bullet_2$ *if* $\min \sigma^+(\lambda^*) = 0$ *increases, then* ϕ_λ *is unstable*

for all $\lambda \in \Lambda_1$*. If* (H_0)*–*(H_3) *are satisfied, then the same holds with* $\mathcal{C} = BC_0$*.*

Proof. One proceeds as in the proof of the discrete version given in Thm. 3.15. Here, both the Props. 3.9 and 3.10 have to be replaced by the corresponding results stated in Prop. 4.9 resp. 4.10. The required strongly center integral manifold result is given in Prop. 4.11. □

Theorem 4.15 (shovel bifurcation II)**.** *Let* $\mathcal{C} = BC$ *and suppose* (H_0)*,* (H_2)*,* (H_3) *hold. If*

$$\min \sigma(\lambda^*) = 0$$

and the dominant spectral interval $\sigma^-(\lambda)$ *has constant multiplicity* $m < \infty$*, then there exists a neighborhood* $\Lambda_1 \subseteq \Lambda$ *of* λ^* *such that:*

(a) Subcritical case*: If* $\min \sigma$ *is increasing at* λ^**, then*
- (a_1) *for* $\lambda < \lambda^*$ *one has*
  - $\bullet_1$ *if* $\max \sigma(\lambda^*) = 0$ *increases, then* ϕ_λ *is uniformly asymptotically stable and the unique entire bounded solution of* $(D)_\lambda$*,*
  - $\bullet_2$ *if* $\max \sigma^+(\lambda^*) = 0$ *increases, then* ϕ_λ *is asymptotically stable, and if also* $\min \sigma^-(\lambda^*) > 0$ *or* $\min \sigma^-(\lambda^*) = 0$ *decreases, then* ϕ_λ *is embedded into an* m*-parameter family of bounded entire solutions to* $(D)_\lambda$ *in* $\mathcal{C}(\mathbb{R}, \Omega)$*,*
  - $\bullet_3$ *if* $\min \sigma^+(\lambda^*) > 0$ *or* $\min \sigma^+(\lambda^*) = 0$ *decreases, then* ϕ_λ *is unstable,*
- (a_2) *for* $\lambda = \lambda^*$ *and* $\min \sigma^+(\lambda^*) > 0$ *the solution* ϕ_λ *is unstable,*
- (a_3) *for* $\lambda > \lambda^*$ *the solution* ϕ_λ *is unstable and the unique entire bounded solution of* $(D)_\lambda$*; in case* $\Sigma(\lambda) \neq \emptyset$ *it has an unstable fiber bundle*

(b) Supercritical case*: If* $\min \sigma$ *is decreasing at* λ^**, then*
- (b_1) *for* $\lambda < \lambda^*$ *the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ*; it is unstable; in case* $\Sigma(\lambda) \neq \emptyset$ *it has an unstable fiber bundle,*
- (b_2) *for* $\lambda = \lambda^*$ *and* $\min \sigma^+(\lambda^*) > 0$ *the solution* ϕ_λ *is unstable,*
- (b_3) *for* $\lambda > \lambda^*$ *one has*
  - $\bullet_1$ *if* $\max \sigma(\lambda^*) = 0$ *decreases, then the unique entire bounded solution of* $(D)_\lambda$ *is* ϕ_λ*; it is uniformly asymptotically stable,*
  - $\bullet_2$ *if* $\max \sigma^+(\lambda^*) = 0$*, then* ϕ_λ *is asymptotically stable, and if additionally* $\min \sigma^-(\lambda^*) > 0$ *or* $\min \sigma^-(\lambda^*) = 0$ *increases, then* ϕ_λ *is embedded into an* m*-parameter family of bounded entire solutions to* $(D)_\lambda$ *in* $\mathcal{C}(\mathbb{R}, \Omega)$*,*
  - $\bullet_3$ *if* $\min \sigma^+(\lambda^*) > 0$ *or* $\min \sigma^+(\lambda^*) = 0$ *increases, then* ϕ_λ *is unstable*

for all $\lambda \in \Lambda_1$*. If* (H_0)*–*(H_3) *are satisfied, then the same holds with* $\mathcal{C} = \mathcal{C}_0$*.*

Proof. The proof is a dual version of the corresponding arguments in Thm. 3.16. Again, one has to replace the Props. 3.9, 3.10, Prop. 3.11 by Prop. 4.9, 4.10 and 4.11, respectively. □

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REFERENCES

[1] R.P. Agarwal, *Difference equations and inequalities*, 2nd ed., Pure and Applied Mathematics 228, Marcel Dekker, New York, 2000.

[2] A.I. Alonso, J. Hong, and R. Obaya, *Exponential dichotomy and trichotomy for difference equations*, Comput. Math. Appl. **38** (1998), 41–49.

[3] L. Arnold, *Random dynamical systems*, Monographs in Mathematics, Springer, Berlin etc., 1998.

[4] B. Aulbach and S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*, J. Difference Equ. Appl. **7** (2001), no. 6, 895–913.

[5] ———, *A spectral theory for nonautonomous difference equations*, Proceedings of the 5th Intern. Conference of Difference Eqns. and Application (Temuco, Chile, 2000) (et al. López-Fenner, J., ed.), Taylor & Francis, London, 2002, pp. 45–55.

[6] B. Aulbach, *A reduction principle for nonautonomous differential equations*, Archiv der Mathematik **39** (1982), 217–232.

[7] B. Aulbach and N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, J. Difference Equ. Appl. **2** (1996), 251–262.

[8] A. Ben-Artzi and I. Gohberg, *Dichotomy, discrete Bohl exponents, and spectrum of block weighted shifts*, Integral Equations Oper. Theory **14** (1991), no. 5, 613–677.

[9] ———, *Dichotomies of perturbed time varying systems and the power method*, Indiana Univ. Math. J. **42** (1993), no. 3, 699–720.

[10] A.G. Baskakov, *Invertibility and the Fredholm property of difference operators*, Mathematical Notes **67** (2000), no. 6, 690–698.

[11] W.A. Coppel, *Stability and asymptotic behavior of differential equations*, D.C. Heath, Boston, 1965.

[12] ———, *Dichotomies in stability theory*, Lect. Notes Math. 629, Springer, Berlin etc., 1978.

[13] J.L. Daleckiĭ and M.G. Kreĭn, *Stability of solutions of differential equations in Banach space*, Translations of Mathematical Monographs 43, American Mathematical Society, Providence, RI, 1974.

[14] L. Dieci and E.S. van Vleck, *Lyapunov and Sacker-Sell spectral intervals*, J. Dyn. Differ. Equations **19** (2007), no. 2, 265–293.

[15] S. Elaydi and O. Hajek, *Exponential trichotomy of differential systems*, J. Math. Anal. Appl. **129** (1988), 362–374.

[16] S. Elaydi and K. Janglajew, *Dichotomy and trichotomy of difference equations*, J. Difference Equ. Appl. **3** (1998), 417–448.

[17] D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. 840, Springer, Berlin etc., 1981.

[18] J.M. Holtzman, *Explicit ε and δ for the implicit function theorem*, SIAM Review **12** (1970), no. 2, 284–286.

[19] T. Hüls, *Computing Sacker-Sell spectra in discrete time dynamical systems*, SIAM J. Numer. Anal. **48** (2010), no. 6, 2043–2064.

[20] R.A. Johnson, P.E. Kloeden and R. Pavani, *Two-step transitions in nonautonomous bifurcations: An explanation*, Stoch. Dyn. **2** (2002), no. 1, 67–92.

[21] T. Kato, *Perturbation theory for linear operators*, corrected 2nd ed., Grundlehren der mathematischen Wissenschaften 132, Springer, Berlin etc., 1980.

[22] K.J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differ. Equations **55** (1984), 225–256.

[23] ———, *Exponential dichotomies for almost periodic equations*, Proc. Am. Math. Soc. **101** (1987), 293–298.

[24] ———, *Exponential dichotomies and Fredholm operators*, Proc. Am. Math. Soc. **104** (1988), no. 1, 149–156.

[25] G. Papaschinopoulos, *Exponential dichotomy for almost periodic linear difference equations*, Ann. Soc. Sci. Bruxelles, Sér. I **102** (1988), no. 1–2, 19–28.

[26] ———, *On exponential trichotomy of linear difference equations*, Appl. Anal. **40** (1991), 89–109.

[27] C. Pötzsche, *Stability of center fiber bundles for nonautonomous difference equations*, Difference and Differential Equations (et al Elaydi, S., ed.), Fields Institute Communications 42, American Mathematical Society, Providence, RI, 2004, pp. 295–304.

[28] ______, *A note on the dichotomy spectrum*, J. Difference Equ. Appl. **15** (2009), no. 10, 1021–1025.

[29] ______, *Nonautonomous bifurcation of bounded solutions I: A Lyapunov-Schmidt approach*, Discrete and Continuous Dynamical Systems (Series B) 14 (2010), no. 2, 739–776.

[30] ______, *Nonautonomous continuation of bounded solutions*, Commun. Pure Appl. Anal. **10** (2011), no. 3, 937–961.

[31] C. Pötzsche and M. Rasmussen, *Local approximation of invariant fiber bundles: An algorithmic approach*, Difference Equations and Discrete Dynamical Systems (et al. Sacker, R.J., ed.), World Scientific, New Jersey, London, 2005, pp. 155–170.

[32] ______, *Taylor approximation of invariant fiber bundles for nonautonomous difference equations*, Nonlin. Analysis (TMA) **60** (2005), no. 7, 1303–1330.

[33] ______, *Taylor approximation of integral manifolds*, J. Dyn. Differ. Equations **18** (2006), no. 2, 427–460.

[34] S. Siegmund, *Dichotomy spectrum for nonautonomous differential equations*, J. Dyn. Differ. Equations **14** (2002), no. 1, 243–258.

[35] R.J. Sacker and G.R. Sell, *A spectral theory for linear differential systems*, J. Differ. Equations **27** (1978), 320–358.

[36] G.R. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences 143, Springer, Berlin etc., 2002.

[37] E. Zeidler, *Nonlinear functional analysis and its applications I (Fixed-points theorems)*, Springer, Berlin etc., 1993.

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