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BIFURCATIONS IN PERIODIC INTEGRODIFFERENCE EQUATIONS IN $C(\Omega)$ II: DISCRETE TORUS BIFURCATIONS

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Abstract. We provide a convenient Neimark-Sacker bifurcation result for time-periodic difference equations in arbitrary Banach spaces. It ensures the bifurcation of "discrete invariant tori" caused by a pair of complex-conjugated Floquet multipliers crossing the complex unit circle. This criterion is made explicit for integrodifference equations, which are infinite-dimensional discrete dynamical systems popular in theoretical ecology, and are used to describe the temporal evolution and spatial dispersal of populations with nonoverlapping generations. As an application, we combine analytical and numerical tools for a detailed bifurcation analysis of a spatial predator-prey model. Since such realistic models can frequently only be studied numerically, we formulate our assumptions in such a fashion as to allow for numerically stable verification.

1. **Motivation and introduction.** For various applications, ranging from the life sciences to engineering and economy, it is well-motivated or even obligatory to study models in an environment changing periodically in time. This is due to the desire for a realistic description of seasonal influences (e.g. in population or economical models), or simply necessary to understand periodic control strategies in engineering. An appropriate tool to describe such scenarios having nonoverlapping generations is nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t, \alpha)} \tag{Δ_α}$$

fulfilling the periodicity condition $\mathcal{F}_t = \mathcal{F}_{t+\theta_0}$, $t \in \mathbb{Z}$ for some basic period $\theta_0 \in \mathbb{N}$ and depending on a real parameter α. More concretely, for our purposes the right-hand sides $\mathcal{F}_t : U_t \times A \to X$ are nonlinear Urysohn integral operators

$$\mathcal{F}_t(u, \alpha) := \int_\Omega f_t(\cdot, y, u(y), \alpha)\, \mathrm{d}\mu(y)$$

defined on subsets U_t of an adequate space X of functions over a set $\Omega \subset \mathbb{R}^\kappa$. These recursions are called integrodifference equations, and appear to be widely popular as models in theoretical spatial ecology [2, 12, 21] to describe populations over a habitat Ω, but also arise in various other fields. Without question, such applications

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C.P. DEDICATES THIS PAPER TO PROFESSOR TOMÁS CARABALLO – FRIEND AND COLLEAGUE – ON THE OCCASION OF HIS 60TH BIRTHDAY require precise criteria indicating qualitative changes in the dynamical behavior of an eqn. (Δ_α) when α is varied.

We investigate the local dynamics of (Δ_α) near a given branch of periodic solutions. Here it is a folklore result that qualitative changes of the dynamical behavior always go hand in hand with Floquet multipliers of solution branches crossing the stability boundary, i.e. the unit circle $\mathbb{S}^1$ in $\mathbb{C}$. Bifurcations of fold, transcritical, pitchfork and period doubling type were studied in the companion paper [1], where geometrically simple and real Floquet multipliers are crossing $\mathbb{S}^1$. This setting allowed to apply abstract branching results for periodic solutions. The paper at hand addresses the complementary situation of having a pair of complex-conjugated Floquet multipliers on $\mathbb{S}^1$. We prove a generalized Neimark-Sacker-like result suitable for periodic problems. The bifurcating objects are not periodic solutions, and a different, more geometrical approach is required. Under appropriate and generic assumptions, this gives rise to a Neimark-Sacker bifurcation in the period map

$$\Pi_{\theta_0}(u, \alpha) := \mathcal{F}_{\theta_0 - 1}(\cdot, \alpha) \circ \ldots \circ \mathcal{F}_0(\cdot, \alpha)(u).$$

Hence, one might argue that stability and bifurcation issues for periodic difference eqns. (Δ_α) are settled, since classical autonomous results excellently summarized in e.g. [17] apply to the period map Π_{θ_0}. We partly back up this opinion, but nevertheless point out conceptual and practical reasons for *not* doing so:

- Conceptually, the geometric intuition on the dynamics of (Δ_α) gets lost when taking up a purely autonomous position. In fact, translated back to the original periodic eqn. (Δ_α), an invariant "discrete torus" bifurcates (see Fig. 1), and the actual bifurcating objects are subsets of the extended state space $\mathbb{Z} \times X$ rather than of X. As (Δ_α) depends on parameters α, whole branches of periodic solutions exist, providing deeper insight than mere fixed points.
- There are numerical reasons underlined by that fact that assumptions can only be checked approximately. Immediate evaluations of compositions like Π_{θ_0} are numerically unstable (see [8]), and particularly lack backward stability (cf. [15]), that is, small perturbations in Π_{θ_0} may correspond to large backward errors in the $\mathcal{F}_t$. This becomes especially important when infinite-dimensional systems are discretized in space. In addition, if (Δ_α) is an integrodifference equation, then a direct evaluation of Π_{θ_0} requires cubature formulas over domains of large dimension $\kappa\theta_0$, which are expensive.

In order to counteract such issues, we formulate our assumptions accordingly, and avoid the evaluation of compositions both for linear and nonlinear equations. Practically, this means lifting the problem into products of X with the number of factors given by the period.

Guided by this philosophy, the structure and contents of the paper are as follows: After describing our global set-up, Sec. 3 provides an abstract Neimark-Sacker bifurcation result for periodic difference eqns. (Δ_α) in general Banach spaces. To verify such bifurcations in arbitrary dimensions, two approaches based on a 2-dimensional center manifold stand to reason: (1) Following a difference equations path, one computes a periodic center manifold $C \subseteq \mathbb{Z} \times X$, obtains a periodic and planar difference equation on C and applies the classical Neimark-Sacker bifurcation criteria from e.g. [29, 9, 17] to the resulting planar period map. (2) From a (discrete) dynamical systems perspective, one starts with the period map Π_{θ_0}, reduces it to a center manifold $C \subseteq X$ and then verifies a Neimark-Sacker bifurcation in the reduced 2-dimensional map. Comparing the two, (2) has the practical advantage that a corresponding bifurcation result is already available — at least in finite dimensions (cf. [17, pp. 185ff]). Yet, we extend the finite-dimensional situation [17] in various aspects: First, the concept of duality pairings (cf. [14, 30]) allows us to work in Banach spaces without precise knowledge of their entire dual space. For instance, when dealing with integrodifference equations (abbreviated as IDEs) on $X = C(\Omega)^d$ we utilize continuous functions, as opposed to Radon measures (cf. [30, pp. 10ff]). Second, we address time-periodic equations. Furthermore, the center manifold theorem involved requires a little care. Restricting to Banach spaces having smooth norms or cut-off functions rules out applications to IDEs on the spaces $C(\Omega)^d$ of continuous functions, whose natural norm is nowhere C^1. An appropriate center manifold theorem is due to [16, 7], which only requires a finite-dimensional center-unstable subspace. The Sec. 4 focuses on a special case relevant in theoretical ecology, where (Δ_α) is a periodic IDE on the Banach space $X = C(\Omega)^d$. In this situation, certain assumptions simplify and become more concrete. Our concept of an IDE is rather flexible, since we work with a general integral induced by a finite measure μ. This enables us to handle classical IDEs (where μ is the Lebesgue-measure on $\mathbb{R}^\kappa$), their Nyström-discretization

$$\mathcal{F}_t(u, \alpha) = \sum_{j=0}^{N-1} \omega_j f_t(\cdot, \eta_j, u(\eta_j), \alpha)$$

with $N \in \mathbb{N}$ weights ω_j and nodes $\eta_j \in \Omega$, as well as finite-dimensional difference equations, all in a common framework. While this conveniently unifies the bifurcation theory of IDEs and their spatial discretizations, we are not concerned with convergence issues of the bifurcating objects as $N \to \infty$ here. The closing Sect. 5 is devoted to applications in the form of specific IDEs. Its first example is a scalar autonomous IDE having a separable kernel; this permits an explicit analysis. More involved is a time-periodic spatial predator-prey model [12, 13, 21], where assumptions can only be verified on the basis of various numerical methods. They include path following and eigenvalue problems, as well as Fredholm integral equations of the second kind, which are solved using Nyström discretizations. For the reader's convenience, we conclude with an appendix on an (autonomous) Neimark-Sacker bifurcation theorem tailor-made for our purposes, as well as information on the Nyström methods used in our simulations and applications.

For related work in discrete time and infinite dimensions we refer to [26, 27]. This reference verifies a Neimark-Sacker bifurcation in the period map of time-periodic delay differential equations. For this purpose, spectral projections are represented by Riesz-Dunford integrals and explicitly constructed. In contrast, we make use of duality pairings [10, 14, 30] and also apply them in the center manifold reduction.

Notation. Let $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ be the complex unit circle. Throughout, X denotes a real Banach space and $X_\mathbb{C}$ is its complexification. The *(Kuratovski) measure of noncompactness* on X is abbreviated as χ_X (cf. [22, 20]).

Given a bounded linear operator $T \in L(X)$, $N(T) := T^{-1}(\{0\})$ is its *kernel* and $R(T) := TX$ the *range*. One denotes $\min\left\{p \in \mathbb{N}_0 : N(T^p) = N(T^{p+1})\right\}$ as the *ascent* and $\min\left\{p \in \mathbb{N}_0 : R(T^p) = R(T^{p+1})\right\}$ as the *descent* of T. We write $\sigma_{\mathrm{ess}}(T)$ for the *essential spectrum* of T in the sense of Browder (cf. [19, 22]), and its radius is abbreviated by $r_{\mathrm{ess}}(T) := \sup_{\lambda \in \sigma_{\mathrm{ess}}(T)} |\lambda|$.

The complexification of $T \in L(X)$ is denoted by $T_{\mathbb{C}} \in L(X_{\mathbb{C}})$. If $\lambda \in \mathbb{C}$, $\xi \in X_{\mathbb{C}}$ satisfy $T_{\mathbb{C}}\xi = \lambda\xi$ with $\xi \neq 0$, then (λ, ξ) is called an *eigenpair* of T and $\operatorname{Eig}_\lambda T$ is the associate generalized eigenspace. For a *simple* eigenpair, $\lambda I_{X_{\mathbb{C}}} - T$ has ascent 1.

With $l \in \mathbb{N}$ and a normed space Y, the linear space of symmetric l-linear mappings $T : X^l \to Y$ is abbreviated as $L_l(X, Y)$, and $L_l(X) := L_l(X, X)$.

If Y is a complex normed space, then one denotes $\langle Y, X_{\mathbb{C}} \rangle$ as a *duality pairing* (see [30, pp. 303ff]) if there exists a sesquilinear form[1] $\langle \cdot, \cdot \rangle : Y \times X_{\mathbb{C}} \to \mathbb{C}$ such that

$$\langle y, x \rangle = 0 \text{ for all } y \in Y \Rightarrow x = 0, \qquad \langle y, x \rangle = 0 \text{ for all } x \in X_{\mathbb{C}} \Rightarrow y = 0.$$

An operator $S' \in L(Y)$ satisfying $\langle y, S_{\mathbb{C}} x \rangle = \langle S' y, x \rangle$ for all $x \in X_{\mathbb{C}}$, $y \in Y$ is called the *dual operator* of $S \in L(X)$. For instance in finite dimensions, when $Y = \mathbb{C}^d$ and $X = \mathbb{R}^d$, a duality pairing $\langle \mathbb{C}^d, \mathbb{C}^d \rangle$ is given by

$$\langle y, x \rangle := \sum_{j=1}^{d} \overline{y_i} x_i \quad \text{for all } x, y \in \mathbb{C}^d; \tag{1.1}$$

it induces the norm $|x| := \sqrt{\langle x, x \rangle}$. W.r.t. this sesquilinear form, $T^\star := (\overline{t_{ji}})_{i,j=1}^d$ is the dual matrix of $T = (t_{ij})_{i,j=1}^d \in \mathbb{C}^{d \times d}$.

2. **Periodic difference equations.** We study θ_0-periodic difference eqns. (Δ_α), that is, the right-hand side $\mathcal{F}_t : U_t \times A \to X$, $t \in \mathbb{Z}$, satisfies

$$\mathcal{F}_t = \mathcal{F}_{t+\theta_0} \quad \text{for all } t \in \mathbb{Z}$$

with a *basic period* $\theta_0 \in \mathbb{N}$. If $\theta_0 = 1$, one speaks of an *autonomous* equation. The sets $U_t \subseteq X$, $t \in \mathbb{Z}$, are assumed to be nonempty and open, while the parameter space $A \subseteq \mathbb{R}$ is an open interval. A *branch* of periodic solutions to (Δ_α) is a function of periodic sequences $\phi(\alpha)$ satisfying the identity $\phi(\alpha)_{t+1} \equiv \mathcal{F}_t(\phi(\alpha)_t, \alpha)$ on $\mathbb{Z} \times A$.

In the remaining section we keep the parameter $\alpha \in A$ fixed. The forward solution to (Δ_α) starting at an initial time $\tau \in \mathbb{Z}$ in an initial state $u_\tau \in U_\tau$ is

$$\varphi_\alpha(t; \tau, u_\tau) = \begin{cases} \mathcal{F}_{t-1}(\cdot, \alpha) \circ \ldots \circ \mathcal{F}_\tau(\cdot, \alpha)(u_\tau), & \tau < t, \\ u_\tau, & t = \tau, \end{cases}$$

as long as the above compositions remain in U_t. One speaks of the *general solution* φ_α, and the *period map* satisfies

$$\Pi_{\theta_0}(u, \alpha) = \varphi_\alpha(\theta_0; 0, u).$$

Let $\mathcal{U} := \{(t, u) \in \mathbb{Z} \times X : u \in U_t\}$ denote the *extended state space* of the periodic difference eqn. (Δ_α). A subset $\mathcal{A} \subseteq \mathcal{U}$ is called *invariant* (w.r.t. (Δ_α)), provided

$$\mathcal{A}(t+1) = \mathcal{F}_t(\mathcal{A}(t), \alpha) \quad \text{for all } t \in \mathbb{Z}$$

holds, and θ-*periodic* for some $\theta \in \mathbb{N}$ if its *fibers* $\mathcal{A}(t) := \{u \in X : (t, u) \in \mathcal{A}\}$ satisfy

$$\mathcal{A}(t) = \mathcal{A}(t + \theta) \quad \text{for all } t \in \mathbb{Z}.$$

Difference eqns. (Δ_α) of our interest possess branches $\phi(\alpha)$ of θ_1-periodic solutions, $\theta_1 \in \mathbb{N}$. If one introduces the common period

$$\theta := \operatorname{lcm}\{\theta_0, \theta_1\},$$

[1] Differing from [14, pp. 48ff] we follow the convention that $\langle \cdot, \cdot \rangle$ is antilinear in the first and linear in the second argument.

then a θ-periodic branch is determined by the nonlinear cyclic system

$$\begin{cases} u_0 = \mathcal{F}_{\theta-1}(u_{\theta-1}, \alpha), \\ u_1 = \mathcal{F}_0(u_0, \alpha), \\ \quad \vdots \\ u_{\theta-1} = \mathcal{F}_{\theta-2}(u_{\theta-2}, \alpha) \end{cases} \tag{2.1}$$

in $U_0 \times \ldots \times U_{\theta-1}$. Indeed, under the convenient notation for θ-tuples,

$$\hat{u} = (u_0, \ldots, u_{\theta-1}) \in X^\theta,$$

if $\hat{u}$ solves (2.1), then its θ-periodic continuation $(\cdots, u_{\theta-1}, \underline{u_0}, \ldots, u_{\theta-1}, u_0, \ldots)$ (the underline indicates the index 0 entry) yields a θ-periodic solution $\phi(\alpha)$ of (Δ_α) and vice versa.

A given branch $\phi(\alpha)$ of θ_1-periodic solutions can be reduced to a trivial one: Rather than (Δ_α), one considers the θ-periodic *equation of perturbed motion*

$$u_{t+1} = \mathcal{F}_t(u_t + \phi(\alpha)_t, \alpha) - \mathcal{F}_t(\phi(\alpha)_t, \alpha) =: \tilde{\mathcal{F}}_t(u_t, \alpha). \tag{$\tilde{\Delta}_\alpha$}$$

Then the subsequent results apply to $(\tilde{\Delta}_\alpha)$ with θ_0 and $\mathcal{F}_t$ replaced by θ and $\tilde{\mathcal{F}}_t$, resp., and the trivial solution as constant solution branch.

If the partial derivatives $D_1\mathcal{F}_t$ exist, then one defines the *variational equation*

$$\boxed{u_{t+1} = D_1\mathcal{F}_t(\phi(\alpha)_t, \alpha)u_t} \tag{V_α}$$

along $\phi(\alpha)$ and the *transition operator* $\Phi_\alpha : \{(t,s) \in \mathbb{Z}^2 : s \le t\} \to L(X)$ as

$$\Phi_\alpha(t,s) := \begin{cases} D_1\mathcal{F}_{t-1}(\phi(\alpha)_{t-1}, \alpha) \cdots D_1\mathcal{F}_s(\phi(\alpha)_s, \alpha), & s < t, \\ I_X, & t = s. \end{cases}$$

The variational eqn. (V_α) is θ-periodic. From a conventional perspective, stability properties of (V_α) as well as of $\phi(\alpha)$ are determined by the *period operator*

$$\Xi_\theta(\alpha) := \Phi_\alpha(\theta, 0) = D_1\mathcal{F}_{\theta-1}(\phi(\alpha)_{\theta-1}, \alpha) \cdots D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha).$$

Its eigenvalues are denoted as *Floquet multipliers* of $\phi(\alpha)$, and the *Floquet spectrum* of $\phi(\alpha)$ is the set

$$\sigma_\theta(\alpha) := \sigma(\Xi_\theta(\alpha)) \subseteq \mathbb{C}.$$

The *multiplicity* of a Floquet multiplier λ is the dimension of the associate eigenspace $N(\lambda I_X - \Xi_\theta(\alpha)) \subseteq X$, and a *simple* Floquet multiplier has multiplicity 1.

Our entire approach avoids imposing assumptions on the period map, the period operator and the Floquet spectrum, since these notions are based on compositions of mappings. As ambient alternative serve the cyclic system (2.1) and, as suggested by numerical analysis [15, 8], the linear block cyclic operators

$$\hat{\mathcal{F}}_1(\alpha) := D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha),$$

$$\hat{\mathcal{F}}_\theta(\alpha) := \begin{pmatrix} 0 & 0 & \ldots & 0 & D_1\mathcal{F}_{\theta-1}(\phi(\alpha)_{\theta-1}, \alpha) \\ D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha) & 0 & \ldots & 0 & 0 \\ & \ddots & & & \vdots \\ & & & D_1\mathcal{F}_{\theta-2}(\phi(\alpha)_{\theta-2}, \alpha) & 0 \end{pmatrix}$$

in $L(X^\theta)$ for $\theta > 1$. The spectra are related by

$$\sigma_\theta(\alpha) = \sigma(\hat{\mathcal{F}}_\theta(\alpha))^\theta \tag{2.2}$$

and Floquet multipliers are θth powers of θ spectral points in $\hat{\mathcal{F}}_\theta(\alpha)$.

Lemma 2.1. *If* $(\nu, \hat{\xi})$ *is a simple eigenpair of* $\hat{\mathfrak{F}}_\theta(\alpha)$ *satisfying* $\nu \neq 0$ *and* $\xi_0 \neq 0$, *then* $(\nu^\theta, \xi_0) \in \mathbb{C} \times X_{\mathbb{C}}$ *is a simple eigenpair of* $\Xi_\theta(\alpha)$.

Proof. Follow the arguments from [1, Props. 2.3 and 2.4]. □

Given a duality pairing $\langle Y, X_{\mathbb{C}} \rangle$ between Y and $X_{\mathbb{C}}$, we define

$$\langle \hat{y}, \hat{x} \rangle_\theta := \sum_{t=0}^{\theta-1} \langle y_t, x_t \rangle \quad \text{for all } x \in X_{\mathbb{C}}^\theta,\ y \in Y^\theta \tag{2.3}$$

and also obtain a duality pairing $\langle Y^\theta, X_{\mathbb{C}}^\theta \rangle_\theta$. With respect to this pairing, one has

Lemma 2.2. *If each* $D_1\mathcal{F}_t(\phi(\alpha)_t, \alpha)$ *has a dual operator* $D_1\mathcal{F}_t(\phi(\alpha)_t, \alpha)' \in L(Y)$, $0 \leq t < \theta$, *then the dual operator of* $\hat{\mathfrak{F}}_\theta(\alpha)$ *exists and is given by*

$$\hat{\mathfrak{F}}_1(\alpha)' = D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha)',$$

$$\hat{\mathfrak{F}}_\theta(\alpha)' = \begin{pmatrix} 0 & D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha)' & & \\ \vdots & & \ddots & \\ 0 & & & D_1\mathcal{F}_{\theta-2}(\phi(\alpha)_{\theta-2}, \alpha)' \\ D_1\mathcal{F}_{\theta-1}(\phi(\alpha)_{\theta-1}, \alpha)' & 0 & \dots & 0 \end{pmatrix}$$

in $L(Y^\theta)$ *for* $\theta > 1$. *Moreover, if* $(\nu, \hat{\eta})$ *is a simple eigenpair of* $\hat{\mathfrak{F}}_\theta(\alpha)'$ *satisfying* $\nu \neq 0$ *and* $\eta_0 \neq 0$, *then* $(\nu^\theta, \eta_0) \in \mathbb{C} \times Y$ *is a simple eigenpair of* $\Xi_\theta(\alpha)'$.

Proof. It results from a straight forward calculation that the dual operator $\hat{\mathfrak{F}}_\theta(\alpha)'$ possesses the claimed form. Then $\hat{\mathfrak{F}}_\theta(\alpha)'\hat{\eta} = \nu\hat{\eta}$ implies

$$\nu\eta_t = D_1\mathcal{F}_t(\phi(\alpha)_t, \alpha)'\eta_{t+1} \text{ for all } 0 \leq t < \theta - 1, \quad \nu\eta_{\theta-1} = D_1\mathcal{F}_{\theta-1}(\phi(\alpha)_{\theta-1}, \alpha)'\eta_0$$

and

$$\begin{aligned}\nu^\theta\eta_0 &= \nu^{\theta-1} D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha)'\eta_1 = \ldots = D_1\mathcal{F}_0(\phi(\alpha)_0, \alpha)' \cdots D_1\mathcal{F}_{\theta-1}(\phi(\alpha)_{\theta-1}, \alpha)'\eta_0 \\ &= \Xi_\theta(\alpha)'\eta_0\end{aligned}$$

shows the assertion; simplicity of the eigenpair (ν, η_0) is due in [1, Prop. 2.5]. □

Theorem 2.3. *A* θ_1*-periodic solution* $\phi(\alpha)$ *to a* θ_0*-periodic eqn.* (Δ_α) *is*

(a) exponentially stable if $\sigma(\hat{\mathfrak{F}}_\theta(\alpha))^\theta \subseteq B_1(0)$,

(b) unstable if there exists a component of $\sigma(\hat{\mathfrak{F}}_\theta(\alpha))^\theta$ *disjoint from* $\bar{B}_1(0)$.

Proof. A proof based on the dichotomy spectrum, which presently reduces to the set $\sqrt[\theta]{|\sigma_\theta(\alpha)|} \setminus \{0\} = \sigma(\hat{\mathfrak{F}}_\theta(\alpha)) \setminus \{0\}$ (see (2.2)), can be found in [25, Thm. 2.1]. □

3. **Discrete torus bifurcations.** Central for our analysis is the situation where

$$\sigma_\theta(\alpha^*) \cap \mathbb{S}^1 \neq \emptyset$$

holds for some *critical parameter* $\alpha^* \in A$. While simple Floquet multipliers ± 1 were tackled in [23, 1], we now address the effects of a complex-conjugated pair on $\mathbb{S}^1$. Since the implicit function theorem still applies to (Δ_{α^*}), an invariant and periodic set $\mathcal{T}_\alpha \subset \mathcal{U}$ bifurcates at $\alpha = \alpha^*$, rather than periodic solutions to (Δ_α).

For $5 < m < \infty$ we consider a θ_0-periodic difference eqn. (Δ_α) with C^m-right-hand side $\mathcal{F}_t : U_t \times A \to X$ such that (Δ_{α^*}) has a θ_1-periodic solution ϕ^*. With

$$\theta := \operatorname{lcm}\{\theta_0, \theta_1\}$$

suppose that the following hold true:

(NS$_1$) Given a duality pairing $\langle Y, X_{\mathbb{C}}\rangle$, assume every $D_1\mathcal{F}_t(\phi_t^*, \alpha^*) \in L(X)$ has a dual operator $D_1\mathcal{F}_t(\phi_t^*, \alpha^*)' \in L(Y)$, and there exist reals $\gamma_t, \gamma_t' \geq 0$ such that

$$\begin{aligned} \chi_X(D_1\mathcal{F}_t(\phi_t^*, \alpha^*)B) &\leq \gamma_t \chi_X(B) \quad \text{for all bounded } B \subset X, \\ \chi_Y(D_1\mathcal{F}_t(\phi_t^*, \alpha^*)'B') &\leq \gamma_t' \chi_Y(B') \quad \text{for all bounded } B' \subset Y \end{aligned} \tag{3.1}$$

holds for all $0 \leq t < \theta$, where $\max\left\{\prod_{s=0}^{\theta-1}\gamma_s, \prod_{s=0}^{\theta-1}\gamma_s'\right\} < 1$,

(NS$_2$) $(\nu_*, \hat{\xi}^*) \in \mathbb{C} \times X_{\mathbb{C}}^\theta$ is a simple eigenpair of $\hat{\mathcal{F}}_\theta(\alpha^*)$ with $\|\xi_0^*\| = 1$,

(NS$_3$) $(\tilde{\nu}_*, \hat{\eta}^*) \in \mathbb{C} \times Y^\theta$ is a simple eigenpair of $\hat{\mathcal{F}}_\theta(\alpha^*)'$ with

$$\langle \eta_0^*, \xi_0^* \rangle = 1,$$

(NS$_4$) $\nu_* \in \mathbb{S}^1$ with $\operatorname{Im} \nu_*^\theta > 0$, the eigenvalues are related by $\tilde{\nu}_*^\theta = \overline{\nu_*^\theta}$, and the remaining spectrum $\sigma(\hat{\mathcal{F}}_\theta(\alpha^*))^\theta \setminus \{\nu_*^\theta, \tilde{\nu}_*^\theta\}$ is disjoint from $\mathbb{S}^1$.

In order to impose assumptions for our main result allowing a numerically stable verification [8, 15], we introduce the block matrix operators

$$\hat{\mathcal{S}}_1 := 0,$$

$$\hat{\mathcal{S}}_\theta := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ D_1\mathcal{F}_1(\phi_1^*, \alpha^*) & 0 & \dots & 0 & 0 \\ & \ddots & & & \vdots \\ & & & D_1\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*, \alpha^*) & 0 \end{pmatrix} \in L(X^\theta)$$

for periods $\theta > 1$. It is straightforward to verify that $I_{X^\theta} - \hat{\mathcal{S}}_\theta$ is invertible. This allows us to gradually define the following l-linear mappings $V_l \in L_l(X, X^\theta)$: Given $v_1, v_2, v_3 \in X$, introduce

$$V_1 v_1 = \left[I_{X^\theta} - \hat{\mathcal{S}}_\theta\right]^{-1} \begin{pmatrix} D_1\mathcal{F}_0(\phi_0^*, \alpha^*)v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.2}$$

$$V_2 v_1 v_2 = \left[I_{X^\theta} - \hat{\mathcal{S}}_\theta\right]^{-1} \begin{pmatrix} D_1^2\mathcal{F}_0(\phi_0^*, \alpha^*)v_1 v_2 \\ D_1^2\mathcal{F}_1(\phi_1^*, \alpha^*)(V_1v_1)_1(V_1v_2)_1 \\ \vdots \\ D_1^2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*, \alpha^*)(V_1v_1)_{\theta-1}(V_1v_2)_{\theta-1} \end{pmatrix}, \tag{3.3}$$

$$V_3 v_1 v_2 v_3 = \left[I_{X^\theta} - \hat{\mathcal{S}}_\theta\right]^{-1} \begin{pmatrix} D_1^3\mathcal{F}_0(\phi_0^*, \alpha^*)v_1 v_2 v_3 \\ V_3^1 v_1 v_2 v_3 \\ \vdots \\ V_3^{\theta-1} v_1 v_2 v_3 \end{pmatrix} \tag{3.4}$$

with components for $1 \leq t < \theta$ defined as

$$\begin{aligned} V_3^t v_1 v_2 v_3 := {} & D_1^2\mathcal{F}_t(\phi_t^*, \alpha^*)(V_1v_1)_t(V_2v_2v_3)_t + D_1^2\mathcal{F}_t(\phi_t^*, \alpha^*)(V_1v_2)_t(V_2v_1v_3)_t \\ & + D_1^2\mathcal{F}_t(\phi_t^*, \alpha^*)(V_1v_3)_t(V_2v_1v_2)_t + D_1^3\mathcal{F}_t(\phi_t^*, \alpha^*)(V_1v_1)_t(V_1v_2)_t(V_1v_3)_t. \end{aligned}$$

These preparations guide us to the main result, which reduces to Thm. A.1 when $\phi^* = 0$ is the trivial solution of an autonomous eqn. (Δ_α) (that is, $\theta = 1$).

Theorem 3.1 (discrete torus bifurcation)**.** *Suppose that beyond (NS$_1$-NS$_4$) also*

(NS$_5$) $\nu_*^{l\theta} \neq 1$ *for all* $l \in \{1, 2, 3, 4\}$ (nonresonance condition),

(NS_6) the transversality condition

$$\begin{aligned}\rho^* =&\theta\,\mathrm{Re}\left(\overline{\nu_*}\frac{\langle\eta_0^*, D_1^2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\psi_{\theta-1}\xi_{\theta-1}^*+D_1D_2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\xi_{\theta-1}^*\rangle}{1+\sum_{t=1}^{\theta-1}\langle\eta_t^*,\xi_t^*\rangle}\right)\\ &+\theta\,\mathrm{Re}\left(\overline{\nu_*}\frac{\sum_{t=0}^{\theta-2}\langle\eta_{t+1}^*, D_1^2\mathcal{F}_t(\phi_t^*,\alpha^*)\psi_t\xi_t^*+D_1D_2\mathcal{F}_t(\phi_t^*,\alpha^*)\xi_t^*\rangle}{1+\sum_{t=1}^{\theta-1}\langle\eta_t^*,\xi_t^*\rangle}\right)\neq 0\end{aligned}$$

holds, where $\hat\psi\in X^\theta$ is the unique solution of

$$[I_{X^\theta}-\hat{\mathcal{F}}_\theta(\alpha^*)]\hat\psi=\begin{pmatrix}D_2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\\ D_2\mathcal{F}_0(\phi_0^*,\alpha^*)\\ \vdots\\ D_2\mathcal{F}_{\theta-2}(\phi_{\theta-2}^*,\alpha^*)\end{pmatrix}, \tag{3.5}$$

(NS_7) given the unique solutions $\hat\xi^1,\hat\xi^2\in X_{\mathbb{C}}^\theta$ of

$$[I_{X_{\mathbb{C}}^\theta}-\hat{\mathcal{F}}_\theta(\alpha^*)_{\mathbb{C}}]\hat\xi^1=\begin{pmatrix}(V_2\xi_0^*\overline{\xi_0^*})_\theta\\ 0\\ \vdots\\ 0\end{pmatrix}, \tag{3.6}$$

$$[I_{X_{\mathbb{C}}^\theta}-\nu^{-2}\hat{\mathcal{F}}_\theta(\alpha^*)_{\mathbb{C}}]\hat\xi^2=\begin{pmatrix}\frac{1}{\nu_*^{2\theta}}(V_2\xi_0^*\xi_0^*)_\theta\\ 0\\ \vdots\\ 0\end{pmatrix}, \tag{3.7}$$

the subsequent constant is nonzero

$$\delta^*:=\tfrac12\,\mathrm{Re}\Big(\overline{\nu_*}^\theta\langle\eta_0^*,(V_3\xi_0^*\xi_0^*\overline{\xi_0^*})_\theta+2(V_2\xi_0^*\xi_0^1)_\theta+(V_2\overline{\xi_0^*}\xi_0^2)_\theta\rangle\Big)$$

are satisfied. Then the solution ϕ^* of (Δ_{α^*}) can be continued to a C^m-branch $\phi(\alpha)$ of θ-periodic solutions to (Δ_α). Defining $\beta(\alpha):=\rho^*(\alpha-\alpha^*)$, the following holds in a neighborhood of $\phi(\alpha)$ for all $\alpha\in A$ near α^*:

(a) Supercritical case: If $\delta^*<0$, then for $\beta(\alpha)\le 0$ the unique invariant set of (Δ_α) is $\phi(\alpha)$, while for $\beta(\alpha)>0$ there exists a θ-periodic invariant set $\mathcal{T}_\alpha\neq\phi(\alpha)$ (cf. Fig. 1).

(b) Subcritical case: If $\delta^*>0$, then for $\beta(\alpha)<0$ there exists a θ-periodic invariant set $\mathcal{T}_\alpha\neq\phi(\alpha)$, while for $\beta(\alpha)\ge 0$ the unique invariant set of (Δ_α) is $\phi(\alpha)$.

Each fiber $\mathcal{T}_\alpha(t)\subseteq U_t$ is C^{m-2}-diffeomorphic to $\mathbb{S}^1$.

The above Thm. 3.1 deserves some remarks:

Remark 3.1. (1) In terms of saturated operator algebras, [10, p. 119, Prop. 25.5] provides sufficient conditions for the existence of dual operators in (NS_1). Of course, these exist when Y is the dual space X' of X (see [10, p. 86, Exam. 16.2]).

(2) The assumption (3.1) requires $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)$ and $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)'$ to be set contractions. In particular, $\gamma_t=0$ holds if one of the mappings $\mathcal{F}_t(\cdot,\alpha)$ is completely continuous (see [20, p. 89, Prop. 6.5]). Similarly, compactness of merely one $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)'$ yields $\gamma_t'=0$. The situation simplifies for $Y=X'$, where compactness of $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)$ implies that $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)'$ is compact (see [10, p. 174, Prop. 42.2]).

(3) Since simple eigenpairs $(\nu_*,\hat\xi^*)$ of $\hat{\mathcal{F}}_\theta(\alpha^*)$ can be continued to a neighborhood of α^* inheriting the C^{m-1}-smoothness of $\alpha\mapsto\hat{\mathcal{F}}_\theta(\alpha)$ (e.g. [3, p. 38, Prop. 3.6.1]),

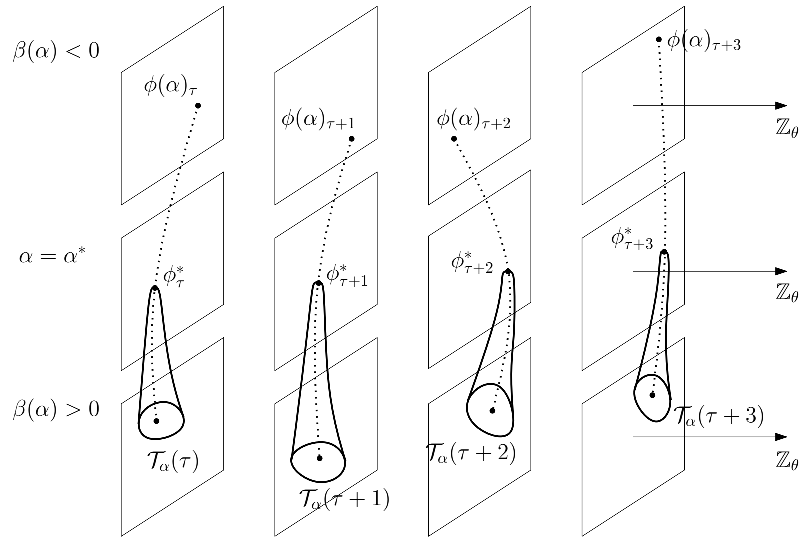

FIGURE 1. Supercritical discrete torus bifurcation from a branch of θ-periodic solutions $\phi(\alpha)$ (dotted) to (Δ_α) into an θ-periodic invariant set $\mathcal{T}_\alpha \subset \mathcal{U}$ (solid lines), where $\theta = 4$

one obtains the disjoint spectral decomposition

$$\sigma(\hat{\mathfrak{F}}_\theta(\alpha)) = \{\nu_+(\alpha), \nu_-(\alpha)\} \,\dot{\cup}\, \Sigma(\alpha) \tag{3.8}$$

with a closed set $\Sigma(\alpha) \subseteq \mathbb{C}$ and $\nu_+(\alpha^*) = \nu_*$, $\nu_-(\alpha^*) = \tilde{\nu}_*$.

(4) The result does not depend on our choice of ν_* and $\tilde{\nu}_*$, but rather on the constant ν_*^θ. While $\hat{\xi}$ and $\hat{\eta}$ are affected by ν_* and $\tilde{\nu}_*$, the eigenfunctions ξ_0^* of $\Xi_\theta(\alpha^*)$ and η_0^* of $\Xi_\theta(\alpha^*)'$ are not. Many of the intermediate computations performed in the proof rely on our choice, but the only quantities affecting the conclusion are ρ^* and δ^*. The former is computed via (3.12) below, itself dependent only on ν_*^θ; the latter is computed via $\overline{\nu_*^\theta}$ and

$$\langle \eta_0^*, (V_3\xi_0^*\xi_0^*\overline{\xi_0^*})_\theta + 2(V_2\xi_0^*\xi_0^1)_\theta + (V_2\overline{\xi_0^*}\xi_0^2)_\theta \rangle.$$

None of V_1, V_2 and V_3 depend in any way on ν_*. While ξ^1 and ξ^2 change as a result of the chosen ν_*, ξ_0^1 and ξ_0^2 do not, as they are the unique solutions of the equations (3.15) and (3.16), where the latter depends only on $\nu_*^{2\theta}$. The interested reader can verify that even the condition $\operatorname{Im} \nu_*^\theta > 0$ is artificial; indeed, interchanging the roles of ν_* and $\tilde{\nu}_*$ does not alter the values ρ^* and δ^*. However, this is less immediate than the above, as the proof would require one to replace ξ_0^* and η_0^* with $\overline{\xi_0^*}$ and $\overline{\eta_0^*}$, respectively.

Remark 3.2 (on the discrete torus). Let $\mathbb{Z}_\theta$ denote the Abelian group of integers $\mathbb{Z}$ under the addition $t +_\theta s := t + s \mod \theta$. Since each fiber of $\mathcal{T}_\alpha$ is diffeomorphic to $\mathbb{S}^1$, we can identify the bifurcating invariant set $\mathcal{T}_\alpha$ with $\mathbb{Z}_\theta \times \mathbb{S}^1$ and in this sense, Thm. 3.1 describes a bifurcation into the "discrete torus" $\mathbb{Z}_\theta \times \mathbb{S}^1$. As seen in the subsequent proof, the fiber $\mathcal{T}_\alpha(0)$ consist of points whose distance to $\phi(\alpha)_0$ behaves like $O\big(\sqrt{|\beta(\alpha)/\delta^*|}\big)$ as $\alpha \to \alpha^*$. More precisely, for elements $u \in \mathcal{T}_\alpha(0)$ one even has the representation $|\langle \eta_0^*, u \rangle| = \sqrt{\left|\frac{\beta(\alpha)}{\delta^*}\right|} + O(|\alpha - \alpha^*|)$ as $\alpha \to \alpha^*$. Such asymptotic estimates also hold for points on the remaining fibers $\mathcal{T}_\alpha(\tau)$ for $\tau \neq 0$. Their distance to $\phi(\alpha)_\tau$ is of order $O\big(\sqrt{|\beta(\alpha)/\delta^*_\tau|}\big)$, where δ^*_τ is the bifurcation indicator δ^* obtained from the shifted difference equation $u_{t+1} = \mathcal{F}_{t+\tau}(u_t, \alpha)$.

Proof of Thm. 3.1. The nonresonance condition (NS_5) for $l = 1$ allows us to apply the implicit function theorem in order to show that (Δ_α) has a C^m-branch $\phi(\alpha)$ of θ-periodic solutions with $\phi(\alpha^*) = \phi^*$, whose derivative at $\alpha = \alpha^*$ is uniquely determined by (3.5), i.e.

$$\hat{\psi} = (\dot{\phi}(\alpha^*)_0, \dots, \dot{\phi}(\alpha^*)_{\theta-1}). \tag{3.9}$$

Because ϕ^* is a θ-periodic solution of (Δ_{α^*}) there is a $\rho_0 > 0$ so that $B_{\rho_0}(\phi^*_t) \subseteq U_t$ for all $t \in \mathbb{Z}$. The continuity of $\alpha \mapsto \phi(\alpha)$ furthermore guarantees the existence of a $\delta_0 > 0$ with $|\phi(\alpha)_t - \phi^*_t| < \frac{\rho_0}{2}$ for all $t \in \mathbb{Z}$ and $\alpha \in B_{\delta_0}(\alpha^*) \subseteq A$; hence we conclude $u + \phi(\alpha)_t \in U_t$ for all $u \in B_{\rho_0/2}(0)$, $\alpha \in B_{\delta_0}(\alpha^*)$. Therefore, the mappings $u \mapsto \mathcal{F}_t(u + \phi(\alpha)_t, \alpha)$ are well-defined for arguments $u \in B_{\rho_0/2}(0)$ and $\alpha \in B_{\delta_0}(\alpha^*)$. This preparation allows us to pass to the eqn. ($\tilde{\Delta}_\alpha$) of perturbed motion, whose right-hand side is also of class C^m, $m > 5$, with derivatives

$$D_1^l \tilde{\mathcal{F}}_t(u, \alpha) = D_1^l \mathcal{F}_t(u + \phi(\alpha)_t, \alpha) \quad \text{for all } 1 \leq l \leq 3, \tag{3.10}$$

$$D_1 D_2 \tilde{\mathcal{F}}_t(u, \alpha) = D_1^2 \mathcal{F}_t(u + \phi(\alpha)_t, \alpha)\dot{\phi}(\alpha)_t + D_1 D_2 \mathcal{F}_t(u + \phi(\alpha)_t, \alpha) \tag{3.11}$$

for $u \in U$. We denote the period map of ($\tilde{\Delta}_\alpha$) by Π_θ, which may be defined on $U := B_{\rho_0/2}(0)$ and for parameters $\alpha \in B_{\delta_0}(\alpha^*)$. Finally, let us abbreviate $\lambda_* := \nu_*^\theta$.

(I) Claim: *Thm. A.1 applies to the period map Π_θ as Π.*

By construction, $\Pi_\theta(\cdot, \alpha)$ has the trivial fixed point, i.e. the identity (A.1) holds, and is a C^m-function, $m > 5$. We verify the further assumptions of Thm. A.1:

ad (A_1): We obtain that $D_1\Pi_\theta(0, \alpha) = \Phi_\alpha(\theta, 0)$. If $B \subseteq X$ is a bounded subset, then (3.1) implies $\chi_X(D_1\Pi_\theta(0, \alpha^*)B) \leq \gamma_{\theta-1} \cdots \gamma_0 \chi_X(B)$ by induction and [22, Thm. 1] proves that the essential spectral radius of $D_1\Pi_\theta(0, \alpha^*)$ is less than 1, since

$$r_{\mathrm{ess}}(D_1\Pi_\theta(0, \alpha^*)) = \lim_{n\to\infty} \sqrt[n]{\inf\left\{\gamma \geq 0 : \left| \begin{array}{l} \chi_X(D_1\Pi_\theta(0, \alpha^*)^n B) \leq \gamma \chi_X(B) \\ \text{for all bounded sets } B \subset X \end{array}\right.\right\}}$$

Using the duality pairing from (NS_1) we observe that the dual operator

$$D_1\Pi(0, \alpha^*)' = D_1\mathcal{F}_0(\phi^*_0, \alpha^*)' \cdots D_1\mathcal{F}_{\theta-1}(\phi^*_{\theta-1}, \alpha^*)'$$

exists. As above, the assumption (3.1) yields $r_{\mathrm{ess}}(D_1\Pi(0, \alpha^*)') < 1$.

ad (A_2–A_5): With Lemma 2.1, we observe that (NS_2) implies (A_2) with $\xi^* := \xi^*_0$. Concerning (NS_3), we note that the dual operator $\hat{\mathcal{F}}_\theta(\alpha^*)'$ exists due to Lemma 2.2, which also ensures that $(\tilde{\nu}_*^\theta, \eta^*_0)$ is a simple eigenpair of $D_1\Pi_\theta(0, \alpha^*)'$. Because of ($\mathrm{NS}_4$) one has $\tilde{\nu}_*^\theta = \overline{\nu_*^\theta} = \overline{\lambda_*}$, and we set $\eta^* := \eta^*_0$. Due to (NS_3), $\langle \eta^*, \xi^* \rangle = 1$ holds. From the relation (2.2) and (NS_4) also $\sigma(D_1\Pi_\theta(0, \alpha^*))$ contains no spectrum on $\mathbb{S}^1$ besides the critical eigenvalues $\lambda_*, \overline{\lambda_*}$. Finally, the nonresonance condition (A_5) is inherited from (NS_5).

ad (A_6): Since $(\nu_*, \hat{\xi}^*)$ is a simple eigenpair of $\hat{\mathcal{F}}_\theta(\alpha^*)$ by assumption (NS_2), we obtain from for instance [3, p. 38, Prop. 3.6.1] that the mentioned eigenpair can be continued in the parameter α, that is, there exist C^{m-1}-functions $\alpha \mapsto \nu(\alpha) \in \mathbb{C}$ and $\alpha \mapsto \hat{\xi}(\alpha) \in X^\theta_{\mathbb{C}}$ satisfying $\nu(\alpha^*) = \nu_*$, $\hat{\xi}(\alpha^*) = \hat{\xi}^*$ and

$$\hat{\mathcal{F}}_\theta(\alpha)\hat{\xi}(\alpha) \equiv \nu(\alpha)\hat{\xi}(\alpha), \qquad \langle \hat{\eta}^*, \hat{\xi}(\alpha) \rangle_\theta \equiv \langle \hat{\eta}^*, \hat{\xi}^* \rangle_\theta.$$

If we differentiate the first identity w.r.t. α and set $\alpha := \alpha^*$, then (3.9) and the representation $N(\langle\hat\eta^*, \cdot\rangle_\theta) = R(\nu_* I_{X_{\mathbb{C}}^\theta} - \hat{\mathcal{F}}_\theta(\alpha^*))$ imply

$$\begin{aligned}\dot\nu(\alpha^*) =& \frac{\langle\hat\eta^*, D\hat{\mathcal{F}}_\theta(\alpha^*)\hat\xi^*\rangle_\theta}{\langle\hat\eta^*, \hat\xi^*\rangle_\theta}\\ \overset{(2.3)}{=}& \frac{\langle\eta_0^*, D_1D_2\tilde{\mathcal{F}}_{\theta-1}(0,\alpha^*)\xi_{\theta-1}^*\rangle + \sum_{t=0}^{\theta-2}\langle\eta_{t+1}^*, D_1D_2\tilde{\mathcal{F}}_t(0,\alpha^*)\xi_t^*\rangle}{1+\sum_{t=1}^{\theta-1}\langle\eta_t^*,\xi_t^*\rangle}\\ \overset{(3.11)}{=}& \frac{\langle\eta_0^*, D_1^2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\psi_{\theta-1}\xi_{\theta-1}^* + D_1D_2\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\xi_{\theta-1}^*\rangle}{1+\sum_{t=1}^{\theta-1}\langle\eta_t^*,\xi_t^*\rangle}\\ &+\frac{\sum_{t=0}^{\theta-2}\langle\eta_{t+1}^*, D_1^2\mathcal{F}_t(\phi_t^*,\alpha^*)\psi_t\xi_t^* + D_1D_2\mathcal{F}_t(\phi_t^*,\alpha^*)\xi_t^*\rangle}{1+\sum_{t=1}^{\theta-1}\langle\eta_t^*,\xi_t^*\rangle}.\end{aligned}$$

Differentiating $|\nu(\alpha)|^\theta \equiv |\lambda_+(\alpha)|$ (cf. (2.2)), where $\lambda_+(\alpha)$ is introduced prior to Thm. A.1, combined with the observation $\frac{\mathrm{d}}{\mathrm{d}\alpha}|\nu(\alpha)|\,|_{\alpha=\alpha^*} = \operatorname{Re}(\overline{\nu_*}\dot\nu(\alpha^*))$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}|\lambda_+(\alpha)|\,|_{\alpha=\alpha^*} = \theta\,|\nu_*|^{\theta-1}\frac{\mathrm{d}}{\mathrm{d}\alpha}|\nu(\alpha)|\,|_{\alpha=\alpha^*} = \theta\operatorname{Re}(\overline{\nu_*}\dot\nu(\alpha^*)), \tag{3.12}$$

and consequently (NS_6) implies the transversality condition (A_6) with the given ρ^*.

ad (A$_7$): Let $\tilde\varphi_\alpha$ denote the general solution of ($\tilde\Delta_\alpha$). Given $l \in \{1,2,3\}$, we establish that the partial derivatives $D_3^l\tilde\varphi_\alpha(\cdot;\tau,u)$ solve linear difference equations. Indeed, by differentiating the identities

$$\tilde\varphi_\alpha(t+1;\tau,u) = \tilde{\mathcal{F}}_t(\tilde\varphi_\alpha(t;\tau,u),\alpha), \qquad \tilde\varphi_\alpha(\tau;\tau,u) = u \tag{3.13}$$

w.r.t. u and keeping an eye on (3.10), one observes that the sequence of derivatives $D_3\tilde\varphi_{\alpha^*}(\cdot;0,0)v_1$, $v_1 \in X$, solves the initial value problem

$$u_{t+1} = D_1\mathcal{F}_t(\phi_t^*,\alpha^*)u_t, \qquad u_0 = v_1,$$

which means

$$\begin{cases} u_0 = v_1,\\ u_1 = D_1\mathcal{F}_0(\phi_0^*,\alpha^*)u_0,\\ \quad\vdots\\ u_\theta = D_1\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)u_{\theta-1}.\end{cases}$$

Setting $(u_1,\ldots,u_\theta) = V_1v_1$, this is equivalent to (3.2), and beyond that

$$D_1\Pi_\theta(0,\alpha^*)v_1 = D_3\tilde\varphi_{\alpha^*}(\theta;0,0)v_1 = (V_1v_1)_\theta;$$

similarly, differentiating (3.13) further yields

$$\begin{aligned} D_1^2\Pi_\theta(0,\alpha^*)v_1v_2 &= D_3^2\tilde\varphi_{\alpha^*}(\theta;0,0)v_1v_2 = (V_2v_1v_2)_\theta,\\ D_1^3\Pi_\theta(0,\alpha^*)v_1v_2v_3 &= D_3^3\tilde\varphi_{\alpha^*}(\theta;0,0)v_1v_2v_3 = (V_3v_1v_2v_3)_\theta\end{aligned} \tag{3.14}$$

for all $v_1, v_2, v_3 \in X$. Formulating (3.6) explicitly yields

$$\begin{aligned}\xi_0^1 &= D_1\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*,\alpha^*)\xi_{\theta-1}^1 + (V_2\xi_0^*\overline{\xi_0^*})_\theta,\\ \xi_{t+1}^1 &= D_1\mathcal{F}_t(\phi_t^*,\alpha^*)\xi_t^1 \text{ for all } 0 \le t < \theta-1,\end{aligned}$$

from which a simple recursion yields

$$[I_{X_{\mathbb{C}}} - D_1\Pi_\theta(0,\alpha^*)]\xi_0^1 = (V_2\xi_0^*\overline{\xi_0^*})_\theta \overset{(3.14)}{=} D_1^2\Pi_\theta(0,\alpha^*)\xi_0^*\overline{\xi_0^*}, \tag{3.15}$$

i.e. (A.2) holds with $\xi^1 = \xi_0^1$. In a similar argument, we derive from (3.7) that

$$\xi_0^2 = \nu_*^{-2} D_1\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*, \alpha^*)\xi_{\theta-1}^2 + \nu_*^{-2\theta}(V_2\xi_0^*\xi_0^*)_\theta,$$
$$\xi_{t+1}^2 = \nu_*^{-2} D_1\mathcal{F}_t(\phi_t^*, \alpha^*)\xi_t^2 \text{ for all } 0 \le t < \theta - 1,$$

and a similar recursion yields

$$[\nu_*^{2\theta} I_{X_\mathbb{C}} - D_1\Pi_\theta(0, \alpha^*)]\xi_0^2 = (V_2\xi_0^*\xi_0^*)_\theta \overset{(3.14)}{=} D_1^2\Pi_\theta(0, \alpha^*)\xi_0^*\xi_0^*, \tag{3.16}$$

that is, (A.3) holds with $\xi^2 = \xi_0^2$. Eventually, our construction of the θ-tuples V_1v_1, $V_2v_1v_2$ and $V_3v_1v_2v_3$ implies that also the formula (A_7) for the bifurcation indicator δ^* can be derived from (NS_7).

(II) For each $\tau \in \mathbb{Z}$ and appropriate open neighborhoods $\tilde{U}_\tau \subseteq X$ of 0, $A_0 \subseteq A$ of α^* we define C^m-mappings $\Pi_\theta^\tau : \tilde{U}_\tau \times A \to X$ via the τ-shifted period map $\Pi_\theta^\tau(u, \alpha) := \tilde{\varphi}_\alpha(\tau + \theta; \tau, u)$. Due to the θ-periodicity of ($\tilde{\Delta}_\alpha$), $\Pi_\theta^{\tau+\theta} = \Pi_\theta^\tau$ (cf. [23, (2.1)]). By construction, $\Pi_\theta^\tau(\cdot, \alpha)$ has the trivial fixed point, and the spectra (excluding 0) of $D_1\Pi_\theta^\tau(0, \alpha)$ are independent of τ, as seen in [23]. Because of step (I), an invariant curve $T_\alpha(0) \subset X$ of $\Pi_\theta^0 = \Pi_\theta$ bifurcates from ϕ_0^* at α^*, and thanks to the cyclic structure of our assumptions, for every τ also an invariant curve $T_\alpha(\tau) \subseteq X$ of $\Pi_\theta^\tau(\cdot, \alpha)$ bifurcates from ϕ_τ^* at the same critical parameter value α^*. In particular, $\tilde{\varphi}_\alpha(t; \tau, T_\alpha(\tau))$ is an invariant curve w.r.t. $\Pi_\theta^t(\cdot, \alpha)$ whose (local) uniqueness property from (I) yields $\tilde{\varphi}_\alpha(t; \tau, T_\alpha(\tau)) = T_\alpha(t)$ for $\tau \le t$. In a skew-product formulation, the set $\mathbb{Z}_\theta \times \bigcup_{\tau \in \mathbb{Z}_\theta} T_\alpha(\tau)$ is invariant under

$$\mathcal{P}_\alpha : \mathbb{Z}_\theta \times \tilde{U}_\tau \to \mathbb{Z}_\theta \times X, \qquad \mathcal{P}_\alpha(t, u) := \begin{pmatrix} \tau + t \mod \theta \\ \tilde{\varphi}_\alpha(t; \tau, u) \end{pmatrix} \quad \text{for all } \tau \le t.$$

The claims follow if we transfer the above results back to the original eqn. (Δ_α) and define the discrete torus fiber-wise by $\mathcal{T}_\alpha(t) := \phi(\alpha)_t + T_\alpha(t)$ for all $t \in \mathbb{Z}$. □

While Thm. 3.1 was based on a general decomposition (3.8), the additional assumption of a spectral remainder $\Sigma(\alpha^*)$ in the open unit disk yields stability results:

Corollary 3.2 (stability)**.** *Under the additional assumption*

$$\Sigma(\alpha^*) \subset B_1(0),$$

the following holds for α near α^*:

- *(a) If $\delta^* < 0$, then ϕ^* is asymptotically stable. In the case of $\beta(\alpha) < 0$, the θ-periodic solution $\phi(\alpha)$ is exponentially stable, while for $\beta(\alpha) > 0$ the invariant set $\mathcal{T}_\alpha$ is attractive and $\phi(\alpha)$ is repulsive.*
- *(b) If $\delta^* > 0$, then ϕ^* is repulsive. In the case of $\beta(\alpha) < 0$, the θ-periodic solution $\phi(\alpha)$ is repulsive and $\mathcal{T}_\alpha$ is attractive, while for $\beta(\alpha) > 0$ the solution $\phi(\alpha)$ is exponentially stable.*

Proof. Due to the upper semi-continuous dependence of the spectrum on parameters (see [11, pp. 209ff]), (3.8) implies the inclusion $\Sigma(\alpha) \subset B_1(0)$ for α in a neighborhood of α^*. Therefore, the extended system (e.g. [17, pp. 165ff])

$$\begin{cases} u_{t+1} = \Pi_\theta(u_t, \alpha_t), \\ \alpha_{t+1} = \alpha_t \end{cases}$$

possesses a 3-dimensional center manifold C, and Π_θ reduced to C becomes a planar map near $0 \in C$. The radial component of its dynamics reads as $r \mapsto (1 + \beta(\alpha) + \delta^* r^2)r + O(r^4)$ (cf. [9, 17, 18, 29]), and the reduction principle yields the claimed stability properties. The continuity of $\mathcal{F}_t$ allows to extend these results from the fixed points or invariant circles of $\Pi_\theta(\cdot,\alpha)$ to the periodic solutions resp. invariant set of (Δ_α). □

4. **Integrodifference equations.** This section applies our abstract bifurcation criteria to periodic IDEs (Δ_α), whose right-hand sides $\mathcal{F}_t$ are nonlinear integral operators of Urysohn type. In this framework, several assumptions simplify, become more explicit or can be verified at least numerically. For instance, abstract linear equations turn into Fredholm integral equations (of the second kind).

Assume that $(\Omega, \mathfrak{A}, \mu)$ is a measure space satisfying $\mu(\Omega) \in (0,\infty)$, but (Ω, d) is also a compact metric space. It is convenient to abbreviate (when $U \subseteq \mathbb{R}^d$)

$$C(\Omega, U) := \{u : \Omega \to U | \, u \text{ is continuous}\}, \qquad C_d := C(\Omega, \mathbb{R}^d)$$

and $\|u\|_\infty := \max_{x\in\Omega} |u(x)|$. Since $(C_d)_\mathbb{C} = C(\Omega, \mathbb{C}^d)$ is the complexification of C_d, as in [30, pp. 303ff] one establishes $\langle\!\langle C(\Omega, \mathbb{C}^d), C(\Omega, \mathbb{C}^d)\rangle\!\rangle$ as duality pairing based on the sesquilinear form

$$\langle\!\langle u, v\rangle\!\rangle := \int_\Omega \langle u(x), v(x)\rangle \, \mathrm{d}\mu(x) \quad \text{for all } u, v \in C(\Omega, \mathbb{C}^d), \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ is given in (1.1). The Cauchy-Schwarz inequality guarantees

$$|\langle\!\langle u, v\rangle\!\rangle| \le \int_\Omega |u(x)|\,|v(x)| \, \mathrm{d}\mu(x) \le \mu(\Omega) \|u\|_\infty \|v\|_\infty \quad \text{for all } u, v \in C(\Omega, \mathbb{C}^d),$$

and hence $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is bounded.

Let us investigate Urysohn integrodifference equations

$$u_{t+1} = \int_\Omega f_t(\cdot, y, u_t(y), \alpha) \, \mathrm{d}\mu(y)$$

fitting in the framework of (Δ_α) with right-hand sides

$$\mathcal{F}_t(u, \alpha)(x) = \int_\Omega f_t(x, y, u(y), \alpha) \, \mathrm{d}\mu(y) \quad \text{for all } x \in \Omega \tag{4.2}$$

and α from an open interval $A \subseteq \mathbb{R}$. We assume that there exists a $\theta_0 \in \mathbb{N}$ such that $f_t = f_{t+\theta_0}$ for all $t \in \mathbb{Z}$; then $\mathcal{F}_t = \mathcal{F}_{t+\theta_0}$, $t \in \mathbb{Z}$, results. With $5 < m < \infty$, the following standing assumptions are supposed to hold for all $0 \le t < \theta_0$:

(H) $f_t : \Omega^2 \times U_t^1 \times A \to \mathbb{R}^d$ is continuous with an open, nonempty and convex set $U_t^1 \subseteq \mathbb{R}^d$, and the derivatives $D^j_{(3,4)} f_t : \Omega^2 \times U_t^1 \times A \to L_j(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)$, $1 \le j \le m$ exist as continuous functions. Furthermore, for every $\varepsilon > 0$, $r > 0$ and $\alpha \in A$ there exists a $\delta > 0$ such that

$$|z_1 - z_2| < \delta \quad \Rightarrow \quad \left| D^m_{(3,4)} f_t(x, y, z_1, \alpha) - D^m_{(3,4)} f_t(x, y, z_2, \alpha)\right| < \varepsilon$$

for all $x, y \in \Omega$, $z_1, z_2 \in U_t^1 \cap B_r(0)$.

Note that for each $t \in \mathbb{Z}$, the domains

$$U_t := \left\{ u \in C(\Omega, U_t^1) : \inf_{x\in\Omega} \operatorname{dist}_{\partial U_t^1} u(x) > 0 \right\}$$

are open. If the functions f_t are real-valued, one speaks of a *scalar* IDE.

The assumptions (H) have immediate consequences (see [24][2] or [20]) concerning the right-hand sides of (Δ_α):

[2]This reference assumes a globally defined operator $\mathcal{F}_t$, i.e. $U_t = C_d$, but the reader is invited to verify that the proofs merely require the domains U_t^1 to open (as assumed above).

(P$_1$) $\mathcal{F}_t(\cdot,\alpha):U_t\to C_d$ is completely continuous (see [24, Cor. 2.2]) for all $\alpha\in A$,
(P$_2$) $\mathcal{F}_t\in C^m(U_t\times A,C_d)$ (see [24, Prop. 2.7]).

Let $\alpha^*\in A$ and an associate θ_1-periodic solution ϕ^* of (Δ_{α^*}) be fixed. Thanks to (P$_2$) the following partial derivatives exist and read as

$$D_1\mathcal{F}_t(\phi_t^*,\alpha^*)v \overset{(4.2)}{=} \int_\Omega D_3f_t(\cdot,y,\phi_t^*(y),\alpha^*)v(y)\,\mathrm{d}\mu(y) \quad \text{for all } v\in C_d.$$

Note that the sequence $(D_1\mathcal{F}_t(\phi_t^*,\alpha^*))_{t\in\mathbb{Z}}$ has the period $\theta=\operatorname{lcm}\{\theta_0,\theta_1\}$.

Lemma 4.1. *The dual operator of $D_1\mathcal{F}_t(\phi_t^*,\alpha^*)\in L(C_d)$ exists and is given by*

$$[D_1\mathcal{F}_t(\phi_t^*,\alpha^*)'v](x)=\int_\Omega D_3f_t(y,x,\phi_t^*(x),\alpha^*)^\star v(y)\,\mathrm{d}\mu(y) \quad \textit{for all } x\in\Omega \tag{4.3}$$

and $t\in\mathbb{Z}$, $v\in C(\Omega,\mathbb{C}^d)$.

Proof. Let $t\in\mathbb{Z}$. It is convenient to neglect the dependence on the parameter α^* in $f_t,\mathcal{F}_t$. Given the sesquilinear form (1.1), for all $v,w\in C(\Omega,\mathbb{C}^d)$ we now have

$$\begin{aligned}
\langle\!\langle v,D\mathcal{F}_t(\phi_t^*)_{\mathbb{C}}w\rangle\!\rangle &= \int_\Omega\langle v(x),[D\mathcal{F}_t(\phi_t^*)w](x)\rangle\,\mathrm{d}\mu(x)\\
&= \int_\Omega\langle v(x),\int_\Omega D_3f_t(x,y,\phi_t^*(y))w(y)\,\mathrm{d}\mu(y)\rangle\,\mathrm{d}\mu(x)\\
&\overset{(1.1)}{=} \int_\Omega\int_\Omega\langle v(x),D_3f_t(x,y,\phi_t^*(y))w(y)\rangle\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&= \int_\Omega\int_\Omega\langle D_3f_t(x,y,\phi_t^*(y))^\star v(x),w(y)\rangle\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)
\end{aligned}$$

and Fubini's theorem (e.g. [4, pp. 159–160, Thm. 5.2.2]) guarantees that

$$\begin{aligned}
\langle\!\langle v,D\mathcal{F}_t(\phi_t^*)_{\mathbb{C}}w\rangle\!\rangle &= \int_\Omega\int_\Omega\langle D_3f_t(x,y,\phi_t^*(y))^\star v(x),w(y)\rangle\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\overset{(1.1)}{=} \int_\Omega\langle\int_\Omega D_3f_t(x,y,\phi_t^*(y))^\star v(x)\,\mathrm{d}\mu(x),w(y)\rangle\,\mathrm{d}\mu(y)\\
&= \int_\Omega\langle\int_\Omega D_3f_t(y,x,\phi_t^*(x))^\star v(y)\,\mathrm{d}\mu(y),w(x)\rangle\,\mathrm{d}\mu(x)\\
&= \int_\Omega\langle[D\mathcal{F}_t(\phi_t^*)'v](x),w(x)\rangle\,\mathrm{d}\mu(x)=\langle\!\langle D\mathcal{F}_t(\phi_t^*)'v,w\rangle\!\rangle.
\end{aligned}$$

Since dual operators are unique [14, p. 46, Thm. 4.6], this implies the claim. □

With $v_1,v_2,v_3\in C_d$, the needed higher-order derivatives of $\mathcal{F}_t$ read as

$$D_1D_2\mathcal{F}_t(\phi_t^*,\alpha^*)v_1=\int_\Omega D_3D_4f_t(\cdot,y,\phi_t^*(y),\alpha^*)v_1(y)\,\mathrm{d}\mu(y), \tag{4.4}$$

$$D_1^2\mathcal{F}_t(\phi_t^*,\alpha^*)v_1v_2=\int_\Omega D_3^2f_t(\cdot,y,\phi_t^*(y),\alpha^*)v_1(y)v_2(y)\,\mathrm{d}\mu(y), \tag{4.5}$$

$$D_1^3\mathcal{F}_t(\phi_t^*,\alpha^*)v_1v_2v_3=\int_\Omega D_3^3f_t(\cdot,y,\phi_t^*(y),\alpha^*)v_1(y)v_2(y)v_3(y)\,\mathrm{d}\mu(y),$$

$$D_2\mathcal{F}_t(\phi_t^*,\alpha^*)=\int_\Omega D_4f_t(\cdot,y,\phi_t^*(y),\alpha^*)\,\mathrm{d}\mu(y) \quad \text{for all } t\in\mathbb{Z}, \tag{4.6}$$

which yields that the operators $\hat{\mathcal{F}}_\theta(\alpha^*)$ and $\hat{\mathcal{S}}_\theta$ become

$$\hat{\mathcal{F}}_\theta(\alpha^*)\hat{v} := \int_\Omega \begin{pmatrix} D_3 f_{\theta-1}(\cdot, y, \phi^*_{\theta-1}(y), \alpha^*) v_{\theta-1}(y) \\ D_3 f_0(\cdot, y, \phi^*_0(y), \alpha^*) v_0(y) \\ \vdots \\ D_3 f_{\theta-2}(\cdot, y, \phi^*_{\theta-2}(y), \alpha^*) v_{\theta-2}(y) \end{pmatrix} \mathrm{d}\mu(y),$$

$$\hat{\mathcal{S}}_1 := 0, \qquad \hat{\mathcal{S}}_\theta \hat{v} := \int_\Omega \begin{pmatrix} 0 \\ D_3 f_1(\cdot, y, \phi^*_1(y), \alpha^*) v_0(y) \\ \vdots \\ D_3 f_{\theta-1}(\cdot, y, \phi^*_{\theta-1}(y), \alpha^*) v_{\theta-2}(y) \end{pmatrix} \mathrm{d}\mu(y);$$

the components $V_3^t \in L_3(C_d)$ needed to compute (3.4) explicitly read as

$$\begin{aligned} V_3^t v_1 v_2 v_3 := \int_\Omega \Big(& D_3^2 f_t(\cdot, y, \phi_t^*(y), \alpha^*)[V_1 v_1]_t(y)[V_2 v_2 v_3]_t(y) \\ & + D_3^2 f_t(\cdot, y, \phi_t^*(y), \alpha^*)[V_1 v_2]_t(y)[V_2 v_1 v_3]_t(y) \\ & + D_3^2 f_t(\cdot, y, \phi_t^*(y), \alpha^*)[V_1 v_3]_t(y)[V_2 v_1 v_2]_t(y) \\ & + D_3^3 f_t(\cdot, y, \phi_t^*(y), \alpha^*)[V_1 v_1]_t(y)[V_1 v_2]_t(y)[V_1 v_3]_t(y) \Big) \mathrm{d}\mu(y). \end{aligned}$$

With V_1, V_2, V_3 as defined in (3.2–3.4), one arrives at

Theorem 4.2 (discrete torus bifurcation for IDEs)**.** *Let $\theta_0, \theta_1 \in \mathbb{N}$, suppose the right-hand side in (Δ_α) is given by (4.2), $X = C_d$ and (H) holds. If $\theta := \operatorname{lcm}\{\theta_0, \theta_1\}$ and there exists a parameter $\alpha^* \in A$ such that the assumptions*

(o) (Δ_{α^*}) has a θ_1-periodic solution ϕ^*,

(i) $\xi_0^* \in C(\Omega, \mathbb{C}^d)$ is derived from the cyclic eigenvalue problem

$$\int_\Omega \begin{pmatrix} D_3 f_{\theta-1}(\cdot, y, \phi^*_{\theta-1}(y), \alpha^*)\xi^*_{\theta-1}(y) \\ D_3 f_0(\cdot, y, \phi^*_0(y), \alpha^*)\xi^*_0(y) \\ \vdots \\ D_3 f_{\theta-2}(\cdot, y, \phi^*_{\theta-2}(y), \alpha^*)\xi^*_{\theta-2}(y) \end{pmatrix} \mathrm{d}\mu(y) = \nu_* \begin{pmatrix} \xi_0^* \\ \xi_1^* \\ \vdots \\ \xi^*_{\theta-1} \end{pmatrix} \tag{4.7}$$

with $\|\xi_0^*\|_\infty = 1$, and $\nu_* \in \mathbb{S}^1$ has multiplicity 1,

(ii) $\eta_0^* \in C(\Omega, \mathbb{C}^d)$ is derived from the cyclic eigenvalue problem

$$\int_\Omega \begin{pmatrix} D_3 f_0(y, x, \phi^*_0(y), \alpha^*)^\star \eta_1^*(y) \\ \vdots \\ D_3 f_{\theta-2}(y, x, \phi^*_{\theta-2}(x), \alpha^*)^\star \eta^*_{\theta-1}(y) \\ D_3 f_{\theta-1}(y, x, \phi^*_{\theta-1}(x), \alpha^*)^\star \eta^*_0(y) \end{pmatrix} \mathrm{d}\mu(y) = \tilde{\nu}_* \begin{pmatrix} \eta_0^*(x) \\ \eta_1^*(x) \\ \vdots \\ \eta^*_{\theta-1}(x) \end{pmatrix} \tag{4.8}$$

for all $x \in \Omega$, $\tilde{\nu}_* \in \mathbb{S}^1$ has multiplicity 1 and $\int_\Omega \langle \eta_0^*(x), \xi_0^*(x)\rangle \,\mathrm{d}\mu(x) = 1$,

(iii) $\tilde{\nu}_*^\theta = \overline{\nu_*^\theta}$ with $\operatorname{Im}\nu_*^\theta > 0$, a remaining spectrum $\sigma(\hat{\mathcal{F}}_\theta(\alpha^*))^\theta \setminus \{\nu_*^\theta, \tilde{\nu}_*^\theta\}$ disjoint from $\mathbb{S}^1$, and one has the nonresonance condition

$$\nu_*^{l\theta} \neq 1 \quad \textit{for all } l \in \{1, 2, 3, 4\}, \tag{4.9}$$

(iv) the transversality condition

$$\begin{aligned} \rho^* = & \theta \operatorname{Re}\left(\overline{\nu_*} \frac{\int_\Omega \langle \eta_0^*(x), [D_1^2 \mathcal{F}_{\theta-1}(\phi^*_{\theta-1}, \alpha^*)\psi_{\theta-1}\xi^*_{\theta-1}](x) + [D_1 D_2 \mathcal{F}_{\theta-1}(\phi^*_{\theta-1}, \alpha^*)\xi^*_{\theta-1}](x)\rangle \,\mathrm{d}\mu(x)}{1 + \sum_{t=1}^{\theta-1} \int_\Omega \langle \eta_t^*(x), \xi_t^*(x)\rangle \,\mathrm{d}\mu(x)} \right) \\ & + \theta \operatorname{Re}\left(\overline{\nu_*} \frac{\sum_{t=0}^{\theta-2} \int_\Omega \langle \eta^*_{t+1}(x), [D_1^2 \mathcal{F}_t(\phi_t^*, \alpha^*)\psi_t \xi_t^*](x) + [D_1 D_2 \mathcal{F}_t(\phi_t^*, \alpha^*)\xi_t^*](x)\rangle \,\mathrm{d}\mu(x)}{1 + \sum_{t=1}^{\theta-1} \int_\Omega \langle \eta_t^*(x), \xi_t^*(x)\rangle \,\mathrm{d}\mu(x)} \right) \neq 0 \end{aligned}$$

holds, where the derivatives are explicitly stated in (4.4–4.6) and $\hat{\psi} \in C_d^\theta$ *is the unique solution of* (3.5),

(v) *given the unique solutions* $\hat{\xi}^1, \hat{\xi}^2 \in C(\Omega, \mathbb{C}^d)^\theta$ *of* (3.6) *resp.* (3.7), *the subsequent real constant is nonzero*

$$\delta^* := \tfrac{1}{2} \operatorname{Re}\Big(\tilde{\nu}_*^\theta \int_\Omega \langle \eta_0^*(x), [V_3 \xi_0^* \xi_0^* \overline{\xi_0^*}](x)_\theta + 2[V_2 \xi_0^* \xi_0^1](x)_\theta + [V_2 \overline{\xi_0^*} \xi_0^2](x)_\theta \rangle \, \mathrm{d}\mu(x)\Big)$$

are fulfilled, then the assertions of Thm. 3.1 on the C^m*-solution branch* $\phi(\alpha)$*, the discrete torus* $\mathcal{T}_\alpha \subseteq \mathbb{Z}_\theta \times C_d$ *and the stability results in Cor. 3.2 hold.*

Proof. Our aim is to apply Thm. 3.1 to a θ-periodic IDE (Δ_α) with right-hand side (4.2), $Y = C(\Omega, \mathbb{C}^d)$ and the sesquilinear form from (4.1).

First of all, the nonresonance condition (4.9) implies $1 \notin \sigma_\theta(\alpha^*)$, and therefore [1, Thm. 3.2] guarantees that the θ_1-periodic solution ϕ^* of (Δ_{α^*}) from (o) can be continued to a C^m-branch $\alpha \mapsto \phi(\alpha)$ of θ-periodic solutions to (Δ_α), where $D\phi(\alpha^*)$ can be determined as the solution $\hat{\psi} \in C_d^\theta$ of the cyclic Fredholm equation (3.5).

ad (NS$_1$): With the duality pairing induced by (4.1), it results from Lemma 4.1 that $D_1\mathcal{F}_t(\phi_t^*, \alpha^*) \in L(C_d)$ has a dual operator, which is of the form (4.3). Both operators are Fredholm integral operators and hence compact. Therefore, (3.1) are satisfied with $\gamma_t = \gamma_t' = 0$ for all $0 \le t < \theta$. Thus, (NS$_1$) is fulfilled.

ad (NS$_2$–NS$_5$): In the present set-up, the eigenvalue problem for $\hat{\mathcal{F}}_0(\alpha^*)$ becomes (4.7), and consequently assumption (i) is equivalent to (NS$_2$). Similarly, the assumption (ii) is necessary and sufficient for (NS$_3$). Finally, (iii) implies both (NS$_4$) and the nonresonance condition (NS$_5$).

ad (NS$_6$): Given the sesquilinear form (4.1) and the derivatives (4.4), (4.5) one immediately sees that (iv) implies (NS$_6$).

ad (NS$_7$): This readily follows from our assumption (v). □

5. **Applications.** In realistic models, the assumptions of Thm. 4.2 can only be verified approximately using numerical tools. As a first step, this requires one to replace integrals (in (4.1) and (4.2)) by numerical quadrature formulas. Therefore, the resulting problems become finite-dimensional and can be approached via computers. The periodic solution branch $\phi(\alpha)$ is obtained from a continuation scheme in the scalar parameter α applied to (2.1) with a Newton solver as corrector and e.g. a tangential predictor. During this continuation process, one monitors the spectrum of $\hat{\mathcal{F}}_\theta(\alpha)$ in order to detect those critical parameters α^* where eigenvalues leave $\mathbb{S}^1$. For such values, one verifies the assumptions Thm. 4.2(i–v), which requires to solve eigenvalue problems (4.7–4.8), as well as the Fredholm integral equations (of the second kind) (3.2–3.4), (3.6–3.7) and (3.5).

We begin with an example allowing an explicit analysis:

5.1. **A Neimark-Sacker bifurcation in a scalar integrodifference equation.** We consider an autonomous scalar IDE (Δ_α) with Hammerstein right-hand side

$$\mathcal{F}(u, \alpha) := \frac{\sqrt{2}}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot, y, \alpha) u(y)(1 - u(y)) \, \mathrm{d}y \tag{5.1}$$

and logistic nonlinearity on the space $C_1 = C[-\frac{1}{2}, \frac{1}{2}]$. This yields the derivative

$$D_1\mathcal{F}(u, \alpha)v_1 = \frac{\sqrt{2}}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot, y, \alpha)(1 - 2u(y))v_1(y) \, \mathrm{d}y \quad \text{for all } u, v_1 \in C_1,\ \alpha \in \mathbb{R}.$$

A detailed bifurcation analysis of (Δ_α) is possible for the degenerate kernel function

$$k : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \qquad k(x, y, \alpha) := \sum_{j=1}^{2} a_j(y, \alpha)e_j(x)$$

with linearly independent functions $a_1, a_2 : [-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R} \to \mathbb{R}$, $e_1, e_2 : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$,

$$a_1(y, \alpha) := \cos(\alpha + \pi y), \qquad a_2(y, \alpha) := (1 + \alpha^2)\sin(\alpha + \pi y),$$
$$e_1(x) := \sqrt{2}\cos(\pi x), \qquad e_2(x) := \sqrt{2}\sin(\pi x),$$

because $\mathcal{F}(u, \alpha) \in \operatorname{span}\{e_1, e_2\}$ holds for every $u \in C_1$, $\alpha \in \mathbb{R}$. Thus, (Δ_α) reduces to a difference equation in $\mathbb{R}^2$ and we use this observation in the subsequent analysis. For this, we note that the complexification of C_1 is $C[-\frac{1}{2}, \frac{1}{2}, \mathbb{C}]$, and

$$\langle u, v\rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{u(y)}v(y)\,\mathrm{d}y$$

serves as our bounded sesquilinear form. The functions e_1, e_2 now satisfy

$$\langle e_i, e_j\rangle = \delta_{ij} \quad \text{for all } 1 \le i, j \le 2.$$

In general, $k(\cdot, \alpha)$ is not symmetric, but becomes the symmetric kernel discussed in [12, 2] for $\alpha = 0$. Linearizing (5.1) along the trivial solution yields

$$D_1\mathcal{F}(0, \alpha)v = \frac{\sqrt{2}}{2}\sum_{j=1}^{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_j(y, \alpha)v(y)\,\mathrm{d}y e_j \quad \text{for all } v \in C_1$$

as derivative, which has the pair of simple eigenvalues

$$\nu_\pm(\alpha) = \frac{1}{4}\left((2 + \alpha^2)\cos\alpha \pm \sqrt{\alpha^4\cos^2\alpha - 4(1 + \alpha^2)\sin^2\alpha}\right).$$

They are complex-conjugated for α in the interval $(0, 2.324)$ (see Fig. 2 (left)), which particularly contains the critical value $\alpha^* = \sqrt{3}$, because of

$$|\nu_+(\alpha^*)| = \sqrt{\nu_+(\alpha^*)\nu_-(\alpha^*)} = \frac{\sqrt{1 + (\alpha^*)^2}}{2} = 1. \tag{5.2}$$

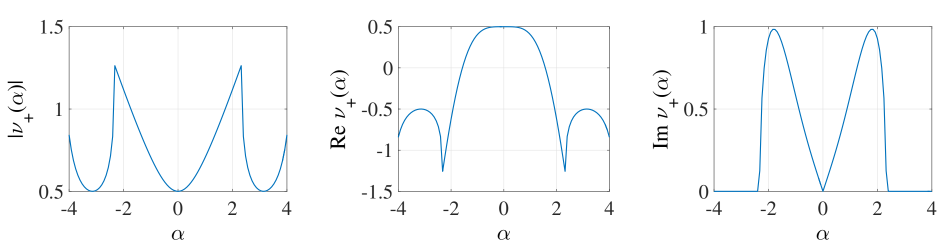

FIGURE 2. Absolute value, real part and imaginary part of $\nu_+(\alpha)$

In order to verify a Neimark-Sacker bifurcation according to Thm. 4.2 (with $\theta = 1$, in fact Thm. A.1) we observe the critical eigenvalue

$$\nu_* = \tfrac{1}{4}\left(5\cos\sqrt{3} + \iota\omega\right), \qquad \omega := \sqrt{16 - 25\cos^2\sqrt{3}} \approx 3.919.$$

TABLE 1. The powers of ν_* are verifying the nonresonance condition 4.2(i)

l	ν_*^l
1	$-0.201-0.980\iota$
2	$-0.919+0.393\iota$
3	$0.570+0.822\iota$
4	$0.691-0.723\iota$

Whence, both the nonresonance condition (4.9) (see Tab. 1), as well as the transversality condition Thm. 4.2(iv) with $\rho^* = \frac{\mathrm{d}}{\mathrm{d}\alpha}|\nu_+(\alpha)| \overset{(5.2)}{=} \frac{\sqrt{3}}{4} > 0$ are satisfied. Furthermore, the eigenfunction ξ^* of $D_1\mathcal{F}(0,\alpha^*)$ corresponding to ν_* with $\|\xi^*\|_\infty = 1$ reads as

$$\xi^* := \frac{2\sin\sqrt{3}e_1 - \big(3\cos\sqrt{3}+\iota\omega\big)e_2}{2\sqrt{\sin\sqrt{3}\,(3+5\sin\sqrt{3})}} \in C[-\tfrac{1}{2},\tfrac{1}{2},\mathbb{C}].$$

The operator dual to $D_1\mathcal{F}(0,\alpha)$ is given by

$$D_1\mathcal{F}(0,\alpha)'v = \frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{k(y,\cdot,\alpha)}v(y)\,\mathrm{d}y = \frac{\sqrt{2}}{2}\sum_{j=1}^{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e_j(y)v(y)\,\mathrm{d}y a_j(\cdot,\alpha).$$

For $\alpha=\alpha^*$ it possesses the eigenvalue $\overline{\nu_*}$ with associate (dual) eigenfunction

$$\eta^* := 8\,s\sin\sqrt{3}e_1 + s\Big(3\cos\sqrt{3}-\iota\omega\Big)e_2 \in C[-\tfrac{1}{2},\tfrac{1}{2},\mathbb{C}]$$

involving a nonzero scaling factor $s\in\mathbb{C}$. The normalization $\langle\eta^*,\xi^*\rangle = 1$ requires

$$s = \frac{2\sqrt{\sin\sqrt{3}\,(3+5\sin\sqrt{3})}}{2(\omega^2+3\cos\sqrt{3}\omega\iota)}.$$

After these preparations, we tackle the problem to compute the crucial value δ^*. For this purpose, observe that

$$D_1^2\mathcal{F}(0,\alpha)v_1v_2 = -\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot,y,\alpha)v_1(y)v_2(y)\,\mathrm{d}y, \qquad D_1^3\mathcal{F}(0,\alpha)=0$$

for $v_1,v_2\in C[-\frac{1}{2},\frac{1}{2}]$ and the Fredholm integral equations (3.6–3.7) simplify to

$$\left[I_{C_1} - D_1\mathcal{F}(0,\alpha^*)\right]\xi^1 = D_1^2\mathcal{F}(0,\alpha^*)\xi^*\overline{\xi^*},$$
$$\left[\nu_*^2 I_{C_1} - D_1\mathcal{F}(0,\alpha^*)\right]\xi^2 = D_1^2\mathcal{F}(0,\alpha^*)\xi^*\xi^*,$$

that is

$$\xi^1 - \frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot,y,\alpha^*)\xi^1(y)\,\mathrm{d}y = -\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot,y,\alpha^*)\xi^*(y)\overline{\xi^*(y)}\,\mathrm{d}y,$$
$$\nu_*^2\xi^2 - \frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot,y,\alpha^*)\xi^2(y)\,\mathrm{d}y = -\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} k(\cdot,y,\alpha^*)\xi^*(y)^2\,\mathrm{d}y.$$

The ansatz $\xi^i = \xi_1^i e_1 + \xi_2^i e_2$ for $i=1,2$ and scalars $\xi_i^j\in\mathbb{C}$ yet to be determined leads to the linear algebraic equations

$$\begin{pmatrix} 1-\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_1(y,\alpha^*)e_1(y)\,\mathrm{d}y & -\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_1(y,\alpha^*)e_2(y)\,\mathrm{d}y \\ -\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_2(y,\alpha^*)e_1(y)\,\mathrm{d}y & 1-\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_2(y,\alpha^*)e_2(y)\,\mathrm{d}y \end{pmatrix}\begin{pmatrix}\xi_1^1\\ \xi_2^1\end{pmatrix}$$

$$= -\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi^*(y)\overline{\xi^*(y)} \begin{pmatrix} a_1(y,\alpha^*) \\ a_2(y,\alpha^*) \end{pmatrix} \mathrm{d}y$$

for the coefficients $\binom{\xi_1^1}{\xi_2^1} \in \mathbb{C}^2$ in the function ξ^1, resp.

$$\begin{pmatrix} \nu_*^2 - \frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_1(y,\alpha^*)e_1(y)\,\mathrm{d}y & -\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_1(y,\alpha^*)e_2(y)\,\mathrm{d}y \\ -\frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_2(y,\alpha^*)e_1(y)\,\mathrm{d}y & \nu_*^2 - \frac{\sqrt{2}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} a_2(y,\alpha^*)e_2(y)\,\mathrm{d}y \end{pmatrix} \begin{pmatrix} \xi_1^2 \\ \xi_2^2 \end{pmatrix}$$
$$= -\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi^*(y)^2 \begin{pmatrix} a_1(y,\alpha^*) \\ a_2(y,\alpha^*) \end{pmatrix} \mathrm{d}y$$

yielding the coefficients $\binom{\xi_1^2}{\xi_2^2} \in \mathbb{C}^2$ in ξ^2. Due to the nonresonance condition (4.9), these solutions are uniquely determined. This equips us with the functions ξ^1, ξ^2, and we can finally evaluate the bifurcation indicator

$$\begin{aligned} \delta^* &= \tfrac{1}{2}\operatorname{Re}\Big(\overline{\nu_*}\big\langle \eta^*, 2D_1^2\mathcal{F}(0,\alpha^*)\xi^*\xi^1 + D_1^2\mathcal{F}(0,\alpha^*)\overline{\xi^*}\xi^2\big\rangle\Big) \\ &= -\frac{\sqrt{2}}{2}\operatorname{Re}\Big(\overline{\nu_*}\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\eta^*(x)} \int_{-\frac{1}{2}}^{\frac{1}{2}} k(x,y,\alpha^*)\left(2\xi^*(y)\xi^1(y) + \overline{\xi^*(y)}\xi^2(y)\right)\mathrm{d}y\,\mathrm{d}x\Big) \\ &\approx -0.9589 < 0. \end{aligned}$$

Summing up, Thm. 4.2 guarantees a supercritical Neimark-Sacker bifurcation. The trivial solution of (Δ_α) loses its exponential stability at $\alpha^* = \sqrt{3}$ and bifurcates into an attractive curve. This confirms our numerical simulation from Fig. 3 (right). It illustrates the projected values

$$\pi_i u := \int_{-\frac{1}{2}}^{\frac{1}{2}} u(y)e_i(y)\,\mathrm{d}y \quad \text{for all } u \in C_1,\ i = 1,2$$

of u_t after a transient time. An analogous computation yields a subcritical Neimark-Sacker bifurcation at the parameter $\alpha^* = -\sqrt{3}$ with $\delta^* \approx -0.9589$ and $\rho^* = -\frac{\sqrt{3}}{4}$, as illustrated in Fig. 3 (left).

5.2. **A predator-prey model.** As a more realistic and concrete example, we consider a simplistic predator-prey model from [13, 21]. It is given in terms of an IDE (Δ_α) in the space C_2 of $\mathbb{R}^2$-valued continuous functions with the right-hand side

$$\mathcal{F}_t(u,\alpha)(x) := \int_\Omega K(x-y)\begin{pmatrix} r_t u^1(y)e^{\alpha - u^1(y) - u^2(y)} \\ c_t u^1(y)u^2(y) \end{pmatrix} \mathrm{d}y \quad \text{for all } x \in \Omega, \tag{5.3}$$

and a kernel $K : \mathbb{R}^\kappa \to \mathbb{R}^{2\times 2}$, whose values are assumed to be diagonal matrices.

Here $u^1 : \Omega \to \mathbb{R}$ describes the spatial distribution of the prey over a habitat Ω and $u^2 : \Omega \to \mathbb{R}$ captures the predator. Their total populations are given by

$$\tau u^i := \int_\Omega u^i(y)\,\mathrm{d}y \in \mathbb{R} \quad \text{for all } i = 1,2.$$

In absence of a predator u^2, the prey population u^1 has Ricker-like dynamics, as studied in [1, Sect. 5.3], including a sequence of period doubling bifurcations ultimately yielding chaotic behavior (see [5] for details). Without a prey u^1, the predators u^2 vanish at once and the dynamics become trivial.

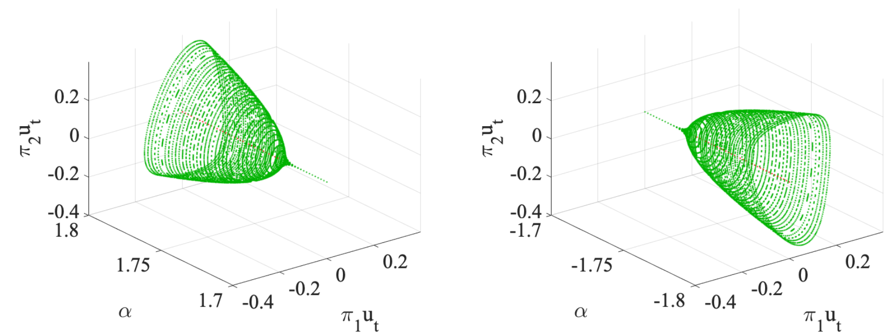

FIGURE 3. Invariant circles displaying total populations from a Neimark-Sacker bifurcation in the autonomous IDE (Δ_α) with right-hand side (5.1) at $\alpha^* = \sqrt{3}$ (left) and $\alpha^* = -\sqrt{3}$ (right). Attractive objects are in green, repulsive ones in red

Let us suppose that the growth rates $(r_t)_{t\in\mathbb{Z}}$, $(c_t)_{t\in\mathbb{Z}}$ are θ-periodic sequences of positive reals. The partial Frechét derivatives of $\mathcal{F}_t$ read as

$$[D_1\mathcal{F}_t(u,\alpha)v_1](x) = \int_\Omega K(x-y)\begin{pmatrix} r_t((1-u^1)v_1^1 - u^1v_1^2)e^{\alpha-u^1-u^2} \\ c_tv_1^1u^2 + c_tv_1^2u^1 \end{pmatrix}\mathrm{d}y,$$

$$[D_1^2\mathcal{F}_t(u,\alpha)v_1v_2](x) = \int_\Omega K(x-y)\begin{pmatrix} r_t\big((u^1-2)v_1^1v_2^1 + (u^1-1)v_1^2v_2^1 \\ \quad +(u^1-1)v_1^1v_2^2 + u^1v_1^2v_2^2\big) \\ \quad \cdot e^{\alpha-u^1-u^2} \\ c_tv_1^2v_2^1 + c_tv_1^1v_2^2 \end{pmatrix}\mathrm{d}y,$$

$$[D_1^3\mathcal{F}_t(u,\alpha)v_1v_2v_3](x) = \int_\Omega K(x-y)\begin{pmatrix} r_t\big((3-u^1)v_1^1v_2^1v_3^1 + (2-u^1)v_1^2v_2^1v_3^1 \\ \quad +(2-u^1)v_1^1v_2^2v_3^1 + (1-u^1)v_1^2v_2^2v_3^1 \\ \quad +(2-u^1)v_1^1v_2^1v_3^2 + (1-u^1)v_1^2v_2^1v_3^2 \\ \quad +(1-u^1)v_1^1v_2^2v_3^2 - u^1v_1^2v_2^2v_3^2\big) \\ \quad \cdot e^{\alpha-u^1-u^2} \\ 0 \end{pmatrix}\mathrm{d}y,$$

$$D_2\mathcal{F}_t(u,\alpha)(x) = \int_\Omega K(x-y)\begin{pmatrix} r_tu^1e^{\alpha-u^1-u^2} \\ 0 \end{pmatrix}\mathrm{d}y$$

for all $t \in \mathbb{Z}$, $x \in \Omega$, $\alpha \in \mathbb{R}$ and $u, v_1, v_2, v_3 \in C_2$, where we neglected the dependence of the functions u^j, v_i^j on the variable $y \in \Omega$.

Further information on (Δ_α) requires us to specify various data and to apply numerical tools. For this purpose, we rely on Nyström methods, replacing the integral in e.g. (Δ_α) by a quadrature rule. Since the integrands in (5.3) are smooth functions, we use Gauß quadrature formulas of 4th order ($N = 100$) and 6th order ($N = 99$, see App. B). In the numerical values presented below, we provide the digits that coincide in the approximations from these two methods.

Example 5.1. On the compact domain $\Omega = [-2, 2]$, consider the Gauß kernel

$$K(x) := \frac{1}{\sqrt{\pi}}\begin{pmatrix} 5\exp(-25x^2) & 0 \\ 0 & \exp(-x^2) \end{pmatrix} \quad \text{for all } x \in [-2, 2]$$

i	α_i^*	ρ_i^*	δ_i^*
1	0.91831	1.9260	-0.859
2	1.28936	1.5721	-0.395
3	2.17617	1.0357	-0.318

TABLE 2. Critical parameters α_i^* where Floquet multipliers along $\phi(\alpha)$ cross $\mathbb{S}^1$, the transversality condition ρ_i^* and the bifurcation indicator δ_i^*

in the 4-periodic predator-prey model (Δ_α) with right-hand side (5.3) and

$$r_t := \begin{cases} \frac{17}{20}, & t \mod 4 = 0, \\ \frac{20}{17}, & t \mod 4 = 2, \\ 1, & \text{else}, \end{cases} \qquad c_t := \tfrac{5}{2}.$$

Then a combination of analytical and numerical techniques allows us to obtain the following bifurcation behavior schematically captured by Fig. 4.

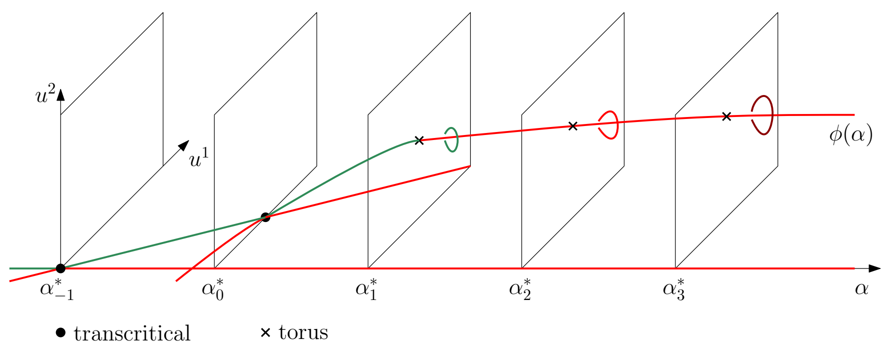

FIGURE 4. Schematic bifurcation diagram for the predator-prey model (Δ_α) given by (5.3). For instance, non-primary bifurcations along the trivial solution are ignored

First, the trivial solution of (Δ_α) loses its exponential stability at a parameter $\alpha_{-1}^* \in [0.0055, 0.0057]$ and [1, Prop. 4.7] guarantees that it transcritically bifurcates into a 4-periodic, exponentially stable branch consisting of prey population u^1 only. At $\alpha_0^* \in [0.4525, 0.4550]$ occurs another bifurcation, again of transcritical type, into the coexistence branch $\phi(\alpha) = \binom{\phi^1}{\phi^2}(\alpha)$ of 4-periodic solutions. Using a numerical path-following algorithm we compute $\phi(\alpha)$ over the parameter range $\alpha \in [0.5, 2.3]$ and Fig. 5 depicts the graphs of the resulting functions $\phi^j(\alpha)_t : [-2, 2] \to \mathbb{R}$ for $j = 1, 2$, $t \in \{0, 1, 2, 3\}$. Simulations illustrated in Fig. 6 indicate that a 4-periodic discrete torus bifurcates supercritically from $\phi(\alpha)$ at a parameter $\alpha_1^* \in [0.9, 0.95]$. Next we confirm this observation using Thm. 4.2. Path-following this solution branch, the three dominant Floquet multipliers were determined, i.e. those Floquet multipliers having the largest moduli (cf. Fig. 7). As a result, Tab. 2 contains the numerical values of the smallest critical parameters α_i^* when multipliers along the 4-periodic solution branch $\phi(\alpha)$ cross the stability boundary $\mathbb{S}^1$, as well as the associate transversality numbers ρ_i^* and bifurcation indicators δ_i^*. For the powers of $\lambda_* = \nu_*^4$ needed in the corresponding nonresonance condition, we refer to Tab. 3.

In conclusion, the branch $\phi(\alpha)$ of 4-periodic solutions loses its exponential stability at $\alpha_1^* \approx 0.92$, and an attractive 4-periodic discrete torus bifurcates supercritically

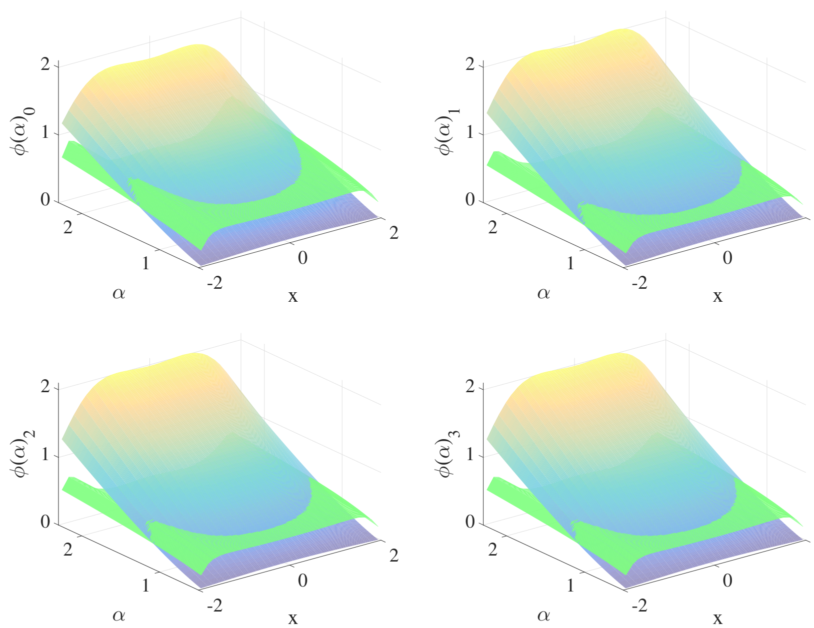

FIGURE 5. 4-periodic solution branch $\phi(\alpha)$ to the IDE (Δ_α) with right-hand side (5.3) for $\alpha \in [0.5, 2.3]$. The distribution of the prey $\phi^1(\alpha)$ is marked in green, while the predators $\phi^2(\alpha)$ vary from blue to yellow

i	$\lambda_+(\alpha_i^*)$	$\lambda_+(\alpha_i^*)^2$	$\lambda_+(\alpha_i^*)^3$	$\lambda_+(\alpha_i^*)^4$
1	$-0.937 + 0.350\iota$	$0.755 - 0.656\iota$	$-0.478 + 0.878\iota$	$0.140 - 0.990\iota$
2	$-0.970 + 0.243\iota$	$0.881 - 0.472\iota$	$-0.740 + 0.673\iota$	$0.554 - 0.833\iota$
3	$-0.428 + 0.904\iota$	$-0.633 - 0.774\iota$	$0.971 - 0.241\iota$	$-0.198 + 0.980\iota$

TABLE 3. The powers of $\lambda_+(\alpha_i^*)$, verifying the nonresonance condition in Thm. 4.2(iii)

(as seen in Fig. 6). Beyond this, Tab. 2 indicates further nonhyperbolic situations. As a secondary bifurcation, at $\alpha_2^* \approx 1.29$ another invariant 4-periodic torus branches off from $\phi(\alpha)$, again supercritically. The same scenario repeats itself at $\alpha_3^* \approx 2.18$. Throughout, Tab. 2 yields that $\delta_i^* < 0 < \rho_i^*$ for $i = 1, 2, 3$.

6. **Concluding remarks.** The companion paper [1] permitted a more general class

$$u_{t+1}(x) = G_t\left(x, \int_\Omega f_t(x, y, u_t(y), \alpha)\,\mathrm{d}\mu(y), \alpha\right) \quad \text{for all } x \in \Omega$$

of IDEs. Although our methods readily extend to this situation, we restricted to right-hand sides (4.2) of Urysohn type, as this was the framework required for our applications in Sect. 5, and also allowed us to simplify the notation. It should be noted that our abstract bifurcation Thm. 3.1 does in fact apply to IDEs

$$u_{t+1}(x) = G_t\left(x, u_t(x), \int_\Omega f_t(x, y, u_t(y), \alpha)\,\mathrm{d}\mu(y), \alpha\right) \quad \text{for all } x \in \Omega,$$

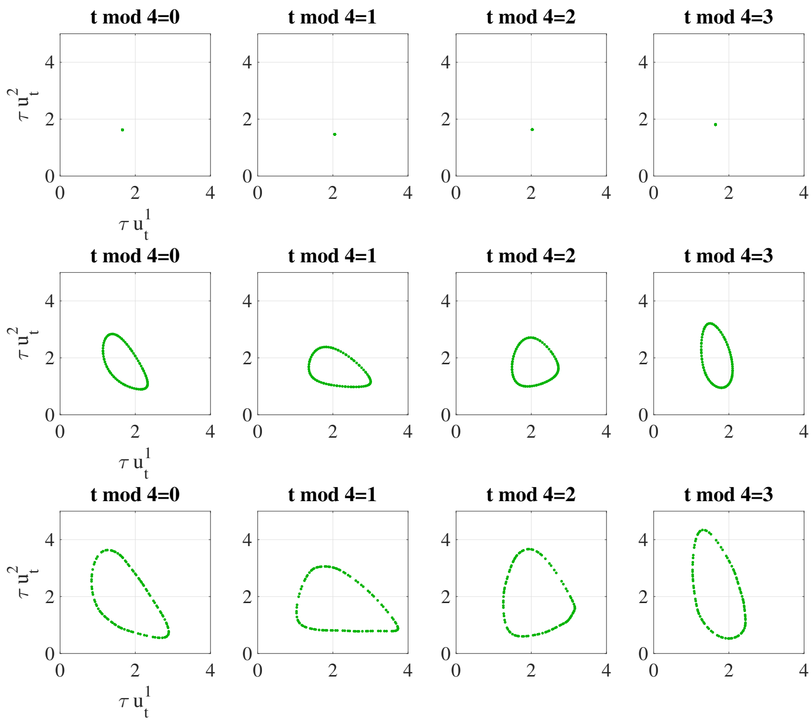

FIGURE 6. 4-periodic invariant circles displaying total populations from a Neimark-Sacker bifurcation in the IDE (Δ_α) with right-hand side (5.3) for $\alpha = 0.9$ (top), $\alpha = 0.95$ (center), $\alpha = 1$ (bottom)

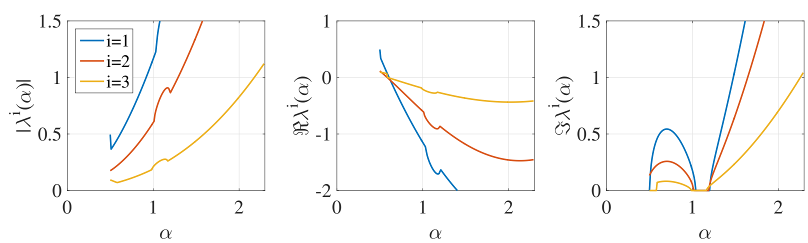

FIGURE 7. Floquet multipliers $\lambda^i(\alpha)$ along the 4-periodic coexistence solution branch $\phi(\alpha)$ of (Δ_α) indicating three critical parameter values α_i^* in the interval $[0.5, 2.3]$

as long as their right-hand sides are set-contractions.

Appendix A. A Neimark-Sacker theorem for maps. Suppose that $U \subseteq X$ is an open neighborhood of 0 in a real Banach space X and $A \subseteq \mathbb{R}$ is an open interval. We assume that a mapping $\Pi : U \times A \to X$ is of class C^m, $5 < m < \infty$,

$$\Pi(0, \alpha) \equiv 0 \quad \text{on } A \tag{A.1}$$

and that for some critical parameter $\alpha^* \in A$ the following hold (see Fig. 8):

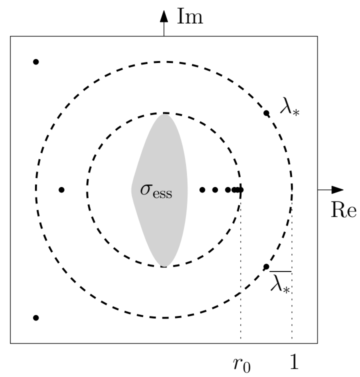

FIGURE 8. Assumptions on the spectrum $\sigma(D_1\Pi(0,\alpha^*)) \subset \mathbb{C}$ with essential radius r_0 in Thm. A.1

(A_1) There exists a duality pairing $\langle Y, X_{\mathbb{C}}\rangle$ under which $D_1\Pi(0,\alpha^*) \in L(X)$ has a dual operator $D_1\Pi(0,\alpha^*)' \in L(Y)$ and their essential spectral radii satisfy

$$r_0 := \max\{r_{\text{ess}}(D_1\Pi(0,\alpha^*)), r_{\text{ess}}(D_1\Pi(0,\alpha^*)')\} < 1,$$

(A_2) $(\lambda_*, \xi^*) \in \mathbb{C} \times X_{\mathbb{C}}$ is a simple eigenpair of $D_1\Pi(0,\alpha^*)$ with $\|\xi^*\| = 1$,

(A_3) $(\overline{\lambda_*}, \eta^*) \in \mathbb{C} \times Y$ is a simple eigenpair of $D_1\Pi(0,\alpha^*)'$ with $\langle \eta^*, \xi^*\rangle = 1$,

(A_4) $\lambda_* \in \mathbb{S}^1$, with $\operatorname{Im}\lambda_* > 0$ and a remaining spectrum $\sigma(D_1\Pi(0,\alpha^*)) \setminus \{\lambda_*, \overline{\lambda_*}\}$ disjoint from $\mathbb{S}^1$.

It is well-known that the pair $\lambda_*, \overline{\lambda_*} \in \mathbb{S}^1$ of simple, complex-conjugated eigenvalues can be continued to C^{m-1}-branches $\lambda_\pm : A_0 \to \mathbb{C}$ of simple eigenvalues of $D_1\Pi(0,\alpha)$, $A_0 \subseteq A$ being an open neighborhood of α^* (see e.g. [3, p. 38, Prop. 3.6.1]), satisfying the spectral decomposition $\sigma(D_1\Pi(0,\alpha)) = \{\lambda_-(\alpha), \lambda_+(\alpha)\} \dot\cup \Sigma(\alpha)$ for all $\alpha \in A_0$ with $\lambda_+(\alpha^*) = \lambda_*$ and a disjoint closed set $\Sigma(\alpha)$ containing the remaining spectrum of $D_1\Pi(0,\alpha)$.

Theorem A.1 (Neimark-Sacker bifurcation)**.** *Suppose that beyond (A_1-A_4) also*

(A_5) $\lambda_*^l \neq 1$ *for all* $l \in \{1,2,3,4\}$ (nonresonance condition)*,*

(A_6) $\rho^* := \frac{\mathrm{d}}{\mathrm{d}\alpha}|\lambda_+(\alpha)|\,|_{\alpha=\alpha^*} = \operatorname{Re}\big(\overline{\lambda_*}\langle \eta^*, D_1D_2\Pi(0,\alpha^*)\rangle\big) \neq 0$ (transversality condition)*,*

(A_7) given the unique solutions $\xi^1, \xi^2 \in X_{\mathbb{C}}$ *of*

$$[I_{X_{\mathbb{C}}} - D_1\Pi(0,\alpha^*)]\xi^1 = D_1^2\Pi(0,\alpha^*)\xi^*\overline{\xi^*}, \tag{A.2}$$

$$[\lambda_*^2 I_{X_{\mathbb{C}}} - D_1\Pi(0,\alpha^*)]\xi^2 = D_1^2\Pi(0,\alpha^*)\xi^*\xi^*, \tag{A.3}$$

it holds that

$$\delta^* := \tfrac{1}{2}\operatorname{Re}\Big(\overline{\lambda_*}\langle \eta^*, D_1^3\Pi(0,\alpha^*)\xi^*\xi^*\overline{\xi^*} + 2D_1^2\Pi(0,\alpha^*)\xi^*\xi^1 + D_1^2\Pi(0,\alpha^*)\overline{\xi^*}\xi^2\rangle\Big) \neq 0$$

are satisfied. With $\beta(\alpha) := \rho^*(\alpha - \alpha^*)$*, the following holds in a neighborhood* $U_0 \subseteq U$ *of* 0 *for all* $\alpha \in A$ *near* α^**:*

(a) Supercritical case: If $\delta^* < 0$*, then for* $\beta(\alpha) \leq 0$ *the unique invariant set of* $\Pi(\cdot,\alpha)$ *is* 0*, while for* $\beta(\alpha) > 0$ *there exists an invariant set* $T_\alpha \subset U_0 \setminus \{0\}$*.*

(b) Subcritical case: If $\delta^* > 0$*, then for* $\beta(\alpha) < 0$ *there exists an invariant set* $T_\alpha \subset U_0 \setminus \{0\}$*, while for* $\beta(\alpha) \geq 0$ *the unique invariant set of* $\Pi(\cdot,\alpha)$ *is* 0*.*

Moreover, T_α *is the unique invariant set being* C^{m-2}*-diffeomorphic to* $\mathbb{S}^1$ *and consists of points having a distance of order* $O\Big(\sqrt{\big|\frac{\beta(\alpha)}{\delta^*}\big|}\Big)$ *from the origin as* $\alpha \to \alpha^*$*.*

Already [28, p. 74, 13.1 Thm.] contains a Neimark-Sacker bifurcation result for maps on Banach spaces, however without explicit expressions for δ^*, ρ^*. We thus present a proof suitable for our purposes:

Proof. The classical situation $X = \mathbb{R}^2$ (and $Y = \mathbb{C}^2$ with (1.1) as inner product) was settled in [29], [18, Thm. 2], [9, p. 33, Thm. 1] or [17, p. 134, Thm. 4.5]. In order to extend the finite-dimensional scenario $X = \mathbb{R}^d$ from [17, pp. 185ff] to a general real Banach space X, both a Fredholm alternative for duality pairings $\langle Y, X_\mathbb{C}\rangle$ and a suitable center manifold theory in Banach spaces are due. Above all, it is convenient to introduce the l-linear maps $\mathcal{D}_l := D_1^l\Pi(0,\alpha^*) \in L_l(X)$, $l = 1, 2, 3$. By assumption (A_4), the only spectral points of $\mathcal{D}_1$ on $\mathbb{S}^1$ are a pair of simple eigenvalues $\lambda_*, \overline{\lambda_*}$. If $\xi^* \in X_\mathbb{C}$ is an eigenvector associate to λ_* from (A_2), then

$$\mathcal{D}_1\xi^* = \lambda_*\xi^*, \qquad\qquad \mathcal{D}_1\overline{\xi^*} = \overline{\lambda_*}\,\overline{\xi^*}.$$

Thanks to assumption (A_1) the dual operator $\mathcal{D}_1' \in L(Y)$ of $\mathcal{D}_1$ exists; due to [14, p. 46, Thm. 4.6] it is also uniquely determined. The corresponding dual eigenvector $\eta^* \in Y$ from (A_3) has the properties

$$\mathcal{D}_1'\eta^* = \overline{\lambda_*}\eta^*, \qquad\qquad \mathcal{D}_1'\overline{\eta^*} = \lambda_*\overline{\eta^*}.$$

Eventually, the critical (and real) central eigenspace of $\mathcal{D}_1$ can be represented as $X^0 = \operatorname{span}\{\operatorname{Re}\xi^*, \operatorname{Im}\xi^*\}$ and has a complement $X^\perp$ of codimension 2.

(I) <u>Claim</u>: *$u \in X^\perp \Leftrightarrow \langle\eta^*, u\rangle = 0$ for all $u \in X$.*
Since $|\lambda_*| = 1 > r_0$ holds by assumption (A_1), we obtain from [22, Lemma 5] that both $\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1$ and $\overline{\lambda_*} I_Y - \mathcal{D}_1'$ are Fredholm of index 0. Because λ_* is a simple eigenvalue of $\mathcal{D}_1$, the mapping $\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1$ has ascent 1, and [19, Thm. 1] guarantees that the index-0-operator $\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1$ also possesses finite descent. Referring to [10, p. 209, Prop. 50.2], this implies the decomposition

$$X_\mathbb{C} = N(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1) \oplus R(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1).$$

Now the Fredholm alternative from [30, p. 304, Thm. 5.G] ensures that the elements $v \in R(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1)$ are characterized by $\langle\eta, v\rangle = 0$ for all $\eta \in N(\overline{\lambda_*} I_Y - \mathcal{D}_1')$. In particular,

$$R(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1) = N(\langle\eta^*, \cdot\rangle), \tag{A.4}$$

which yields $\langle\eta^*, \overline{\xi^*}\rangle = 0$, as

$$(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1)\big(\tfrac{1}{2\operatorname{Im}\lambda_*}\overline{\xi^*}\big) = \big((\overline{\lambda_*} I_{X_\mathbb{C}} - \mathcal{D}_1) + 2\operatorname{Im}\lambda_* I_{X_\mathbb{C}}\big)\big(\tfrac{1}{2\operatorname{Im}\lambda_*}\overline{\xi^*}\big) = 0 + \overline{\xi^*}$$

implies $\overline{\xi^*} \in R(\lambda_* I_{X_\mathbb{C}} - \mathcal{D}_1)$.

Define $X^\perp := \big\{u - \langle\eta^*, u\rangle\xi^* - \langle\overline{\eta^*}, u\rangle\overline{\xi^*} \in X \,|\, u \in X\big\}$, and note that the identities $\langle\eta^*, \xi^*\rangle = 1$ and $\langle\eta^*, \overline{\xi^*}\rangle = 0$ imply $\langle\eta^*, \operatorname{Re}\xi^*\rangle = \frac{1}{2}$ as well as $\langle\eta^*, \operatorname{Im}\xi^*\rangle = -\frac{\imath}{2}$. A basic computation now yields $\langle\eta^*, u\rangle \neq 0$ for all $u \in X_0 \setminus \{0\}$ and $\langle\eta^*, u\rangle = 0$ for all $u \in X^\perp$. As such, $X_0 \cap X^\perp = \{0\}$, and by the definition of $X^\perp$ we thus have $X = X_0 \oplus X^\perp$; the claim follows.

(II) <u>Claim</u>: *$\frac{d}{d\alpha}|\lambda_+(\alpha)|\,|_{\alpha=\alpha^*} = \rho^*$ from (A_6).*
As (λ_*, ξ^*) is a simple eigenpair by assumption (A_2), it follows from [3, p. 38, Prop. 3.6.1] that it can be embedded into a C^1-branch $\alpha \mapsto (\lambda_+(\alpha), \xi(\alpha))$ of eigenpairs to $D_1\Pi(0,\alpha)$ over an open neighborhood $A_0 \subseteq A$ of α^*. If we differentiate the identity $D_1\Pi(0,\alpha)\xi(\alpha) \equiv \lambda_+(\alpha)\xi(\alpha)$ on A_0 and set $\alpha = \alpha^*$, this results in

$$\dot\lambda_+(\alpha^*)\xi^* = D_2\mathcal{D}_1\xi^* + [\mathcal{D}_1 - \lambda_* I_{X_\mathbb{C}}]\dot\xi(\alpha^*),$$

and applying the linear form $\langle \eta^*, \cdot\rangle$ gives us

$$\begin{aligned}\dot\lambda_+(\alpha^*) &\overset{(A_3)}{=} \langle \eta^*, D_1D_2\Pi(0,\alpha^*)\xi^*\rangle + \langle \eta^*, [\mathcal{D}_1 - \lambda_* I_{X_{\mathbb{C}}}]\dot\xi(\alpha^*)\rangle \\ &\overset{(A.4)}{=} \langle \eta^*, D_1D_2\Pi(0,\alpha^*)\xi^*\rangle.\end{aligned}$$

Using this formula and the identity $|\lambda_+(\alpha)| \equiv \sqrt{\overline{\lambda_+(\alpha)}\lambda_+(\alpha)}$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,|\lambda_+(\alpha)|\,|_{\alpha=\alpha^*} = \tfrac{1}{2}\big(\overline{\lambda_*}\dot\lambda_+(\alpha^*) + \lambda_*\overline{\dot\lambda_+(\alpha)}\big) = \operatorname{Re}\big(\overline{\lambda_*}\dot\lambda_+(\alpha^*)\big),$$

which yields the assertion.

(III) Thanks to step (I), the inclusion $v \in X^\perp$ holds if and only if $\langle \eta^*, v\rangle = 0$, which implies two real constraints on v. We decompose $u \in X$ as $u = z\xi^* + \overline{z}\overline{\xi^*} + v$ with $z \in \mathbb{C}$ (and $z\xi^* + \overline{z}\overline{\xi^*} \in X^0$) and $v \in X^\perp$. In combination, this yields

$$z = \langle \eta^*, u\rangle, \qquad\qquad v = u - \langle \eta^*, u\rangle\xi^* - \langle \overline{\eta^*}, u\rangle\overline{\xi^*}$$

as new coordinates, in which the autonomous eqn. $u_{t+1} = \Pi(u_t, \alpha^*)$ becomes

$$\begin{cases} z_{t+1} = \lambda_* z_t + \frac{1}{2}G_{20}z_t^2 + G_{11}z_t\overline{z_t} + \frac{1}{2}G_{02}\overline{z_t}^2 \\ \qquad\quad + \frac{1}{2}G_{21}z_t^2\overline{z_t} + z_t\langle \eta^*, \mathcal{D}_2\xi^* v_t\rangle + \overline{z_t}\langle \eta^*, \mathcal{D}_2\overline{\xi^*}v_t\rangle + \dots, \\ v_{t+1} = \mathcal{D}_1 v_t + \frac{1}{2}z_t^2 H_{20} + z_t\overline{z_t}H_{11} + \frac{1}{2}\overline{z_t}^2 H_{02} + \dots \end{cases} \tag{A.5}$$

with the complex numbers

$$\begin{aligned} &G_{20} := \langle \eta^*, \mathcal{D}_2\xi^*\xi^*\rangle, \quad G_{11} := \langle \eta^*, \mathcal{D}_2\xi^*\overline{\xi^*}\rangle, \\ &G_{02} := \langle \eta^*, \mathcal{D}_2\overline{\xi^*\xi^*})\rangle, \quad G_{21} := \langle \eta^*, \mathcal{D}_3\xi^*\xi^*\overline{\xi^*}\rangle \end{aligned}$$

and the vectors

$$\begin{aligned} H_{20} &= \mathcal{D}_2\xi^*\xi^* - \langle \eta^*, \mathcal{D}_2\xi^*\xi^*\rangle\xi^* - \langle \overline{\eta^*}, \mathcal{D}_2\xi^*\xi^*\rangle\overline{\xi^*}, \\ H_{11} &= \mathcal{D}_2\xi^*\overline{\xi^*} - \langle \eta^*, \mathcal{D}_2\xi^*\overline{\xi^*}\rangle\xi^* - \langle \overline{\eta^*}, \mathcal{D}_2\xi^*\overline{\xi^*}\rangle\overline{\xi^*}, \quad H_{02} = \overline{H_{20}}. \end{aligned}$$

In order to apply the center manifold theorem [7, Thm. 5.4] to (A.5), we show that the center-unstable subspace of $\mathcal{D}_1 \in L(X)$ is finite-dimensional. For this we observe that $\Lambda := \sigma(\mathcal{D}_1) \setminus B_1(0)$ is finite. Otherwise, the set Λ would contain an (injective) sequence, which, as a subset of $\sigma(\mathcal{D}_1)$, is bounded. Thus, the Bolzano-Weierstraß theorem guarantees that Λ contains an accumulation point. Since $\sigma_{\mathrm{ess}}(\mathcal{D}_1) \subseteq \bar{B}_{r_0}(0)$ contains all accumulation points and $r_0 < 1$, this is a contradiction.

Moreover, [19, Thm. 1] established that the elements $\lambda \in \Lambda$ are isolated points of $\sigma(\mathcal{D}_1)$ and poles of the resolvent map $z \mapsto (zI_{X_{\mathbb{C}}} - \mathcal{D}_1)^{-1}$ (of finite rank) with associate generalized eigenspaces satisfying $\dim \operatorname{Eig}_\lambda \mathcal{D}_1 < \infty$. In conclusion, the center-unstable space $\bigoplus_{\lambda\in\Lambda} \operatorname{Eig}_\lambda \mathcal{D}_1$ is finite-dimensional.

The detailed reduction to a 2-dimensional center-unstable manifold C, given as graph of a C^m-function $c : W_0 \to X^\perp$ over a neighborhood $W_0 \subseteq X^0$ of 0, is formally identical to [17, pp. 185ff]. The nonresonance condition (A_5) and the transversality condition (A_6) combined with (II) allow us to apply e.g. [17, p. 134, Thm. 4.5] (and [29] for smoothness) to the difference equation reduced to C. Then points on the bifurcating curve of this equation in X_0 have a distance of order $O\big(\sqrt{|\beta(\alpha)/\delta^*|}\big)$ from the origin. Thanks to $c(0) = 0$ and $Dc(0) = 0$ we hence obtain that the points $z\xi + \overline{z}\overline{\xi} + c(z\xi + \overline{z}\overline{\xi}) \in X$ of the bifurcating circle $T_\alpha \subset X$ allow the estimate

$$\big\|z\xi + \overline{z}\overline{\xi} + c(z\xi + \overline{z}\overline{\xi})\big\| \le \big(1 + O\big(\big\|z\xi + \overline{z}\overline{\xi}\big\|\big)\big)\,\big\|z\xi + \overline{z}\overline{\xi}\big\| \quad \text{as } z \to 0.$$

In conclusion, also the distance of points on T_α to $0 \in X$ behaves as $O\big(\sqrt{|\beta(\alpha)/\delta^*|}\big)$ for $\alpha \to \alpha^*$. □

Appendix B. **Nyström methods.** In Sect. 5 we simulate IDEs with right-hand sides (4.2) using their Nyström discretizations. This means integrals over $\Omega = [a, b]$ are replaced by quadrature methods based on the 4*th order Gauß formula*

$$\int_a^b u = h \sum_{j=0}^{N-1} \left(u\left(\eta_j - \tfrac{h}{\sqrt{3}}\right) + u\left(\eta_j + \tfrac{h}{\sqrt{3}}\right)\right) + \frac{b-a}{270} h^4 u^{(4)}(\xi)$$

and on the 6*th order Gauß formula*

$$\int_a^b u = \frac{h}{9} \sum_{j=0}^{N-1} \left(5u\left(\eta_j - h\sqrt{\tfrac{3}{5}}\right) + 8u(\eta_j) + 5u\left(\eta_j + h\sqrt{\tfrac{3}{5}}\right)\right) + \frac{b-a}{31500} h^6 u^{(6)}(\xi)$$

with nodes $\eta_j := a + (2j+1)h$, $h := \frac{b-a}{2N}$ and some $\xi \in [a, b]$ (see [6, pp. 385ff]).

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