A Continuation Principle for Fredholm maps II: Application to homoclinic solutions

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When studying the behavior of autonomous ordinary differential equations under time-dependent perturbations vanishing for $t \to \pm\infty$, their equilibria generically persist locally as homoclinic solutions. Using an abstract and flexible continuation theorem, we find even global branches of such homoclinic solutions for parametrized nonautonomous ordinary differential equations. Our approach is based on degree-theoretical arguments. In particular, Landesman-Lazer conditions are proposed to obtain the existence of homoclinic solutions by means of a nonzero degree.

1 Introduction

It is both an interesting, as well as a relevant question to ask about the fate of an equilibrium x^* of a parametrized (ordinary) differential equation $\dot{x} = f(x, \lambda)$, when the parameter λ is replaced by a time-variant function $\lambda(t)$? For instance, λ might describe how the environment affects a model, which could be periodic to capture seasonal effects, but will be rather arbitrary in other applications.

From a mathematical perspective, time-dependent parameters give rise to nonautonomous (ordinary) differential equations and therefore one cannot expect that constant solutions (equilibria) exist anymore (cf. [12]). Hence, a wider class of reference solutions is appropriate in order to tackle related continuation or bifurcation problems (cf. [18, 19]). In this context it is a key observation that hyperbolic equilibria of autonomous equations (the linearization has no spectrum on the imaginary axis) persist locally as bounded solutions defined on the entire real axis, when constant parameters become time-varying functions (see [3, 19]). This fact results from a combination of admissibility properties for exponential dichotomies with the implicit function theorem, and is thus local in nature. Information on the global behavior can be deduced on basis of global implicit function theorems, which in turn require degree-theoretical arguments. For instance, in [20] we applied such a result due to [9] to arrive at corresponding statements on the global structure of solution branches (in a discrete time setting of difference equations).

The paper at hand has a slightly different focus. Rather than asking for the behavior of global branches, we tackle the problem to find a solution for every parameter value and whether these solutions can be chosen from a continuum? A corresponding abstract *continuation principle* was shown in [21] for general equations in Banach spaces. Here we present a concrete and nontrivial application to ordinary differential equations. As a showcase, it applies to the following situation: Given an equation having w.l.o.g. the trivial equilibrium, what happens to this zero solution when replacing the parameters by time-variant functions decaying to 0 in forward and backward time? This means we investigate the effect of nonautonomous perturbations vanishing at infinity. Locally, i.e. for

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small perturbations, the zero equilibrium will persist as a solution vanishing at $\pm\infty$ – one speaks of a *homoclinic solution*. We provide sufficient conditions guaranteeing that homoclinic solutions exist over the entire parameter range, which might be unbounded.

In order to be more precise, we investigate time-variant (nonautonomous) ordinary differential equations

$$\dot{x} = f(t, x, \lambda), \tag{D_λ}$$

depending on a parameter λ. Since one cannot expect (D_λ) to have equilibria, an appropriate counterpart are bounded solutions existing on the entire real line. In this spirit, we provide sufficient conditions so that (D_λ) has *homoclinic solutions*, i.e., entire solutions $\phi_\lambda \colon \mathbb{R} \to \mathbb{R}^d$ satisfying

$$\lim_{t\to\pm\infty} \phi_\lambda(t) = 0$$

for every parameter λ from a connected (metrizable Banach) manifold Λ, which moreover form a continuum.

The main tool for this purpose is a continuation principle for parameter-dependent Fredholm maps from [21]. Indeed, the ordinary differential equation (D_λ) is formulated as abstract equation

$$G(x, \lambda) = 0 \tag{O_λ}$$

in an ambient Banach space. In order to be more precise, we consider oriented Fredholm mappings $G \colon \overline{O} \times \Lambda \to Y$ between Banach spaces X, Y, where $O \subseteq X$ is nonempty, open and suppose

- **(A1)** the parameter space Λ is a connected metrizable (Banach) manifold (or more general, a connected absolute neighborhood retract, cf. [8, p. 287, Cor. (5.4)]),
- **(A2)** for any compact set $\Lambda_0 \subseteq \Lambda$ the restriction $G|_{\overline{O}\times\Lambda_0}$ is proper,
- **(A3)** $G^{-1}(0) \cap ((\partial O) \times \Lambda) = \emptyset$ and $G(\cdot, \lambda)|_O$ is Fredholm of index 0 for all $\lambda \in \Lambda$,
- **(A4)** there exists a $\lambda^* \in \Lambda$ such that $\deg(G(\cdot, \lambda^*), O, 0) \neq 0$,

where $\deg(G(\cdot, \lambda^*), O, 0)$ is an ambient topological degree, we are going to specify later. Under these assumptions, our key tool reads as follows:

Theorem 1.1 (global continuation principle, cf. [21, Thm. 1.1]) *If* (A1–A4) *hold, then* $G^{-1}(0) \cap (O \times \Lambda)$ *has a connected component* $\mathcal{C}$ *which for every* $\lambda \in \Lambda$ *contains a solution* $x \in O$ *of* (O_λ).

An actual application of Thm. 1.1 to an abstract formulation (O_λ) of (D_λ) first requires an ambient spatial setting: We work with the continuous limit 0 functions for Y and its C^1-counterpart for X. The corresponding criteria needed to fulfill (A2) are adapted from [13, 15]. Second, various tools from nonautonomous dynamics like exponential dichotomies [12, 13] or the Bebutov flow (cf. [23, 13]) to come into play. In particular, a generalized Landesman-Lazer condition (see for instance, [10]) proves to be helpful — note that this approach is novel because typically Landesman-Lazer conditions were used only for time-periodic problems (D_λ) so far.

The central problem in this (and of course a general) application of Thm. 1.1 is to verify that the degree in assumption (A4) does not vanish. This task requires to choose a degree theory appropriate for the particular problem. Here, we found a combination of the Benevieri-Furi degree [1] (as opposed to that of Fitzpatrick-Pejsachowicz-Rabier [11]) with the Landesman-Lazer condition to be advantageous in calculations, which result in the criterion of Prop. 4.6.

The paper is organized as follows: After introducing our terminology and notation, the following §2 deals with homoclinic solutions for (D_λ) and provides an appropriate functional analytical setting. This includes a recap of the Fredholm theory for linear differential equations, as well as criteria for the properness of substitution operators. On this basis we can formulate and prove our main continuation Thm. 3.1 in §3. Also an immediate example is given. Verifying its assumption involves to show that a topological degree does not vanish. This is a nontrivial task and settled in §4 by means of Landesman-Lazer conditions; in addition, we provide two examples. Finally, our two appendices aim to help readers possibly unfamiliar with single perquisites of this paper, and intend to keep it largely self-contained. Basic concepts of topological dynamics such as the construction of the Bebutov flow are contained in App. A, while App. B contains a crucial tool to compute the Benevieri-Furi degree.

As a further field of applications for Thm. 1.1, although not examined here, we mention boundary value problems over unbounded domains [13, 9, 15].

Notation and preliminaries

In what follows, we use the notations $\mathbb{R}_+ := [0,\infty)$, $\mathbb{R}_- := (-\infty, 0]$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\delta_{ij} \in \{0,1\}$ is the Kronecker symbol. For a metric space X, $B_r(x)$ and $\bar{B}_r(x)$ are the open resp. closed r-balls centered in x, the interior of a set $\Omega \subset X$ is Ω°, the closure is $\overline{\Omega}$ and $\partial\Omega$ the boundary.

For Banach spaces X, Y we denote the space of linear bounded operators from X to Y by $L(X,Y)$, $GL(X,Y)$ are the invertible elements and $\Phi_0(X,Y)$ the linear index 0 Fredholm operators. We briefly write $L(X) := L(X,X)$ (similarly for other spaces) and id_X for the identity map on X. Furthermore, $N(T) := T^{-1}(0)$ and $R(T) := TX$ are the *kernel* resp. the *range* of $T \in L(X,Y)$.

Given a nonempty, open set $O \subseteq X$ and a nonlinear C^1-map $F\colon O \to Y$, one speaks of a *Fredholm map* $F\colon \overline{O} \to Y$ with index 0, if $DF(x) \in \Phi_0(X,Y)$ holds for all $x \in O$.

Typically dealing with mappings $G : O \times \Lambda \to Y$ depending on two variables, it is convenient to abbreviate $G_\lambda := G(\cdot,\lambda) : O \to Y$. The elementary proof of the next result is left to the reader:

Lemma 1.2 *Let Λ be a metric space. If a continuous mapping $G\colon X \times \Lambda \to Y$ satisfies*

(i) *$\{G(x,\cdot) : \Lambda \to Y \mid x \in B\}$ is equicontinuous for all bounded $B \subseteq X$,*

(ii) *$G_\lambda : X \to Y$, $\lambda \in \Lambda$, is proper on every bounded, closed subset of X,*

then G is proper on every product $B \times \Lambda_0$ with $B \subset X$ bounded, closed and $\Lambda_0 \subseteq \Lambda$ compact.

A *generalized Fredholm homotopy* of index 0 is a continuous map $H\colon O \times [0,1] \to Y$ with continuous derivative $(x,t) \mapsto DH_t(x) \in \Phi_0(X,Y)$ for every $t \in [0,1]$.

Finally, norms on finite-dimensional linear spaces are denoted by $|\cdot|$.

2 Homoclinic solutions of ordinary differential equations

The aim of this section is to illustrate Thm. 1.1 by means of a nontrivial application. Let us thereto consider parametrized ordinary differential equations (ODE for short)

$$\dot{x} = f(t,x,\lambda) \tag{D_λ}$$

having a right-hand side $f\colon \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$. Here, $\Omega = \overline{U} \subseteq \mathbb{R}^d$, where U is an open convex neighborhood of 0 and, for the sake of consistency with the above, the parameter space Λ is a connected (metrizable Banach) manifold. An *entire solution* to (D_λ) stands for a differentiable function $\phi\colon \mathbb{R} \to \Omega$ satisfying the solution identity $\dot{\phi}(t) \equiv f(t,\phi(t),\lambda)$ on $\mathbb{R}$.

Assume from now on that $f\colon \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ fulfills for $j \in \{0,1\}$:

(H0) $f\colon \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$, $D_2 f\colon \mathbb{R} \times \Omega^\circ \times \Lambda \to L(\mathbb{R}^d)$ exist as continuous functions and

$$\sup_{t\in\mathbb{R}} \sup_{x \in B\cap\Omega^\circ} \left| D_2^j f(t,x,\lambda) \right| < \infty \text{ for all } \lambda \in \Lambda \text{ and every bounded } B \subseteq \Omega,$$

(H1) for every compact set $K \subset \Omega$, $\varepsilon > 0$ and $\lambda_0 \in \Lambda$ there exists a $\delta > 0$ such that for all $\lambda \in B_\delta(\lambda_0)$ holds

$$|x-y| < \delta \text{ and } |t-s| < \delta \quad \Rightarrow \quad \sup_{t\in\mathbb{R}} \left| D_2^j f(t,x,\lambda) - D_2^j f(s,y,\lambda_0) \right| < \varepsilon,$$

for all $x, y \in K \cap \Omega^\circ$, $t, s \in \mathbb{R}$,

(H2) $\lim_{t\to\pm\infty} f(t,0,\lambda) = 0$ for all $\lambda \in \Lambda$.

Remark 2.1 It is not hard to see that the derivative $D_2 f\colon \mathbb{R}\times\Omega^\circ\times\Lambda \to L(\mathbb{R}^d)$ can be extended to $\mathbb{R}\times\Omega\times\Lambda$ (this extension will be denoted by the same symbol $D_2 f$). In particular, the extended derivative $D_2 f$ is continuous on $\mathbb{R}\times\Omega\times\Lambda$. Moreover, $f\colon \mathbb{R}\times\Omega\times\Lambda \to \mathbb{R}^d$ and $D_2 f\colon \mathbb{R}\times\Omega\times\Lambda \to L(\mathbb{R}^d)$ satisfy (H0) and (H1) on $B \cap \partial\Omega$ and $K \cap \partial\Omega$, respectively.

If M denotes a metric space and $Y \subset E$, where E is a normed space with a norm $\|\cdot\|$, then $\mathcal{C}(M,Y)$ are the continuous and $\mathcal{BC}(M,Y)$ the bounded continuous mappings $g\colon M \to Y$. In particular, we abbreviate $\mathcal{C}(Y) := \mathcal{C}(\mathbb{R},Y)$, $\mathcal{C} := \mathcal{C}(\mathbb{R}^d)$ and similarly for other function spaces. Note that $\|\phi\|_0 := \sup_{x\in M} |\phi(x)|$ defines a norm on $\mathcal{BC}(M,\mathbb{R}^d)$. For any $Y \subseteq E$ containing 0, we define the set

$$\mathcal{C}_0(\mathbb{R}^m, E) := \left\{ g \in \mathcal{C}(\mathbb{R}^m, E) \;\middle|\; \|g(x)\| \xrightarrow[|x|\to\infty]{} 0 \right\}.$$

In particular, for $m = 1$, we will briefly write $\mathcal{C}_0(Y) := \mathcal{C}_0(\mathbb{R}, Y)$ and $\mathcal{C}_0 := \mathcal{C}_0(\mathbb{R}^d)$, respectively. This yields Banach spaces $\mathcal{C}_0 \subset \mathcal{BC}$ w.r.t. the natural norm $\|\cdot\|_0$.

Moreover, $\mathcal{C}^1(\Omega)$ with $Y = \Omega$ are the continuously differentiable functions $\phi\colon \mathbb{R} \to \Omega$ and

$$\mathcal{C}_0^1(\Omega) := \left\{ \phi \in \mathcal{C}^1(\Omega) \mid \phi, \dot\phi \in \mathcal{C}_0 \right\}.$$

We equip $\mathcal{C}_0^1$ with the norm

$$\|\phi\|_1 := \max\left\{ \|\phi\|_0, \|\dot\phi\|_0 \right\}$$

and obtain the continuous embedding $\mathcal{C}_0^1 \hookrightarrow \mathcal{C}_0$. Finally, throughout, we will consider the shift (linear) operator $S^s\colon \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ given by $(S^s\phi)(t) = \phi(t+s)$ for all $\phi \in \mathcal{C}(\Omega)$ and $t \in \mathbb{R}$, where $s \in \mathbb{R}$ is a fixed real number.

2.1 Fredholm properties of linear ODEs

We now develop the required Fredholm theory for our purposes. Consider a linear ODE

$$\dot x = A(t)x \tag{L}$$

with continuous coefficient $A : \mathbb{R} \to L(\mathbb{R}^d)$ and associated *evolution matrix* $U\colon \mathbb{R}^2 \to L(\mathbb{R}^d)$.

Suppose $J \subseteq \mathbb{R}$ is an unbounded interval. An *invariant projector* is a function $\Pi\colon J \to L(\mathbb{R}^d)$ of projections $\Pi(t) \in L(\mathbb{R}^d)$ such that

$$U(t,s)\Pi(s) = \Pi(s)U(t,s) \text{ for all } t,s \in J.$$

A linear ODE (L) has an *exponential dichotomy* (ED for short, [5]) on J with invariant projector Π, if there exist reals $K \geqslant 1$, $\alpha > 0$ such that

$$|U(t,s)\Pi(s)| \leqslant Ke^{-\alpha(t-s)}, \qquad |U(s,t)[\mathrm{id}_{\mathbb{R}^d} - \Pi(t)]| \leqslant Ke^{\alpha(s-t)} \text{ for all } s \leqslant t \tag{2.1}$$

and $t,s \in J$ hold. The constant dimension of $N(\Pi(t))$, $t \in J$, is called *Morse index* of (L). The associate *dichotomy spectrum* (cf. [12, pp. 82ff]) becomes

$$\Sigma_J(A) := \{\gamma \in \mathbb{R} \mid \dot x = [A(t) - \gamma\,\mathrm{id}_{\mathbb{R}^d}]x \text{ admits no ED on } J\}$$

and we conveniently write $\Sigma^+(A) := \Sigma_{\mathbb{R}_+}(A)$, $\Sigma^-(A) := \Sigma_{\mathbb{R}_-}(A)$, $\Sigma(A) := \Sigma_{\mathbb{R}}(A)$ for the *forward*, *backward* and *all time spectrum* of (L). The evolution matrix of the adjoint equation

$$\dot x = -A(t)^T x, \tag{L^*}$$

is given by $U^*(t,s) := U(s,t)^T$ for all $s,t \in \mathbb{R}$. If A is additionally bounded, then we define two linear operators $L_A, L_A^* \in L(\mathcal{C}_0^1, \mathcal{C}_0)$ by

$$(L_A\phi)(t) := \dot\phi(t) - A(t)\phi(t), \qquad (L_A^*\phi)(t) := \dot\phi(t) + A(t)^T\phi(t) \text{ for all } t \in \mathbb{R}. \tag{2.2}$$

Lemma 2.2 *The following statements are equivalent:*

(a) $0 \notin \Sigma^+(A)$ *and* $0 \notin \Sigma^-(A)$ *with corresponding projectors* Π^+ *resp.* Π^-,

(b) L_A *is Fredholm with* $\operatorname{ind} L_A = \operatorname{rk} \Pi^+(0) - \operatorname{rk} \Pi^-(0)$.

Proof. The implication $(a) \Rightarrow (b)$ follows from [16, Lemma 4.2], while the proof of the converse implication $(b) \Rightarrow (a)$ can be found in [17]. □

In applications, (L) will be a variational equation. More detailed, suppose that $\phi^* : \mathbb{R} \to \Omega$ is a continuous function (not necessarily a solution) and consider the variational equation

$$\dot{x} = D_2 f(t, \phi^*(t), \lambda)x, \tag{V_λ}$$

whose half-line dichotomy spectra are denoted by $\Sigma^+(\lambda), \Sigma^-(\lambda)$ for $\lambda \in \Lambda$. One says that ϕ^* is *hyperbolic*, if (V_λ) has an ED on $\mathbb{R}$, i.e. $0 \notin \Sigma(\lambda)$. One speaks of a *weakly hyperbolic* function, if $0 \notin \Sigma^+(\lambda)$, $0 \notin \Sigma^-(\lambda)$ and the corresponding projectors satisfy $\operatorname{rk} \Pi^+_\lambda(0) = \operatorname{rk} \Pi^-_\lambda(0)$. Clearly, hyperbolicity implies weak hyperbolicity, but means a significantly stronger assumption.

2.2 Substitution operators

Rather than as an ODE, we will consider (D_λ) as abstract equation between function spaces. This, first of all, requires to consider substitution operators

$$F : \mathcal{C}_0(\Omega) \times \Lambda \to \mathcal{C}_0, \qquad F(\phi, \lambda) := f(\cdot, \phi(\cdot), \lambda). \tag{2.3}$$

Given a closed and totally disconnected set $Z \subset Y \subset \mathbb{R}^d$ we define

$$\mathcal{C}_Z(Y) := \left\{ \phi \in \mathcal{BC}(\mathbb{R}, Y) \;\middle|\; \lim_{|t| \to \infty} \operatorname{dist}_Z(\phi(t)) = 0 \right\}$$

and borrow the following compactness criterion from [22]:

Lemma 2.3 (compactness in $\mathcal{C}_Z$) *A subset $\mathcal{F} \subset \mathcal{C}_Z(\mathcal{D})$, where $\mathcal{D} \subset \mathbb{R}^d$ is closed with $Z \subset \mathcal{D}$, is relatively compact if and only if*

(i) *$\mathcal{F}$ is bounded,*

(ii) *$\mathcal{F}$ is uniformly equicontinuous, that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|t - s| < \delta$ implies $|\phi(t) - \phi(s)| < \varepsilon$ for all $\phi \in \mathcal{F}$ and $t, s \in \mathbb{R}$,*

(iii) *if there are sequences $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ and $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} |s_n| = \infty$ and $S^{s_n}\phi_n \to \varphi \in \mathcal{BC}(\mathbb{R}, \mathcal{D})$ pointwise on $\mathbb{R}$, then $\varphi(\mathbb{R}) \subset Z$.*

In particular, if $Z \subset \mathbb{R}^d$ is compact, then for all $z \in Z$, $\mathcal{C}_{\{z\}}(\mathcal{D}) \subset \mathcal{C}_Z(\mathcal{D})$ and $\mathcal{F} \subset \mathcal{C}_{\{z\}}(\mathcal{D})$ is relatively compact in $\mathcal{C}_{\{z\}}(\mathcal{D})$ if and only if $\mathcal{F}$ is relatively compact in $\mathcal{C}_Z(\mathcal{D})$.

It is convenient to state the special case $Z = \{0\}$ separately:

Corollary 2.4 (compactness in $\mathcal{C}_0(\Omega)$) *A subset $\mathcal{F} \subset \mathcal{C}_0(\Omega)$ is relatively compact if and only if*

(i) *$\mathcal{F}$ is bounded,*

(ii) *$\mathcal{F}$ is* uniformly equicontinuous,

(iii) *if there are sequences $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ and $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} |s_n| = \infty$ and $S^{s_n}\phi_n \to \varphi \in \mathcal{BC}(\mathbb{R}, \Omega)$ pointwise on $\mathbb{R}$, then $\varphi(t) \equiv 0$.*

The next result ensures that F admits good properties:

Proposition 2.5 (properties of substitution operators) *The operator $F : \mathcal{C}_0(\Omega) \times \Lambda \to \mathcal{C}_0$ is well-defined, continuous and has the following properties for all $\phi, \phi_0 \in \mathcal{C}_0(\Omega)$ and $\lambda \in \Lambda$:*

(a) *$D_1 F : \mathcal{C}_0(\Omega) \times \Lambda \to L(\mathcal{C}_0)$ exists as a continuous function with $D_1 F(\phi, \lambda)\psi(t) = D_2 f(t, \phi(t), \lambda)\psi(t)$ for all $\psi \in \mathcal{C}_0$,*

(b) $D_1F(\phi,\lambda) - D_1F(\phi_0,\lambda) \in L(\mathcal{C}_0)$ *is compact.*

Proof. Well-definedness and continuity of F, as well as assertion (a), were shown in [19, Sec. 3] or [15, Sec. 2].

(b): As sums of compact operators are compact (see [25, p. 278, Thm. (i)]), it suffices to show that the difference $D_1F(\phi,\lambda) - D_1F(\phi_0,\lambda) \in L(\mathcal{C}_0)$ is compact for all $\phi \in \mathcal{C}_0(\Omega)$. Again, $D_1F(\phi,\lambda) - D_1F(\phi_0,\lambda) = M$ holds with the multiplication operator $M \in L(\mathcal{C}_0)$ given by

$$(M\psi)(t) := \underbrace{(D_2f(t,\phi(t),\lambda) - D_2f(t,\phi_0(t),\lambda))}_{=:A(t)}\psi(t) \text{ for all } t \in \mathbb{R}$$

(see (2.3)). In order to show that M is compact, we apply the criterion Cor. 2.4 to the set $\mathcal{F} := MB_1(0)$ with $B_1(0) \subset \mathcal{C}_0$ and define $C := \sup_{t\in\mathbb{R}} |A(t)|$:
ad (i): For every $\phi \in \mathcal{F}$ there is a $\psi \in B_1(0) \subset \mathcal{C}_0$ with

$$|\phi(t)| = |A(t)\psi(t)| \leqslant C \|\psi\|_0 \leqslant C \text{ for } t \in \mathbb{R}.$$

ad (ii): Let $\varepsilon > 0$. Since $A\colon \mathbb{R} \to L(\mathbb{R}^d)$ is continuous and $\lim_{t\to\pm\infty} A(t) = 0$, there exists a $\delta_1 > 0$ so that $|t-s| < \delta_1$ yields $|A(t) - A(s)| < \varepsilon/2$ for $t, s \in \mathbb{R}$. If we set $\delta := \min\{\delta_1, \varepsilon/(2C)\}$, then

$$\begin{aligned} |\phi(t) - \phi(s)| &\leqslant |A(t)(\psi(t) - \psi(s))| + |(A(t) - A(s))\psi(s)| \\ &\leqslant C|\psi(t) - \psi(s)| + |(A(t) - A(s))\psi(s)| \\ &\leqslant C|\psi(t) - \psi(s)| + |A(t) - A(s)| \sup_{t\in\mathbb{R}} |\psi(t)| \\ &\leqslant C|t-s| + |A(t) - A(s)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $t, s \in \mathbb{R}$, $|t-s| < \delta$, $\phi \in \mathcal{F}$, by the mean value estimate. So, $\mathcal{F}$ is uniformly equicontinuous.
ad (iii): This follows from the limit relation

$$0 \leqslant |\phi(t)| = |A(t)\psi(t)| \leqslant |A(t)| \xrightarrow[t\to\pm\infty]{} 0 \text{ for all } \phi \in \mathcal{F}.$$

In conclusion, $\mathcal{F} \subset \mathcal{C}_0$ is relatively compact and thus M is compact. □

As a result of Prop. 2.5 the operator

$$G\colon \mathcal{C}_0^1(\Omega) \times \Lambda \to \mathcal{C}_0, \qquad G(\phi,\lambda) := \dot{\phi} - F(\phi,\lambda) \tag{2.4}$$

is well-defined (see [19, Cor. 3.5]) and we obtain the elementary, yet crucial

Proposition 2.6 ([19, Thm. 3.6(b)]) *Let $\lambda \in \Lambda$ and G be given in* (2.4)*. If $\phi \in \mathcal{C}_0$ solves (D_λ), then $\phi \in \mathcal{C}_0^1(\Omega)$ and (O_λ) hold. Conversely, if $\phi \in \mathcal{C}^1(\Omega) \cap \mathcal{C}_0$ solves the operator equation (O_λ), then ϕ is a homoclinic solution to (D_λ) with $\phi \in \mathcal{C}_0^1$.*

With Prop. 2.5(a) our assumptions imply that $D_1G : \mathcal{C}_0^1(\Omega) \times \Lambda \to L(\mathcal{C}_0^1, \mathcal{C}_0)$ exists and

$$D_1G(\phi,\lambda)\psi = \dot{\psi} - D_1F(\phi,\lambda)\psi \text{ for all } \psi \in \mathcal{C}_0^1 \tag{2.5}$$

is a continuous function satisfying:

Lemma 2.7 *If $\phi_0 \in \mathcal{C}_0^1(\Omega)$ is weakly hyperbolic, then the following holds for $\phi \in \mathcal{C}_0^1(\Omega)$, $\lambda \in \Lambda$:*

(a) $D_1G(\phi_0,\lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$,

(b) $D_1G(\phi_0,\lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0) \Leftrightarrow D_1G(\phi,\lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$.

P r o o f. Above all, the function $A(t) := D_2 f(t, \phi_0(t), \lambda)$ is bounded and continuous.
(a): This is an immediate consequence of Lemma 2.2.
(b): In (a) we established $D_1 G(\phi_0, \lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$. Using (2.5) one shows that

$$D_1 G(\phi, \lambda) = D_1 G(\phi_0, \lambda) + D_1 F(\phi_0, \lambda) - D_1 F(\phi, \lambda) \text{ for all } \phi \in \mathcal{C}_0^1(\Omega)$$

and Prop. 2.5(b) guarantees that $D_1 G(\phi, \lambda)$ is a compact perturbation of an index 0 Fredholm operator. By [24, p. 165, Thm. 6.40(b)] the Fredholm index is not affected and we accordingly arrive at $D_1 G(\phi, \lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$. □

In App. A we collected basic concepts from topological dynamics. Here, the *hull* of a nonautonomous ODE (D_λ) is abbreviated as $\mathcal{H}(\lambda)$ and $\omega(\lambda), \alpha(\lambda) \subseteq \mathcal{H}(\lambda)$ denote the ω- resp. α-limit set; the topology is induced by the metric (A.1). A subset $G \subseteq \mathcal{H}(f)$ is called *admissible*, if:

- $Z_G := \{x \in \Omega \mid \exists g \in G : g(t, x) \equiv 0 \text{ on } \mathbb{R}\}$ is compact and totally disconnected,
- $\left\{\phi \in \mathcal{C}^1(\Omega) \cap \mathcal{BC} \mid \dot\phi(t) \equiv g(t, \phi(t)) \text{ on } \mathbb{R}\right\}$ consists only of constant functions for all $g \in G$.

Remark 2.8 It is not hard to see if $A\colon \mathbb{R} \to L(\mathbb{R}^d)$ is continuous and bounded, then a subset $G \subset \mathcal{H}(A)$ is admissible if and only if for any $B \in G$ all nontrivial solutions of a differential equation $\dot\phi(t) = B(t)\phi(t)$, for all $t \in \mathbb{R}$, are unbounded.

Lemma 2.9 *Let $\lambda \in \Lambda$. If $\alpha(\lambda) \cup \omega(\lambda)$ is admissible, then $G_\lambda\colon \mathcal{C}_0^1(\Omega) \to \mathcal{C}_0$ is proper on all bounded, closed subsets $B \subset \mathcal{C}_0^1(\Omega)$.*

P r o o f. We neglect the dependence on the fixed $\lambda \in \Lambda$ in our notation. Note that the claim, namely $G_\lambda|_B$ is proper on each bounded, closed $B \subseteq \mathcal{C}_0^1(\Omega)$, is equivalent to the implication

$$(G(\phi_n))_{n\in\mathbb{N}} \text{ converges in } \mathcal{C}_0 \text{ for some bounded sequence } (\phi_n)_{n\in\mathbb{N}} \text{ in } \mathcal{C}_0^1(\Omega)$$
$$\Rightarrow (\phi_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}$$

Accordingly, let $(\phi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{C}_0^1(\Omega)$ such that $(G(\phi_n))_{n\in\mathbb{N}}$ converges in $\mathcal{C}_0$ to φ. We thus need to show the existence of a convergent subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ in $\mathcal{C}_0^1(\Omega)$ using Lemma 2.3 with $\mathcal{F} := \{\phi_n\}_{n\in\mathbb{N}}$. Above all, there exist a real $R \geqslant 0$ with

$$\max\{|\phi_n(t)|, |\dot\phi_n(t)|\} \leqslant R \text{ for all } n \in \mathbb{N},\ t \in \mathbb{R}. \tag{2.6}$$

ad (i): From (2.6) one obtains $\{\phi_n(t) \in \mathbb{R}^d \mid n \in \mathbb{N}, t \in \mathbb{R}\} \subseteq B_R(0)$.
ad (ii): The mean value estimate implies

$$|\phi_n(t) - \phi_n(s)| \overset{(2.6)}{\leqslant} R\,|t - s| \text{ for all } n \in \mathbb{N},\ t, s \in \mathbb{R} \tag{2.7}$$

and therefore $\mathcal{F}$ is uniformly equicontinuous.
ad (iii): The set $Z := \{x \in \Omega \mid \exists g \in \alpha(\lambda) \cup \omega(\lambda) : g(t, x) \equiv 0 \text{ on } \mathbb{R}\}$ with $0 \in Z$ is compact and totally disconnected by the admissibility assumption. We choose a sequence in $\mathcal{F} \subset \mathcal{C}_{\{0\}}(\Omega) \subset \mathcal{C}_Z(\Omega)$, which clearly is a subsequence of $(\phi_n)_{n\in\mathbb{N}}$ and w.l.o.g. denoted as $(\phi_n)_{n\in\mathbb{N}}$ again. For a real sequence $(s_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} |s_n| = \infty$, let us suppose that $\psi_n := S^{s_n}\phi_n \in \mathcal{C}_0^1(\Omega)$ converges pointwise to some $\psi \in \mathcal{BC}$. In the following steps, we establish $\psi(\mathbb{R}) \subseteq Z$. Abbreviating $f_n := S^{s_n} f$ and $F_n\colon \mathcal{C}_0^1(\Omega) \to \mathcal{C}_0$, $F_n(\phi) := f_n(\cdot, \phi(\cdot))$, we obtain

$$\dot\psi_n(t) - f_n(t, \psi_n(t)) \equiv \dot\phi_n(t + s_n) - f(t + s_n, \phi_n(t + s_n)) \equiv G(\phi_n)(t + s_n) \text{ on } \mathbb{R}$$

and consequently

$$\dot\psi_n = F_n(\psi_n) + S^{s_n} G(\phi_n) \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

(I) Claim: $(F_n(\psi_n))_{n\in\mathbb{N}}$ *converges compactly to* $F(\psi)$.
First, $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^1(\Omega)$ is bounded and like in (2.7) also uniformly equicontinuous. Hence, on every compact subset $J \subset \mathbb{R}$ the Arzelá-Ascoli theorem (see [25, p. 85]) applies as in the proof of Lemma A.1 and there exists a sequence of functions, again denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges compactly to ψ. Second, due to Lemma A.1 there also exists a subsequence, denoted as $(f_n)_{n\in\mathbb{N}}$, which converges compactly to some $f_0 \in \alpha(\lambda) \cup \omega(\lambda)$. Choose $\rho > 0$ so large that $\overline{B}_\rho(0) \subset \mathbb{R}^d$ contains the ranges of ψ_n and ψ. Since f_n converges uniformly to f_0 on $J \times (\overline{B}_\rho(0) \cap \Omega)$ one has

$$\sup_{t\in J} |f_n(t, \psi_n(t)) - f_0(t, \psi_n(t))| \xrightarrow[n\to\infty]{} 0.$$

Because $(\psi_n)_{n\in\mathbb{N}}$ converges compactly to ψ and since Lemma A.1 ensures compact convergence of $(f_n)_{n\in\mathbb{N}}$ to f_0, we conclude

$$\lim_{n\to\infty} \sup_{t\in J} |f_0(t, \psi_n(t)) - f_0(t, \psi(t))| = 0.$$

Combining the last two limit relations in the inequality

$$|f_n(t, \psi_n(t)) - f_0(t, \psi(t))| \leqslant |f_n(t, \psi_n(t)) - f_0(t, \psi_n(t))| + |f_0(t, \psi_n(t)) - f_0(t, \psi(t))|$$

for all $t \in J$, $n \in \mathbb{N}$ establishes the present claim.
(II) Claim: $(S^{s_n} G(\phi_n))_{n\in\mathbb{N}}$ *converges compactly to* 0.
This follows readily from the inequality

$$|S^{s_n} G(\phi_n)(t)| \leqslant |G(\phi_n)(t + s_n) - \varphi(t + s_n)| + |\varphi(t + s_n)|\,.$$

(III) Passing to $n \to \infty$ in (2.8) shows that $(\dot{\psi}_n)_{n\in\mathbb{N}}$ converges compactly to $F(\psi)$ by step (I) and (II), which in turn implies that $\psi \in \mathcal{BC}(\Omega)$ is differentiable and solves (D_λ). The solution identity $\dot{\psi}(t) \equiv f_0(t, \psi(t))$ on $\mathbb{R}$ even guarantees $\psi \in \mathcal{BC}^1(\Omega)$ by (H2). Thus, the assumed admissibility of $\alpha(\lambda) \cup \omega(\lambda)$ enforces ψ to be constant $x_0 \in \Omega$ and consequently

$$f_0(t, x_0) \equiv f_0(t, \psi(t)) \equiv \dot{\psi}(t) \equiv 0 \text{ on } \mathbb{R}.$$

Hence, by definition it is $\psi(\mathbb{R}) \subseteq Z$.
In summary, we verified the conditions of Lemma 2.3 and thus the sequence $(\phi_n)_{n\in\mathbb{N}}$ has a subsequence $(\phi_n)_{n\in\mathbb{N}}$ (denoted by the same symbol) that converges to a function $\widetilde{\varphi}$ in $\mathcal{C}_0$. It remains to show convergence in the $\mathcal{C}_0^1$-topology: Continuity of $F\colon \mathcal{C}_0(\Omega) \to \mathcal{C}_0$ (see [19, Lemma 3.3]) implies

$$\lim_{n\to\infty} \|F(\phi_n) - F(\widetilde{\varphi})\|_0 = 0.$$

Since $(G(\phi_n))_{n\in\mathbb{N}}$ is assumed to converge uniformly to φ, it follows that $\dot{\phi}_n = F(\phi_n) + G(\phi_n)$ converges to $F(\widetilde{\varphi}) + \varphi$. Hence $\widetilde{\varphi}$ is differentiable and $\dot{\psi} = F(\widetilde{\varphi}) + \varphi$. Consequently, (ϕ_n) converges to $\widetilde{\varphi}$ in the $\mathcal{C}_0^1$-topology. This completes the proof. □

Proposition 2.10 (properness) *If* $\alpha(\lambda) \cup \omega(\lambda)$ *is admissible for all* $\lambda \in \Lambda$*, then* $G\colon \mathcal{C}_0^1(\Omega) \times \Lambda \to \mathcal{C}_0$ *is proper on every product* $B \times \Lambda_0$ *with* $B \subset \mathcal{C}_0^1(\Omega)$ *bounded, closed and* $\Lambda_0 \subseteq \Lambda$ *compact.*

Proof. Thanks to Prop. 2.5 the mapping $G\colon \mathcal{C}_0^1(\Omega) \times \Lambda \to \mathcal{C}_0$ is continuous and it remains to check the assumptions of Lemma 1.2:
ad (i): Given a bounded $B \subset \mathcal{C}_0^1(\Omega)$, $\lambda_0 \in \Lambda$, choose $\varepsilon > 0$. Due to (H2) there exists a $\delta > 0$ so that $|\lambda - \lambda_0| < \delta$ implies

$$\|G(\phi, \lambda) - G(\phi, \lambda_0)\| = \sup_{t\in\mathbb{R}} |f(t, \phi(t), \lambda) - f(t, \phi(t), \lambda_0)| < \varepsilon$$

for all $\lambda \in \Lambda$. Thus, $\{G(\phi, \cdot) \mid \phi \in B\}$ is equicontinuous.
ad (ii): Lemma 2.9 implies that $G_\lambda : \mathcal{C}_0^1(\Omega) \to \mathcal{C}_0$, $\lambda \in \Lambda$, is proper on the bounded, closed subsets of $\mathcal{C}_0^1(\Omega)$. □

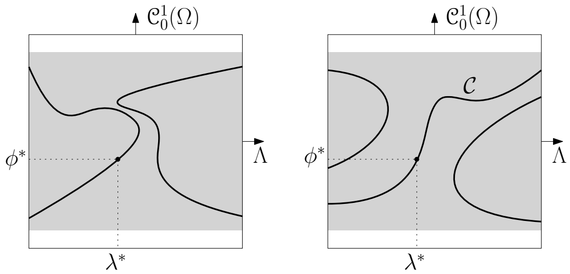

Fig. 1 Illustration of Thm. 3.1, where the grey shaded area symbolizes $O \times \Lambda$: For every parameter value λ there exists a homoclinic solution of (D_λ) (left). Yet, only the solution branch $\mathcal{C}$ in the right figure covers the entire parameter space Λ and is guaranteed by Thm. 3.1.

3 Continuation of homoclinic solutions

After these preparations, we finally arrive at the promised application of Thm. 1.1 showing that (D_λ) admits a continuum of homoclinic solutions. We study the structure of the solution set

$$\mathcal{S} = \left\{(\phi, \lambda) \in \mathcal{C}_0^1(\Omega) \times \Lambda \mid \dot{\phi}(t) \equiv f(t, \phi(t), \lambda) \text{ on } \mathbb{R}\right\}.$$

In comparison to related global implicit function theorems for homoclinic solutions from [20] or [13, 9, 15], we do not assume the existence of a hyperbolic homoclinic solution ϕ^* of (D_{λ^*}) for some parameter $\lambda^* \in \Lambda$. We indeed drop the hyperbolicity assumption, but strengthen the solution property $\dot{\phi}^*(t) \equiv f(t, \phi^*(t), \lambda^*)$ (abstractly, $G(\phi^*, \lambda^*) = 0$) to the *degree condition*

$$\deg(G_{\lambda^*}, O, 0) \neq 0. \tag{3.1}$$

Theorem 3.1 (continua of homoclinic solutions) *Suppose beyond (H0-H2) that the following assumptions hold for all* $\lambda \in \Lambda$*:*

(i) $\alpha(\lambda) \cup \omega(\lambda)$ *is admissible,*

(ii) *there is a weakly hyperbolic function* $\phi_\lambda^* \in \mathcal{C}_0^1(\Omega)$.

If there exists a $\lambda^* \in \Lambda$ *so that the degree condition* (3.1) *holds and* $G^{-1}(0) \cap (\partial O \times \Lambda) = \emptyset$, *then (see Fig. 1(right))*

(a) *for every* $\lambda \in \Lambda$ *there exists a solution* $\phi_\lambda \in \mathcal{C}_0^1(\Omega)$ *of* (D_λ),

(b) *all these homoclinic solutions* ϕ_λ *are contained in a component of* $\mathcal{S}$,

where the mapping G is defined in (2.4) *and* $O = \mathcal{C}_0^1(\Omega)^\circ$.

Remark 3.2 (1) Once the limit sets are known, checking the admissibility assumption (i) requires two aspects: First, verifying that the constant solutions are totally disconnected is basically an algebraic condition on the solutions to a nonlinear equation. Second, more substantial is to show that the bounded entire solutions of the limit systems are constant. This can be done by showing that a limit system has the trivial solution and a unique bounded solution. Conditions for the latter are wide-spread and we only mention the prototypical [6, p. 297, Thm. 4.1].

(2) For the weak hyperbolicity assumption (ii) one needs to verify EDs on both half-lines. Numerical tools for this endeavor were given e.g. in [7].

Proof. The argument is based on Thm. 1.1 with the Banach spaces $X = \mathcal{C}_0^1$, $Y = \mathcal{C}_0$ and the open set $O = \mathcal{C}_0^1(\Omega)^\circ$. The mapping $G\colon \overline{O} \times \Lambda \to \mathcal{C}_0$ from (2.4) characterizes the homoclinic solutions to (D_λ) via Prop. 2.6. The linear mapping $\psi \mapsto \dot{\psi}$ between $\mathcal{C}_0^1$ and $\mathcal{C}_0$ is bounded; thus Prop. 2.5 implies that $G\colon \mathcal{C}_0^1(\Omega) \times \Lambda \to \mathcal{C}_0$ is continuous. By Prop. 2.5(a) the partial derivative $D_1 F$ exists as a continuous function and it results that also $D_1 G$ exists with

$$D_1 G : O \times \Lambda \to L(\mathcal{C}_0^1, \mathcal{C}_0), \qquad D_1 G(\phi, \lambda)\psi = \dot{\psi} - D_1 F(\phi, \lambda)\psi$$

being continuous. We check the further conditions of Thm. 1.1:

ad (A1): By assumption the parameter space Λ is a connected (metrizable Banach) manifold.

ad (A2): Thanks to (i) we can conclude from Prop. 2.10 that G fulfills the desired properness.

ad (A3): Due to (ii) the linear equation (V_λ) has EDs on both half-lines and Lemma 2.2 implies that $D_1G(\phi_\lambda^*, \lambda)$ is Fredholm with index 0. Hence, Lemma 2.7 ensures $D_1G(\phi, \lambda) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$ for all $\phi \in \mathcal{C}_0^1(\Omega)$ and consequently, $G_\lambda|_O$ is a Fredholm operator of index zero.

Finally, since (A4) holds by assumption, Thm. 1.1 implies the assertion. □

The concluding example illustrates how to understand Thm. 3.1 as perturbation result. Following the terminology from Ex. A.3 enables us to state rather explicit assumptions:

Example 3.3 (asymptotically periodic equations) Let $g : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^d$ be continuous with continuous derivative D_2g. We consider an asymptotically periodic ODE $\dot{x} = g(t, x, \mu_0)$ whose right-hand side depends on a fixed parameter $\mu_0 \in \mathbb{R}^p$ and its perturbation (D_λ) given by

$$f(t, x, \lambda) := g(t, x, \mu_0 + \operatorname{diag}(\lambda_1, \dots, \lambda_p)\mu(t))$$

with $\mu \in \mathcal{C}_0(\mathbb{R}^p)$ and parameters $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p =: \Lambda$. In order to apply Thm. 3.1 several further assumptions on g are due:

- If g, D_2g are bounded and continuous in (x, μ) uniformly in t, then (H0-H1) are satisfied.

- $\lim_{t \to \pm\infty} g(t, 0, \mu_0) = 0$ implies the limit condition (H2).

- There exist $p_\pm$-periodic functions $g^\pm : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $A^\pm : \mathbb{R} \to L(\mathbb{R}^d)$ such that

  $$\lim_{t \to \pm\infty} \sup_{x \in B} \left| g(t, x, \mu_0) - g^\pm(t, x) \right| = 0, \qquad \lim_{t \to \pm\infty} \left| D_2 g(t, 0, \mu_0) - A^\pm(t) \right| = 0.$$

  This ensures λ-independent limit sets (cf. Ex. A.3)

  $$\alpha(\lambda) = \left\{ S^t g^- : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \middle| \, t \in [0, p_-] \right\}, \quad \omega(\lambda) = \left\{ S^t g^+ : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \middle| \, t \in [0, p_+] \right\}.$$

  In order to have them admissible, suppose $\{x \in \Omega : \, g^\pm(t, x) \equiv 0 \text{ on } \mathbb{R}\}$ are totally disconnected and that the only bounded entire solutions to $\dot{x} = g^\pm(t, x)$ are constant ones. Whence, assumption (i) is fulfilled.

- If $U^\pm : \mathbb{R}^2 \to L(\mathbb{R}^d)$ denote the evolution matrices of

  $$\dot{x} = A^\pm(t)x, \tag{3.2}$$

  then $\sigma(U^\pm(p_\pm, 0)) \cap \mathbb{S}^1 = \emptyset$ and the corresponding stable subspaces have the same dimension. So both limit systems (3.2) have an ED on $\mathbb{R}$ with the same Morse index. Choosing $\phi_\lambda^* = 0$ for every parameter $\lambda \in \mathbb{R}^p$, this implies the weak hyperbolicity assumption (ii).

Provided the degree condition (3.1) can be satisfied, for every $\lambda \in \mathbb{R}^p$ the equation (D_λ) has a homoclinic solution ϕ_λ which is contained in a continuum.

By imposing conditions both on the ODE (D_λ) and the abstract map G, Thm. 3.1 is plagued by somewhat entangled assumptions. However, one possibility to verify (3.1) is given next:

4 Nonzero degree via a Landesman-Lazer condition

When it comes to applications of Thm. 3.1, sufficient criteria for the degree condition (3.1) are crucial, yet nontrivial. Our subsequent argument is based on the homotopy invariance of the degree and a Landsman-Lazer condition. Throughout, we suppose the assumptions of Thm. 3.1 are fulfilled and for simplicity, let us set $\Omega := \mathbb{R}^d$. Nonetheless, additional assumptions are due:

In this regard suppose there exists a parameter $\lambda^* \in \Lambda$ so that (D_{λ^*}) has a homoclinic solution $\phi^* : \mathbb{R} \to \mathbb{R}^d$; both ϕ^* and λ^* are kept fixed throughout. We consider a family of ODEs

$$\dot{x} = A(t)x + sr(t, x) \tag{D^s}$$

depending on a parameter $s \in [0, 1]$ and

$$\begin{aligned} A(t) &:= D_2 f(t, \phi^*(t), \lambda^*), \\ r(t, x) &:= f(t, x + \phi^*(t), \lambda^*) - f(t, \phi^*(t), \lambda^*) - D_2 f(t, \phi^*(t), \lambda^*)x, \end{aligned} \tag{4.1}$$

for which we observe:

- Every equation (D^s), $s \in [0, 1]$, has the trivial solution, while particularly (D^0) coincides with the variational equation (V_{λ^*}) and (D^1) is nothing but the equation of perturbed motion for the solution ϕ^* of the ODE (D_{λ^*}).
- Due to (H0-H1) the functions $r \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $D_2 r : \mathbb{R} \times \mathbb{R}^d \to L(\mathbb{R}^d)$ are bounded and uniformly continuous on compact subsets of $\mathbb{R}^d$ (in the sense of App. A). Moreover, it holds $r(t, 0) \equiv 0$ and $D_2 r(t, 0) \equiv 0$ on $\mathbb{R}$. Finally, the function $\mathbb{R} \ni t \mapsto A(t) \in L(\mathbb{R}^d)$ is bounded and uniformly continuous on $\mathbb{R}$.

We begin with a summary of the Fredholm theory from [18]. Since ϕ^* is weakly hyperbolic by assumption (ii) of Thm. 3.1 we derive from Lemma 2.2 that the operator L_A from (2.2) is Fredholm of index

$$\operatorname{ind} L_A = \dim R(\Pi^+(0)) - \dim R(\Pi^-(0)) = 0$$

and

$$\begin{aligned} N(L_A) &= \{U(\cdot, 0)\xi \in \mathcal{C}_0^1 \mid \xi \in R(\Pi^+(0)) \cap N(\Pi^-(0))\}, \\ R(L_A) &= \left\{ \psi \in \mathcal{C}_0 \,\middle|\, \int_{\mathbb{R}} \langle \phi(s), \psi(s) \rangle \,\mathrm{d}s = 0 \text{ for all } \phi \in \mathcal{C}_0^1 \text{ solving } (L^*) \right\}, \end{aligned} \tag{4.2}$$

where Π^+, Π^- are the dichotomy projectors for (L). Suppose that

$$\begin{aligned} &R(\Pi^+(0)) \cap N(\Pi^-(0)) = \operatorname{span}\{\xi_1, \dots, \xi_m\}, \\ &m = \dim N(L_A) = \dim\big(R(\Pi^+(0)) \cap N(\Pi^-(0))\big), \\ &\big(R(\Pi^+(0)) + N(\Pi^-(0))\big)^\perp = \operatorname{span}\{\eta_1^*, \dots, \eta_m^*\}, \\ &m = \operatorname{codim} R\big(L_A\big) = \dim(R(\Pi^+(0)) + N(\Pi^-(0)))^\perp \end{aligned}$$

with linearly independent $\xi_i \in \mathbb{R}^d$ (resp. $\eta_j^* \in \mathbb{R}^d$). Without loss of generality one can assume that

$$\langle \xi_i, \xi_j \rangle = \delta_{ij} \text{ and } \langle \eta_i^*, \eta_j^* \rangle = \delta_{ij} \text{ for all } 1 \leqslant i, j \leqslant m, \tag{4.3}$$

since otherwise we can use the Gram-Schmidt procedure to create the corresponding set of orthonormal vectors.

Furthermore (cf. [18, Lemma 3.7]), one has

$$\begin{aligned} N(L_A) &= \operatorname{span}\{U(\cdot, 0)\xi_1, \dots, U(\cdot, 0)\xi_m\}, \\ N(L_A^*) &= \operatorname{span}\{U(0, \cdot)^T \eta_1^*, \dots, U(0, \cdot)^T \eta_m^*\}. \end{aligned} \tag{4.4}$$

Remark 4.1 Any $\phi \in N(L_A)$ admits a (unique) representation

$$\phi = \sum_{i=1}^m \alpha_i U(\cdot, 0)\xi_i \tag{4.5}$$

with coefficients $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and one defines an isomorphism $J \colon N(L_A) \to N(L_A^*)$ mapping the basis elements $U(\cdot, 0)\xi_i$ of $N(L_A)$ to the basis elements $U(0, \cdot)^T \eta_i^*$ of $N(L_A^*)$. In particular,

$$J\left(\sum_{i=1}^m \alpha_i U(\cdot, 0)\xi_i\right) = U(0, \cdot)^T \sum_{i=1}^m \alpha_i \eta_i^*$$

and it is convenient to denote this mapping by $\phi \mapsto \phi^\star$.

The above facts enable us to construct the projector (see also Appendix C)

$$Q \in L(\mathcal{C}_0), \qquad (Q\phi)(t) := \sum_{i=1}^{m} \frac{\omega_i(t) \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s) \rangle \, \mathrm{d}s}{|U(t,0)\eta_i|} \cdot U(t,0)\eta_i^*$$

onto the complement of $R(L_A)$, where $\omega_i \in \mathcal{C}_0((0,\infty))$ satisfies

$$\int_{\mathbb{R}} \omega_i(s) |U(s,0)\eta_i^*|^{-1} \mathrm{d}s = 1.$$

Note that if $U(0,\cdot)^T \eta_i^* \in N(L_A^*)$, then $U(\cdot,0)\eta_i$ is not necessarily bounded, where η_i satisfies the conditions formulated in (4.3). This follows from the identity

$$\langle U(0,s)^T \eta_i^*, U(s,0)\eta_i^* \rangle \equiv 1 \text{ on } \mathbb{R}$$

and $U(0,\cdot)^T \eta_i^* \in \mathcal{C}_0$. Hence, the definition of Q involves a function $\omega_i(\cdot)|U(\cdot,0)\eta_i|^{-1} U(\cdot,0)\eta_i^* \in \mathcal{C}_0$. The space of integrable functions $\phi\colon \mathbb{R} \to \Omega$ is denoted by $\mathcal{L}^1(\Omega)$ and

$$\mathcal{L}^1 := \mathcal{L}^1(\mathbb{R}^d), \qquad \|\phi\|_{\mathcal{L}^1} := \int_{\mathbb{R}} |\phi(s)| \, \mathrm{d}s$$

is a Banach space. It is not hard to see that (2.1) and (4.5) imply the following fact: If $\phi \in N(L_A)$, then $\|\phi\|_{\mathcal{L}^1} < \infty$ (the same holds for $\phi^* \in N(L_A^*)$).

In our considerations we will need a function $r_p\colon \mathbb{R} \times \mathbb{R}^d \times [1,\infty) \to \mathbb{R}^d$ defined by

$$r_p(t,x,\mu) := \frac{r(t,\mu x)}{\mu}. \tag{4.6}$$

The functions in (D^s) are assumed to fulfill:

(lr1) The limit set $\alpha(A) \cup \omega(A)$ is admissible.

(lr2) (**Landesman-Lazer condition**) There exist $C_r > 0$ and $\theta \in \mathcal{C}_0(\mathbb{R}^2, [0,\infty))$ such that

$$|r(t,x) - r(s,x)| \leqslant C_r |t-s| \text{ and } |r(t,x) - r(t,y)| \leqslant \theta(t, |x-y|)|x-y|, \tag{4.7}$$

$$\int_{\mathbb{R}} \liminf_{\substack{x \to \varphi(t) \\ \mu \to \infty}} \langle r_p(t,x,\mu), \varphi^\star(t) \rangle \, \mathrm{d}t > 0, \tag{4.8}$$

for all $x, y \in \mathbb{R}^d$, $t, s \in \mathbb{R}$ and for all $\varphi \in N(L_A) \subset \mathcal{C}_0^1$ with $\|\varphi\|_1 = 1$.

It is clear that (4.8) rules out linear ODEs where $r = 0$. Note that (lr2) has a geometric interpretation: Roughly speaking, $r\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ must intersect the range $R(L_A)$ transversally. In other words, the inequality from (4.8) means that a function $t \mapsto r(t,\phi(t))$ is not contained in the kernel $N(L_A)$. This follows from the well known identity $\langle \phi(t), \psi(t) \rangle \equiv 0$ on $\mathbb{R}$ for all $\phi \in N(L_A)$ and $\psi \in N(L_A^*)$.

Some comments about the functions $r \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and $\theta \in \mathcal{C}_0(\mathbb{R}^2, [0,\infty))$ are due:

Remark 4.2 Notice that a function $r\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying (4.7) induces that

$$\begin{aligned} |r_p(t,x,\mu) - r_p(s,y,\mu)| &\leqslant |r_p(t,x,\mu) - r_p(s,x,\mu)| + |r_p(s,x,\mu) - r_p(s,y,\mu)| \\ &\leqslant C_r |t-s| + \theta(s, \mu|x-y|)|x-y|, \end{aligned} \tag{4.9}$$

for all $\mu \in [1,\infty)$, $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$. Furthermore,

$$|r_p(t,x,\mu)| = |r_p(t,x,\mu) - r_p(t,0,\mu)| \leqslant \theta(t, \mu|x|)|x| \leqslant L_\theta |x|, \tag{4.10}$$

for all $x \in \mathbb{R}^d$ and $\mu \geqslant 1$, where $L_\theta := \sup_{t,u \in \mathbb{R}} |\theta(t,u)|$.

Since $\theta \in \mathcal{C}_0(\mathbb{R}^2, [0,\infty))$, it follows that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(t,u)| > \delta \Rightarrow |\theta(t,u)| < \varepsilon.$$

In particular, if $|u| > \delta$, then $|\theta(t,u)| < \varepsilon$ for all $t \in \mathbb{R}$.

Now we prove a consequence of the Landesman-Lazer assumption:

Lemma 4.3 *If (lr1-lr2) hold, there exists a real $M_r > 0$ such that*

$$\int_{\mathbb{R}} \langle r(t,\varphi(t)), \varphi^\star(t)\rangle \,\mathrm{d}t > 0 \textit{ for all } \varphi \in N(L_A) \subset \mathcal{C}_0^1 \textit{ with } \|\varphi\|_1 \geqslant M_r.$$

Proof. Assume on the contrary that no such $M_r > 0$ exists, i.e., we find $(\varphi_n)_{n\in\mathbb{N}}$ in $N(L_A)$ such that

$$\|\varphi_n\|_1 > 1 \text{ with } \|\varphi_n\|_1 \to \infty \text{ and } \int_{\mathbb{R}} \langle r(t,\varphi_n(t)), \varphi_n^\star(t)\rangle \,\mathrm{d}t \leqslant 0.$$

Because $\psi_n := \|\varphi_n\|_1^{-1}\varphi_n$ (resp. $\psi_n^\star := \|\varphi_n^\star\|_1^{-1}\varphi_n^\star$) define bounded sequences in the finite-dimensional subspaces $N(L_A)$ (resp. $N(L_A^*)$), we can assume w.l.o.g. that $(\psi_n)_{n\in\mathbb{N}}$ (resp. $(\psi_n^\star)_{n\in\mathbb{N}}$) converges to some element $\psi \in N(L_A)$ (resp. to $\psi^\star \in N(L_A^*)$). Furthermore, (4.4) implies that there exist $\alpha_j^n, \alpha_j \in \mathbb{R}$, where $j = 1, \dots, m$ and $n \geqslant 1$, such that

$$\psi_n^\star(t) = \sum_{j=1}^m \alpha_j^n \cdot U(0,t)^T \eta_j^* \quad \text{and} \quad \psi^\star(t) = \sum_{j=1}^m \alpha_j \cdot U(0,t)^T \eta_j^*. \tag{4.11}$$

Let $\gamma\colon \mathbb{R} \to \mathbb{R}$ be defined as follows

$$\gamma(t) = \sum_{j=1}^m \beta_j \cdot \left|U(0,t)^T \eta_j^*\right|, \tag{4.12}$$

where $\beta_j := \sup\{|\alpha_j^n \mid n \in \mathbb{N}\}$, for $j = 1, \dots, m$. Notice that $\beta_j < \infty$ since $\alpha_j^n \to \alpha_0$ as $n \to \infty$, for $j = 1, \dots, m$. What is more, Lemma C.1 implies that $\gamma \in \mathcal{L}^1$ and

$$|\psi_n^\star(t)| \leqslant \gamma(t) \text{ for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

What is more, (lr2) implies that there exists $L_\theta > 0$ such that

$$|r_p(t,\psi_n(t),\mu)| \leqslant L_\theta |\psi_n(t)| \leqslant L_\theta,$$

for all $t \in \mathbb{R}$, $\mu \in [1,\infty)$ and $n \in \mathbb{N}$.

We have

$$\int_{\mathbb{R}} \langle r(t,\varphi_n(t)), \varphi_n^\star(s)\rangle \,\mathrm{d}t \leqslant 0 \quad \Longleftrightarrow \quad \int_{\mathbb{R}} \langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi_n^\star(t)\rangle \,\mathrm{d}t \leqslant 0$$

and

$$\begin{aligned}
&\liminf_{n\to\infty} \langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi_n^\star(t)\rangle = \liminf_{n\to\infty} \langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi^\star(t)\rangle \\
&\geqslant \liminf_{\substack{x\to\psi(t)\\ \mu\to\infty}} \langle r_p(t,x,\mu), \psi^\star(t)\rangle\,.
\end{aligned} \tag{4.13}$$

Furthermore, from (lr1) and the Cauchy-Schwarz inequality it follows that

$$|\langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi_n^\star(t)\rangle| \leqslant |r_p(t,\psi_n(t),\|\varphi_n\|_1)| \cdot |\psi_n^\star(t)| \leqslant L_\theta |\psi_n^\star(t)| \leqslant L_\theta \gamma(t).$$

Consequently, this together with (4.13) enables us to apply Fatou's lemma, which gives the contradiction

$$\begin{aligned}
0 &\geqslant \liminf_{n\to\infty} \int_{\mathbb{R}} \langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi_n^\star(t)\rangle \,\mathrm{d}t \geqslant \int_{\mathbb{R}} \liminf_{n\to\infty} \langle r_p(t,\psi_n(t),\|\varphi_n\|_1), \psi_n^\star(t)\rangle \,\mathrm{d}t \\
&\geqslant \int_{\mathbb{R}} \liminf_{\substack{x\to\psi(t)\\ \mu\to\infty}} \langle r_p(t,x,\mu), \psi^\star(t)\rangle \,\mathrm{d}t \overset{(4.8)}{>} 0
\end{aligned}$$

and completes the proof. □

We consider the substitution operator $R_0 \colon \mathcal{C}_0^1 \to \mathcal{C}_0$ induced by $r \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. According to (H0-H1) it is continuously differentiable with derivative

$$(DR_0(\phi)\psi)(t) = (D_2 f(t, \phi^*(t) + \phi(t)) - D_2 f(t, \phi^*(t)))\, \psi(t) \text{ for all } t \in \mathbb{R},\ \phi, \psi \in \mathcal{C}_0.$$

Furthermore, also the operator

$$G^s \colon \mathcal{C}_0^1 \to \mathcal{C}_0, \qquad G^s(\phi)(t) := \dot{\phi}(t) - A(t)\phi(t) - sr(t, \phi(t)) \text{ for all } t \in \mathbb{R},\ s \in [0,1]$$

is well-defined and allows the representation

$$G^s(\phi) = L_A \phi - s R_0(\phi)$$

with $(L_A\phi)(t) := \dot{\phi}(t) - A(t)\phi(t)$ and $(R_0(\phi))(t) := r(t, \phi(t))$.

Lemma 4.4 *For each $s \in [0,1]$ the operator $G^s \colon \mathcal{C}_0^1 \to \mathcal{C}_0$ is Fredholm of index zero. Moreover, if (lr1) holds, then G^s proper on closed, bounded sets.*

Proof. First, since ϕ^* is weakly hyperbolic, Lemma 2.2 shows $L_A \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0)$ and we have to establish

$$L_A - sDR_0(\phi) \in \Phi_0(\mathcal{C}_0^1, \mathcal{C}_0) \text{ for all } \phi \in \mathcal{C}_0^1.$$

However, as in the proof of Prop. 2.5(b) one shows that the perturbation $DR_0(\phi) = DR_0(\phi) - DR_0(0)$ is compact and therefore has no effect on the Fredholm properties of L_A (cf. [24, p. 165, Thm. 6.40(b)]). Second, Lemma 2.9 and (lr1) yield the properness of G^s on all bounded, closed $B \subset \mathcal{C}_0^1$. □

Lemma 4.5 *If (lr1-lr2) hold, there exists a $\rho \geqslant M_r$ such that*

(a) *$G^s(\phi) \neq 0$ for all $s \in (0,1]$ and $\phi \in \mathcal{C}_0^1$ with $\|\phi\|_1 \geqslant \rho$,*

(b) *$QR_0(\phi) \neq 0$ for all $\phi \in N(L_A) - \{0\}$ with $\|\phi\|_1 \geqslant \rho$.*

Proof. (a): Assume no such $\rho > 0$ exists, i.e., we find $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^1$ and $(s_n)_{n\in\mathbb{N}}$ in $(0,1]$ such that

$$\|\phi_n\|_1 \geqslant 1 \text{ with } \|\phi_n\|_1 \to \infty \text{ and } L_A\phi_n - s_n R_0(\phi_n) = 0. \tag{4.14}$$

Let $\psi_n := \|\phi_n\|_1^{-1} \phi_n$, $\mathcal{F} := \{\psi_n \mid n \in \mathbb{N}\}$ and $Z := Z_{\alpha(A)\cup\omega(A)}$. Notice that $0 \in Z$. This follows directly from the assumption (H2). Furthermore, since $\alpha(A) \cup \omega(A)$ is admissible, ti follows that Z is compact.

We prove that $\mathcal{F}$ is relatively compact in $\mathcal{C}_Z$ (and hence in $\mathcal{C}_0$, see Lemma 2.3). First of all, it is clear that $(\psi_n)_{n\in\mathbb{N}}$ is bounded. Since $(\psi_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}_0^1$, the sequence $(\dot{\psi}_n)_{n\in\mathbb{N}}$ is bounded and therefore all functions in $\mathcal{F}$ are Lipschitz with the same constant, which implies that $\mathcal{F}$ is uniformly equicontinuous. Now we are to show that $\mathcal{F}$ satisfies (iii). For this purpose, take any sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\lim_{n\to\infty} |t_n| = \infty$ and a subsequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ (denoted by the same symbol as the elements of $\mathcal{F}$) such that

$$\chi_n(t) := \psi_n(t + t_n) \xrightarrow[n\to\infty]{} \chi(t) \text{ for all } t \in \mathbb{R} \text{ and for some } \chi \in \mathcal{BC}.$$

Without loss of generality we can assume that either $t_n \to -\infty$ or $t_n \to \infty$ (we will consider both cases simultaneously). We have to show $\chi(\mathbb{R}) \subset Z$ using the Arzelá-Ascoli theorem [25, p. 85]: By passing to a subsequence (if necessary) we may assume

- $(\chi_n)_{n\in\mathbb{N}}$ converges to χ uniformly on compact intervals,
- $(B_n)_{n\in\mathbb{N}}$ converges to some $B \in \alpha(A) \cup \omega(A)$ (w.r.t. (A.5)), where $B_n := S^{t_n}A$ (i.e., $B_n(t) = A(t + t_n)$).

Whence, $B_n\chi_n$ converges to $B\chi$ (with respect to the compact-open topology in $\mathcal{C}_0$) and (4.10) leads to

$$\begin{aligned} |r_p(t + t_n, \chi_n(t), \|\phi_n\|_1) &\leqslant \theta(t + t_n, \|\phi_n\|_1 \cdot |\chi_n(t)|)|\chi_n(t)| \\ &\leqslant \theta(t + t_n, \|\phi_n\|_1 \cdot |\chi_n(t)|) \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

uniformly on any compact interval $J \subset \mathbb{R}$ (this follows from Remark 4.2 and the fact that for any compact interval J and $\delta > 0$ there exists n_0 such that $|t_n + t| > \delta$ for all $n \geqslant n_0$ and for all $t \in J$). Moreover, (4.14) implies that

$$\dot{\chi}_n(t) = B_n(t)\chi_n(t) + s_n r_p(t + t_n, \chi_n(t), \|\phi_n\|_1),$$

where $s_n \in (0, 1]$. Consequently, $(\dot{\chi}_n)_{n\in\mathbb{N}}$ converges to $B\chi$ uniformly on compact intervals of $\mathbb{R}$. This in turn proves that the limit χ is differentiable and that $\{\dot{\chi}_n\}_{n\in\mathbb{N}}$ converges, uniformly on compact intervals, to $\dot{\chi}$; hence, $\dot{\chi}(t) \equiv B(t)\chi(t)$ on $\mathbb{R}$. Since χ is bounded and $\alpha(A) \cup \omega(A)$ is admissible by (lr1), it follows that χ is constant with $\chi(\mathbb{R}) = \{c\} \subseteq Z$, which establishes that $\mathcal{F}$ is relatively compact in $\mathcal{C}_Z$ (and hence in $\mathcal{C}_0$).

Since $\mathcal{F}$ is relatively compact in $\mathcal{C}_0$, we can assume w.l.o.g. that $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ in $\mathcal{C}_0$. Now we will show that $(\dot{\psi}_n)_{n\in\mathbb{N}}$ also converges to $\dot{\psi}$ in $\mathcal{C}_0$. For this aim, observe that (4.14) implies that

$$\dot{\psi}_n(t) = A(t)\psi_n(t) + s_n r_p(t, \psi_n(t), \|\phi_n\|_1) \text{ for all } t \in \mathbb{R}. \tag{4.15}$$

Now, taking into account (4.15) and reasoning as above, we can deduce that ψ_0 is differentiable and that $\{\dot{\psi}_n\}_{n\in\mathbb{N}}$ converges, uniformly on compact intervals, to $\dot{\psi}$ satisfying the equation

$$\dot{\psi}(t) = A(t)\psi(t) \text{ for all } t \in \mathbb{R}.$$

Furthermore, we have

$$\begin{aligned}
|\dot{\psi}_n(t) - \dot{\psi}(t)| &\leqslant |A(t)\psi_n(t) - A(t)\psi(t)| + |r_p(t, \psi_n(t), \|\phi_n\|_1)| \\
&\leqslant \left(\sup_{u\in\mathbb{R}} |A(u)|\right) |\psi_n(t) - \psi(t)| + \theta(t, \|\phi_n\|_1 |\psi_n(t)|)|\psi_n(t)| \\
&\leqslant M_A|\psi_n(t) - \psi(t)| + \mathbb{1}_{[0,\varepsilon)}(|\psi_n(t)|)L_\theta\varepsilon + \mathbb{1}_{[\varepsilon,\infty)}(|\psi_n(t)|)\theta(t, \|\phi_n\|_1|\psi_n(t)|) \\
&\leqslant M_A|\psi_n(t) - \psi(t)| + L_\theta\varepsilon + \mathbb{1}_{[\varepsilon,\infty)}(|\psi_n(t)|)\theta(t, \|\phi_n\|_1|\psi_n(t)|),
\end{aligned} \tag{4.16}$$

for all $t \in \mathbb{R}$, where $\mathbb{1}_S : \mathbb{R} \to \{0, 1\}$ denotes the characteristic function of a set $S \subset \mathbb{R}$, and $\varepsilon > 0$ (recall that $|\psi_n(t)| \leqslant 1$). Now we are ready to show that $(\dot{\psi}_n)_{n\in\mathbb{N}}$ converges to $\dot{\psi}$ in $\mathcal{C}_0$. Let $\varepsilon > 0$. Then there exists n_0 such that

$$\begin{aligned}
&|\psi_n(n)(t) - \psi(t)| < \varepsilon \text{ for all } t \in \mathbb{R} \text{ and } n \geqslant n_0, \\
&\theta(t, \|\phi_n\|_1|z|) < \varepsilon \text{ for all } t \in \mathbb{R}, z \in \mathbb{R}^d \text{ with } |z| \geqslant \varepsilon \text{ and } n \geqslant n_0 \text{ (see Remark 4.2)}.
\end{aligned} \tag{4.17}$$

Hence, in view of (4.16) and (4.16), we have

$$|\dot{\psi}_n(t) - \dot{\psi}(t)| < (M_A + L_\theta + 1)\varepsilon \text{ for all } t \in \mathbb{R} \text{ and } n \geqslant n_0,$$

which proves the desired conclusion. The above considerations can be concluded with the following remark

$$\|\psi_n - \psi\|_1 \xrightarrow[n\to\infty]{} 0,$$

which in turn implies that ψ is a nontrivial solution of (L) in $\mathcal{C}^1_0$. Indeed, for if it were not, then we would have the following contradiction:

$$1 = \left\| \|\phi_n\|_1^{-1}\phi_n \right\|_1 = \|\psi_n\|_1 = \|\psi_n - \psi\|_1 \xrightarrow[n\to\infty]{} 0.$$

Now observe that $N(L_A) = \{0\}$ yields the assertion, because we get a contradiction to $0 \neq \psi \in N(L_A)$. Thus, from now on, we can assume that $N(L_A) \neq \{0\}$.

Hence from (4.15) we obtain

$$\left\langle \dot{\psi}_n(t), \psi^\star(t) \right\rangle \equiv \langle A(t)\psi_n(t), \psi^\star(t)\rangle + s_n \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi^\star(t)\rangle \text{ on } \mathbb{R}$$

for all $n \in \mathbb{N}$. Integrating the above equality shows

$$\int_{\mathbb{R}} \left\langle \dot{\psi}_n(t), \psi^\star(t) \right\rangle \mathrm{d}t = \int_{\mathbb{R}} \langle A(t)\psi_n(t), \psi^\star(t)\rangle \,\mathrm{d}t + s_n \int_{\mathbb{R}} \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi^\star(t)\rangle \,\mathrm{d}t.$$

On the other hand, since $\psi^\star \in \mathcal{C}_0^1$, it follows (via integrating by parts) that

$$\begin{aligned}
&\int_{\mathbb{R}}\left\langle \dot{\psi}_n(t), \psi^\star(t) \right\rangle \mathrm{d}t = -\int_{\mathbb{R}}\left\langle \psi_n(t), \dot{\psi}^\star(t) \right\rangle \mathrm{d}t \\
&= \int_{\mathbb{R}}\left\langle \psi_n(t), A(t)^T\psi^\star(t)\right\rangle \mathrm{d}t = \int_{\mathbb{R}}\langle A(t)\psi_n(t), \psi^\star(t)\rangle \,\mathrm{d}t
\end{aligned}$$

and hence

$$0 = \int_{\mathbb{R}} \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi^\star(t)\rangle \,\mathrm{d}t.$$

Observe that

$$\begin{aligned}
&\liminf_{n\to\infty} \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi_n^\star(t)\rangle = \liminf_{n\to\infty} \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi^\star(t)\rangle \\
&\geqslant \liminf_{\substack{x\to\psi(t)\\ \mu\to\infty}} \langle r_p(t, x, \mu), \psi^\star(t)\rangle\,,
\end{aligned} \tag{4.18}$$

for all $t \in \mathbb{R}$. Finally, repeating the estimations as in (4.13) and using Fatou's lemma, we arrive the contradiction

$$\begin{aligned}
0 &= \liminf_{n\to\infty} \int_{\mathbb{R}} \langle r_p(t, \psi_n(t), \|\phi_n\|_1), \psi_n^\star(t)\rangle \,\mathrm{d}t \geqslant \int_{\mathbb{R}} \liminf_{n\to\infty} \langle r_p(s, \psi_n(s), \|\phi_n\|_1), \psi_n^\star(t)\rangle \,\mathrm{d}t \\
&\geqslant \int_{\mathbb{R}} \liminf_{\substack{x\to\psi(t)\\ \mu\to\infty}} \langle r_p(t, x, \mu), \psi^\star(t)\rangle \,\mathrm{d}t \overset{(4.8)}{>} 0.
\end{aligned}$$

This completes the proof.

(b): Assume once again that there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $N(L_A)$ satisfying

$$\|\phi_n\|_1 \to \infty \text{ and } QR_0(\phi_n) = 0 \text{ for all } n \in \mathbb{N}.$$

Due to the equivalences

$$\begin{aligned}
QR_0(\phi_n) = 0 &\iff \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, r(s, \phi_n(s))\right\rangle \mathrm{d}s = 0 \text{ for all } 1 \leqslant i \leqslant r \\
&\iff \int_{\mathbb{R}} \langle r(s, \phi_n(s)), \varphi^\star(s)\rangle \,\mathrm{d}s = 0 \text{ for all } \varphi^\star \in N(L_A^*)
\end{aligned}$$

we are in the same position as in the proof of (a), which verifies assertion (b).

However, as for the proof of part (b), here we do not need the assumption (lr1). It follows from the fact that the condition $\dim N(L_A) < \infty$ implies that the sequence $\psi_n := \|\phi_n\|_1^{-1}\phi_n \in N(L_A) \subset \mathcal{C}_0^1$ is relatively convergent in $\mathcal{C}_0^1$. □

Consequently, Lemma 4.4 and 4.5 allow us to apply the Benevieri-Furi degree of App. B to G_{λ^*}:

Proposition 4.6 (degree condition with the nontrivial kernel) *Suppose that (lr1-lr2) hold. If $N(L_A) \neq \{0\}$, then the condition* (3.1) *holds on $O = B_\rho(0)$ with $\rho \geqslant M_r$ given in Lemma 4.5.*

Note that $N(L_A) = \{0\}$ implies

$$\deg(L_A - R_0, B_\rho(0), 0) = \deg(L_A, B_\rho(0), 0) \neq 0.$$

This equality follows from Lemma 4.5(a) and the homotopy invariance of the Benevieri-Furi degree from App. B.

P r o o f. Note that $G_{\lambda^*} = L_A - R_0$. First, observe that Prop. B.1 implies

$$\begin{aligned}\deg(L_A - R_0, B_\rho(0), 0) &= \deg_B(-QR_0|_{B_\rho(0)\cap N(L_A)}, B_\rho(0)\cap N(L_A), 0)\\ &= (-1)^{\dim N(L_A)} \deg_B(QR_0|_{B_\rho(0)\cap N(L_A)}, B_\rho(0)\cap N(L_A), 0)\end{aligned}$$

and it remains to show

$$\deg_B(QR_0|_{B_\rho(0)\cap N(L_A)}, B_\rho(0)\cap N(L_A), 0) \neq 0.$$

For this, we will use the homotopy invariance (bf2) from App. B and remark

$$(QR_0\phi)(t) = \sum_{i=1}^m \frac{\omega(t)\int_{\mathbb{R}}\langle U(0,s)^T\eta_i^*, r(s,\phi(s))\rangle \mathrm{d}s}{|U(t,0)\eta_i|} U(t,0)\eta_i \tag{4.19}$$

We represent ϕ as in (4.5) with coefficients $\alpha_1,\dots,\alpha_m \in \mathbb{R}$,

$$\omega_i(t) := \frac{\omega(t)U(t,0)\eta_i}{|U(t,0)\eta_i|} \text{ for all } t\in\mathbb{R} \tag{4.20}$$

and taking (4.19) and (4.5) into account, we define a homotopy $H\colon (\overline{B}_\rho(0)\cap N(L_A))\times[0,1]\to \mathcal{C}_0$ by

$$H(\phi,\tau)(t) := \sum_{i=1}^m \omega_i(t)\int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, h_i(s,\phi(s),\tau)\right\rangle \mathrm{d}s,$$

where

$$h_i\colon \mathbb{R}\times(\overline{B}_\rho(0)\cap N(L_A))\times[0,1]\to\mathcal{C}_0, \qquad h_i(t,\phi,\tau) := (1-\tau)r(t,\phi(t)) + \tau\alpha_i\omega_i(t).$$

This homotopu satisfies

$$H(\phi,0) = QR_0(\phi) \text{ and } H(\phi,1) = \gamma_r\sum_{i=1}^m \alpha_i\omega_i,$$

Clearly, $H(\phi,1)$ is an isomorphism and hence

$$\deg_B(H(\phi,1), B_\rho(0)\cap N(L_A), 0)\neq 0.$$

It remains to show that

$$H^{-1}(0)\cap(\partial\overline{B}_\rho(0)\cap N(L_A)) = \emptyset.$$

Assume the contrary, that is, there are

$$\phi_0 = \sum_{i=1}^m \alpha_i^0 U(\cdot,0)\xi_i \in (\partial\overline{B}_\rho(0)\cap N(L_A)),\ \ \tau_0\in[0,1] \text{ with } H(\phi_0,\tau_0) = 0.$$

Then

$$\langle H(\phi_0,\tau_0)(t), T\phi_0(t)\rangle \equiv 0 \text{ on } \mathbb{R} \text{ with the isomorphism (see (4.20)),}$$

$$T\colon N(L_A)\to \operatorname{span}\{\omega_1(\cdot),\dots,\omega_m(\cdot)\}, \qquad U(\cdot,0)\xi_i \mapsto \|\omega_i\|_0^{-2}\omega_i(\cdot).$$

Thus, if $\phi_0\in\partial\overline{B}_\rho(0)\cap N(L_A)$ has a decomposition as in (4.5), then

$$\langle H(\phi_0,\tau_0)(t), T\phi_0(t)\rangle = 0 \quad\Longleftrightarrow\quad \sum_{i=1}^m \alpha_i^0\int_{\mathbb{R}}\left\langle U(0,s)^T\eta_i^*, h_i(s,\phi_0(s),\tau_0)\right\rangle \mathrm{d}s = 0$$

and

$$\sum_{i=1}^m \alpha_i^0 \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, (1-\tau_0)r(s,\phi_0(s))\right\rangle \mathrm{d}s = -\sum_{i=1}^m \alpha_i^0 \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, \tau_0\alpha_i^0\omega_i(s)\right\rangle \mathrm{d}s.$$

On the other hand, by Lemma 4.3, for each $\tau_0 \in [0,1]$ one has

$$\begin{aligned}
&\sum_{i=1}^m \alpha_i^0 \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, (1-\tau_0)r(s,\phi_0(s))\right\rangle \mathrm{d}s \\
&= (1-\tau_0)\int_{\mathbb{R}} \sum_{i=1}^m \alpha_i^0 \left\langle U(0,s)^T\eta_i^*, r(s,\phi_0(s))\right\rangle \mathrm{d}s \\
&= (1-\tau_0)\int_{\mathbb{R}} \langle r(s,\phi_0(s)), \phi_0^\star(s)\rangle \,\mathrm{d}s = \begin{cases} >0, & \text{if } \tau_0 \in [0,1), \\ 0, & \text{if } \tau_0 = 1 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&-\sum_{i=1}^m \alpha_i^0 \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, \tau_0\alpha_i^0\omega_i(s)\right\rangle \mathrm{d}s \\
&= -\tau_0 \sum_{i=1}^m (\alpha_i^0)^2 \int_{\mathbb{R}} \left\langle U(0,s)^T\eta_i^*, \omega_i(s)\right\rangle \mathrm{d}s \\
&= -\tau_0 \sum_{i=1}^m (\alpha_i^0)^2 = \begin{cases} <0, & \text{if } \tau_0 \in (0,1], \\ 0, & \text{if } \tau_0 = 0; \end{cases}
\end{aligned}$$

a contradiction. □

We close this section with two examples illustrating the applicability of our results and particularly the Landesman-Lazer conditions. It is supposed that the ODE (D_λ) is globally defined, such that $O = \mathcal{C}_0^1(\Omega)$ has empty boundary, Λ is a connected (metrizable Banach) manifold and $\lambda^* \in \Lambda$ is fixed.

Example 4.7 Let $\Omega = \mathbb{R}$ and $m \in \mathbb{N} \setminus \{1,2\}$. Consider a scalar ODE (D_λ) whose right-hand side $f\colon \mathbb{R} \times \mathbb{R} \times \Lambda \to \mathbb{R}$ fulfills (H0-H2) and moreover

- $f(t,0,\lambda^*) \equiv 0$ on $\mathbb{R}$, i.e. (D_{λ^*}) has the trivial solution $\phi^* = 0$,
- the limits $\beta_k^\pm(\lambda) := \lim_{t\to\pm\infty} D_2^k f(t,0,\lambda)$ for $0 \leqslant k < m$ and

  $$\beta_m^\pm(\lambda) := \lim_{t\to\pm\infty} \int_0^1 \frac{(1-h)^{m-1}}{(m-1)!} D_2^m f(t,hx,\lambda)\,\mathrm{d}h$$

  satisfy $\beta_k^\pm(\lambda) \neq 0$ for all odd $k \geqslant 1$, $\beta_k^\pm(\lambda) = 0$ for all even $k \geqslant 0$ and

  $$\frac{\beta_k^\pm(\lambda)}{\beta_1^\pm(\lambda)} \geqslant 0 \text{ for all } \lambda \in \Lambda \text{ and odd } k \geqslant 3. \tag{4.21}$$

Hence, (D_λ) is asymptotically autonomous. In particular, Taylor's Formula yields

$$f(t,x,\lambda) = \sum_{k=0}^{m-1} \frac{D_2^k f(t,0,\lambda)}{k!} x^k + \int_0^1 \frac{(1-h)^{m-1}}{(m-1)!} D_2^m f(t,hx,\lambda)\,\mathrm{d}h x^m$$

and consequently we arrive at

$$\lim_{t\to\pm\infty} f(t,x,\lambda) = \beta_1^\pm(\lambda)x + \beta_3^\pm(\lambda)x^3 + \ldots + \beta_m^\pm(\lambda)x^m$$

$$= \beta_1^{\pm}(\lambda)x\left(1 + \tfrac{\beta_3^{\pm}(\lambda)}{\beta_1^{\pm}(\lambda)}x^2 + \ldots + \tfrac{\beta_m^{\pm}(\lambda)}{\beta_1^{\pm}(\lambda)}x^{m-1}\right).$$

Thus, the limit functions are autonomous ODEs, whose only equilibrium is zero (cf. (4.21)) and the limit set union $\alpha(\lambda)\cup\omega(\lambda)$ is admissible. We next check that $\phi^* = 0$ is weakly hyperbolic:
Case 1: $\beta_1^{\pm}(\lambda) < 0$ shows that (V_λ) has an ED on $\mathbb{R}$ with $\Pi_\lambda(t) \equiv 1$, thus $\phi^* = 0$ is hyperbolic.
Case 2: $\beta_1^{-}(\lambda) < 0 < \beta_1^{+}(\lambda)$ leads to an ED on $\mathbb{R}_+$ with $\Pi_\lambda^+(t) \equiv 0$ and an ED on $\mathbb{R}_-$ with $\Pi_\lambda^-(t) \equiv 1$. Since the projectors have different ranks, ϕ^* is not weakly hyperbolic.
Case 3: $\beta_1^{+}(\lambda) < 0 < \beta_1^{-}(\lambda)$ gives the dual situation of an ED on $\mathbb{R}_+$ with $\Pi_\lambda^+(t) \equiv 1$ and an ED on $\mathbb{R}_-$ with $\Pi_\lambda^-(t) \equiv 0$; again ϕ^* is not weakly hyperbolic
Case 4: $0 < \beta_1^{\pm}(\lambda)$ guarantees and ED on $\mathbb{R}$ with $\Pi_\lambda(t) \equiv 0$, $\phi^* = 0$ is even hyperbolic.
As a consequence, the assumption (ii) of Thm. 3.1 requires $\beta_1^+(\lambda)\beta_1^-(\lambda) > 0$. We furthermore get

$$\begin{aligned} A(t) &= D_2 f(t,0,\lambda^*), \\ r(t,x) &= \sum_{k=2}^{m-1} \frac{D_2^k f(t,0,\lambda^*)}{k!}x^k + \int_0^1 \frac{(1-h)^{m-1}}{(m-1)!} D_2^m f(t,hx,\lambda^*)\,\mathrm{d}h x^m. \end{aligned}$$

It remains to verify the triviality of the kernel $N(L_A)$:
Case 1: $\beta_1^{\pm}(\lambda^*) < 0$ implies $N(L_A) = \{0\}$ and (4.8) hold trivially.
Case 4: $0 < \beta_1^{\pm}(\lambda^*)$ leads to an ED on $\mathbb{R}$ with $\Pi_\lambda(t) \equiv 0$ and $N(L_A) = \{0\}$ implies (4.8).

In conclusion, Thm. 3.1 applies and not only yields a homoclinic solution ϕ_λ of (D_λ) for all $\lambda \in \Lambda$, but also states that they are contained in a continuum.

Example 4.8 Given $\Omega = \mathbb{R}^2$ and $m \in \mathbb{N}\setminus\{1\}$ with $\Lambda = \mathbb{R}^k$, now consider a planar ODE (D_λ) with

$$f(t,x,\lambda) := A(t)x + b(t,x,\lambda) + c(t,\lambda),$$

where $A\colon \mathbb{R} \to L(\mathbb{R}^2)$ is defined by

$$A(t) := \begin{pmatrix} -a(t) & 0 \\ 0 & a(t) \end{pmatrix}, \quad a(t) := \begin{cases} -1, & t < -1, \\ t, & -1 \leqslant t \leqslant 1, \\ 1, & 1 < t, \end{cases} \tag{4.22}$$

and $b\colon \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^k \to \mathbb{R}^2$ and $c\colon \mathbb{R}\times\mathbb{R}^k \to \mathbb{R}^2$ satisfy the following conditions:

(i) b, and $D_2 b$ (resp. c and $D_1 c$) are uniformly continuous and bounded on $\mathbb{R}\times K$ for any compact set $K \subset \mathbb{R}^2\times\mathbb{R}^k$ (resp. for any compact set $K \subset \mathbb{R}^k$).

(ii) $b(t,0,\lambda) \equiv 0$ on $\mathbb{R}\times\{0\}\times\mathbb{R}^k$ and $c_\lambda \in \mathcal{C}_0(\mathbb{R},\mathbb{R}^2)$ for all $\lambda \in \mathbb{R}^k$.

In addition, we assume that there exists $\lambda^* \in \Lambda = \mathbb{R}^k$ with the following properties:

(iii) Db_{λ^*} exists with $Db_{\lambda^*} \in \mathcal{C}_0(\mathbb{R}\times\mathbb{R}^2, L(\mathbb{R}\times\mathbb{R}^2,\mathbb{R}^2))$, $D_2 b(t,0,\lambda^*) = 0$ and $c(t,\lambda^*) \equiv 0$, for all $t \in \mathbb{R}$.

(iv) There exists a measurable and bounded function $b_\infty\colon \mathbb{R}\times[-1,1]\times[-1,1] \to \mathbb{R}^2$ such that

$$\begin{gathered} \liminf_{\substack{x\to z \\ \mu\to\infty}} \frac{\pi_i(b(t,\mu x,\lambda^*))}{\mu} = \pi_i(b_\infty(t,z)) \text{ for all } t \in \mathbb{R}, z \in [-1,1]^2 \text{ and } i = 1,2, \text{ and} \\ \pi_2(b_\infty(\mathbb{R}\times(0,1]\times\{0\})) \subset (0,\infty) \text{ and } \pi_2(b_\infty(\mathbb{R}\times[-1,0)\times\{0\})) \subset (-\infty,0). \end{gathered} \tag{4.23}$$

It is not hard to see that the above assumptions (i.e., (4.22) together with (i)) imply that f satisfies (H0-H1). Note that

$$\lim_{t\to\pm\infty} f(t,0,\lambda) = \lim_{t\to\pm\infty} c(t,\lambda) = 0$$

guarantees (H2). These assumptions also lead to autonomous, linear and hyperbolic limit systems $\dot{x} = f^{\pm}(x)$ with

$$f^+(x) := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x, \qquad f^-(x) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x$$

and limit set union $\alpha(\lambda) \cup \omega(\lambda) = \{f^-, f^+\}$, which is not only independent of λ, but also admissible; that is (i) holds. Concerning the weak hyperbolicity assumption (ii), for every $\lambda \in \Lambda$ we choose $\phi_\lambda^* := 0$ and readily obtain the identity

$$D_2 f(t, \phi_\lambda^*(t), \lambda) \equiv D_2 f(t, 0, \lambda) \equiv A(t) \text{ on } \mathbb{R}.$$

Now (L) has the evolution matrix

$$U(t,s) = \begin{cases} \begin{pmatrix} \exp(t-s) & 0 \\ 0 & \exp(s-t) \end{pmatrix}, & t, s < -1, \\[2ex] \begin{pmatrix} \exp\left((s^2 - t^2)/2\right) & 0 \\ 0 & \exp\left((t^2 - s^2)/2\right) \end{pmatrix}, & t, s \in [-1, 1], \\[2ex] \begin{pmatrix} \exp(s-t) & 0 \\ 0 & \exp(t-s) \end{pmatrix}, & 1 < t, s, \end{cases}$$

whose cocycle property $U(t,\tau) = U(t,s)U(s,\tau)$ for all $\tau, s, t \in \mathbb{R}$ leads to explicit representations of $U(t,s)$ for arbitrary $t, s \in \mathbb{R}$. Then (L) has an ED on both half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors

$$\Pi^+(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Pi^-(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and ϕ_λ^* is weakly hyperbolic, i.e. (ii) is verified. It remains to satisfy the degree condition (3.1), for which we employ Prop. 4.6. It follows from the above assumptions that (D_{λ^*}) has the trivial solution $\phi_0 = 0$ yielding the nonlinearity

$$r(t, x) = b(t, x, \lambda^*).$$

What is more, by using the mean value theorem for the function $b_{\lambda^*} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$ and the fact that its derivative Db_{λ^*} belongs to the space $\mathcal{C}_0(\mathbb{R} \times \mathbb{R}^2, L(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2))$, one can show that the function r satisfies the first condition in (4.7) in (lr2). As for the second condition in (lr2), we observe

$$R(\Pi^+(0)) \cap N(\Pi^-(0)) = R(\Pi^+(0)) + N(\Pi^-(0)) = \mathbb{R}e_1$$

and choose $\xi_1 = \xi_1^* = e_1$, $\eta_1^* = e_2$; e_i denote the canonical unit vectors. The explicit form

$$U(t,0) = \begin{cases} \begin{pmatrix} \exp((1-2|t|)/2) & 0 \\ 0 & \exp((2|t|-1)/2) \end{pmatrix}, & |t| > 1, \\[2ex] \begin{pmatrix} \exp(-t^2/2) & 0 \\ 0 & \exp(t^2/2) \end{pmatrix}, & |t| \leqslant 1, \end{cases}$$

$$U(0,t)^T = \begin{cases} \begin{pmatrix} \exp((2|t|-1)/2) & 0 \\ 0 & \exp((1-2|t|)/2) \end{pmatrix}, & |t| > 1, \\[2ex] \begin{pmatrix} \exp(t^2/2) & 0 \\ 0 & \exp(-t^2/2) \end{pmatrix}, & |t| \leqslant 1 \end{cases}$$

leads to $N(L_A) = \mathbb{R}ue_1$, $N(L_A^*) = \mathbb{R}ue_2$ with the positive function $u\colon \mathbb{R} \to [0,1]$ given by

$$u(t) := \begin{cases} \exp((1-2\,|t|)/2), & |t| > 1, \\ \exp(-t^2/2), & |t| \leqslant 1. \end{cases} \tag{4.24}$$

Therefore, we obtain $\|\varphi\|_1 = 1$ for all functions $\varphi_\pm = u_\pm e_1 \in N(L_A)$, while Rem. 4.1 leads to $\varphi_\pm^\star = u_\pm e_2$, where $u_\pm := \pm u$. Finally, observe that (4.23) implies that

$$\pi_2(b_\infty(t, (\pm u(t), 0)))(\pm u(t)) > 0 \text{ for all } t \in \mathbb{R}$$

and hence

$$\begin{aligned} &\int_{\mathbb{R}} \liminf_{\substack{x \to \varphi_\pm(t) \\ \mu \to \infty}} \langle r_p(t,x,\mu), \varphi_\pm^\star(t)\rangle \,\mathrm{d}t = \int_{\mathbb{R}} \liminf_{\substack{x \to \varphi_\pm(t) \\ \mu \to \infty}} \pi_2(r_p(t,x,\mu))\pi_2(\varphi_\pm^\star(t))\mathrm{d}t \\ &= \int_{\mathbb{R}} \liminf_{\substack{x \to \varphi_\pm(t) \\ \mu \to \infty}} \pi_2(r_p(t,x,\mu))u_\pm(t)\mathrm{d}t = \int_{\mathbb{R}} \pi_2(b_\infty(t,(\pm u(t),0)))(\pm u(t))\,\mathrm{d}s > 0. \end{aligned} \tag{4.25}$$

Thus, (4.25) guarantees (4.8). In conclusion, Thm. 3.1 applies to our planar ODE (D_λ) and yields a continuum of homoclinic solutions covering $\mathbb{R}^k$.

5 Perspectives

First, if Λ is locally compact, then Assumption (H1) can be weakened to the following:

(H1') there exists $\gamma > 0$ such that for every $\varepsilon > 0$, $\lambda_0 \in \Lambda$ there exists a $\delta > 0$ such that for all $\lambda \in B_\delta(\lambda_0)$ holds

$$|x-y| < \delta \text{ and } |t-s| < \delta \quad \Rightarrow \quad \left|D_2^j f(t,x,\lambda) - D_2^j f(s,y,\lambda_0)\right| < \varepsilon,$$

for all $x, y \in B_\gamma(0)$, $t, s \in \mathbb{R}$ and $j \in \{0,1\}$.

Nevertheless, the substitution operator $F\colon \mathcal{C}_0(\Omega) \times \Lambda \to \mathcal{C}_0$ from (2.3) and $G\colon \mathcal{C}_0^1(\Omega) \times \Lambda \to \mathcal{C}_0$ from (2.4) are still well-defined and continuous. In conclusion, for locally compact Λ and Assumption (H1) replaced by (H1'), all results from this paper remain true. The proof relies on the subsequent lemma, which appears to be of independent interest:

Lemma 5.1 *Suppose that Λ is locally compact and that (H0) and (H1') hold. Let $\phi \in \mathcal{C}_0^1(\Omega)$ and $\lambda_0 \in \Lambda$. Then for every $\varepsilon > 0$ there exists a $\xi > 0$ such that*

$$|t-s| < \xi \quad \Rightarrow \quad \sup_{t\in\mathbb{R}} \left|D_2^j f(s,\psi(s),\lambda) - D_2^j f(t,\phi(t),\lambda_0)\right| < \varepsilon, \tag{5.1}$$

for all $\psi \in B_\xi(\phi)$, $\lambda \in B_\xi(\lambda_0)$, $t, s \in \mathbb{R}$ and $j \in \{0,1\}$.

Proof. Take $\varepsilon > 0$ and $\lambda_0 \in \Lambda$. Since $\phi \in \mathcal{C}_0^1(\Omega)$, it follows that there exists $T > 0$ such that $\phi(s) \in B_{\gamma/2}(0)$ for all $|s| \geqslant T$. Take $\delta > 0$ as in Assumption (H1') for $\varepsilon/2$ and $\lambda_0 \in \Lambda$. Without loss of generality one can assume that $\delta < \gamma/2$ and one can assume that $\bar{B}_{\delta/2}(\lambda_0)$ is compact. Observe that if $\psi \in B_\delta(\phi)$, then $\psi(s) \in B_\gamma(0)$ for all $|s| \geqslant T$. Since $\phi([-T,T]) \subset \Omega$ is compact, there exists an open subset $\Omega' \subset \Omega$ such that $\overline{\Omega'} \subset \Omega$ is compact with $\phi([-T,T]) \subset \Omega'$. Hence there exists $\delta_1 > 0$ such that if $\psi \in B_{\delta_1}(\phi)$, then $\psi(s) \in \Omega'$ for all $s \in [-T,T]$. Since $\phi \in \mathcal{C}_0^1$, it follows that the derivative of ϕ is bounded on $\mathbb{R}$ and hence ϕ is Lipschitz on $\mathbb{R}$ with some constant L_ϕ. Let $\delta_2 := (1/2)\min\{\delta, \delta_1, \delta_1/L_\phi\}$. Then

$$|t-s| < \delta_2 \text{ and } \psi \in B_{\delta_2}(\phi) \Rightarrow |\psi(s) - \phi(t)| < \delta.$$

Consequently from Assumption (H1') it follows that

$$|t-s| < \delta_2 \Rightarrow \left|D_2^j f(s,\psi(s),\lambda) - D_2^j f(t,\phi(t),\lambda_0)\right| < \varepsilon \tag{5.2}$$

for all $|t|, |s| \geqslant T, \lambda \in B_{\delta_2}(\lambda_0), \psi \in B_{\delta_2}(\phi), j \in \{0,1\}$.

On the other hand, since $[-T-\delta_2, T+\delta_2] \times \overline{\Omega'} \times \overline{B}_{\delta_2/2}(\lambda_0)$ is compact, we deduce from Assumption (H0) that the corresponding restrictions

$$f\colon [-T-\delta_2, T+\delta_2] \times \overline{\Omega'} \times \overline{B}_{\delta_2/2}(\lambda_0) \to \mathbb{R}^d,$$
$$D_2 f\colon [-T-\delta_2, T+\delta_2] \times \overline{\Omega'} \times \overline{B}_{\delta_2/2}(\lambda_0) \to L(\mathbb{R}^d)$$

are uniformly equicontinuous, and therefore there exists $\rho < \delta_2/2$ such that

$$|t-s| < \rho \Rightarrow \left| D_2^j f(s, \psi(s), \lambda) - D_2^j f(t, \phi(t), \lambda_0) \right| < \varepsilon \tag{5.3}$$

for all $|s|, |t| \leqslant T + \delta_2, \lambda \in B_\rho(\lambda_0), \psi \in B_\rho(\phi), j \in \{0,1\}$. Finally, putting $\xi := \rho$ and taking into account (5.2) and (5.3), we obtain the desired conclusion from (5.1). □

Second, our overall approach can be generalized to Carathéodory differential equations (D_λ), whose right-hand side f is only assumed to be measurable in the time variable. Here solutions are merely absolutely continuous and not of class C^1 anymore. Hence, the appropriate functional analytical setting is to work with the spaces $\mathcal{W}_0^{1,\infty}$ (weakly differentiable functions with limit 0) and $\mathcal{L}_0^\infty$ (essentially bounded functions decaying to 0). It remains to apply compactness criteria in these spaces in order to arrive at corresponding conditions for the properness of $G\colon \mathcal{W}_0^{1,\infty} \times \Lambda \to \mathcal{L}_0^\infty$, that is, a counterpart to Prop. 2.10.

Third, it would be an interesting endeavor to obtain global continuation results or a global continuation principle, which applies to bounded entire solutions, rather than homoclinic solutions. A crucial difficulty in this setting is to obtain Fredholm properties of G acting between the spaces $\mathcal{C}^1(\Omega)$ and $\mathcal{C}$.

Appendices

A Topological dynamics

This final appendix collects preliminaries from topological dynamics (cf. [23, 4]) and properties of the Bebutov flow. Let $\Omega = \overline{U} \subseteq \mathbb{R}^d$, where U is an open convex neighborhood of 0. Given a continuous function $f\colon \mathbb{R}\times\Omega \to \mathbb{R}^d$ satisfying (H0-H1),

$$\mathcal{H}(f) := \overline{\{f(\cdot + s, \cdot)\colon \mathbb{R}\times\Omega \to \mathbb{R}^d |\, s \in \mathbb{R}\}} \subseteq \mathcal{C}(\mathbb{R}\times\Omega, \mathbb{R}^d)$$

defines its *hull*, where the closure is taken in $\mathcal{C}(\mathbb{R}\times\Omega, \mathbb{R}^d)$ with the compact-open topology. Recall that the compact-open topology in $\mathcal{H}(f)$ is induced by the following metric:

$$d(g, \bar g) := \sum_{k,l=1}^\infty \frac{1}{2^{k+l}} |g - \bar g|_{k,l}, \tag{A.1}$$

where $|g-\bar g|_{k,l} := \sup_{|t|\leqslant k} \sup_{|x|\leqslant l} |g(t,x) - \bar g(t,x)|$. This allows to introduce the mentioned *Bebutov flow*

$$S^s\colon \mathcal{H}(f) \to \mathcal{H}(f), \qquad S^s g := g(\cdot + s, \cdot) \text{ for all } s \in \mathbb{R} \tag{A.2}$$

induced by f. Thus, (A.2) defines a continuous dynamical system $\mathbb{R}\times\mathcal{H}(f) \ni (s,g) \mapsto S^s g \in \mathcal{H}(f)$ (cf. [4]).

Moreover, the construction of the Bebutov flow equips us with tools from dynamical systems and, e.g.

$$\omega(f) := \{g \in \mathcal{H}(f) |\, \exists s_n \to \infty \mid \lim_{n\to\infty} d(f(\cdot + s_n, \cdot), g) = 0\}$$

defines the ω*-limit set* of f, while the α*-limit set* becomes

$$\alpha(f) := \{g \in \mathcal{H}(f) |\, \exists s_n \to \infty \mid \lim_{n\to\infty} d(f(\cdot - s_n, \cdot), g) = 0\}.$$

The following lemma is well-known (see [14, Lemma 4.5] and [13, 20]):

Lemma A.1 *The set $\mathcal{H}(f)$ admits the following properties:*

(a) *$\mathcal{H}(f)$ is nonempty and compact. In particular, $\alpha(g), \omega(g) \neq \emptyset$ are compact for all $g \in \mathcal{H}(f)$,*

(b) *If $g \in \mathcal{H}(f)$, then g is bounded and uniformly continuous on $\mathbb{R} \times K$, for any compact set $K \subset \Omega$.*

(c) *For every sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\lim_{n\to\infty} |s_n| = \infty$ there exists $g \in \mathcal{H}(f)$ and a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ so that $(S^{s_{n_k}} f)_{k\in\mathbb{N}}$ converges to g (with respect to the compact-open topology).*

Example A.2 Almost periodic and almost automorphic functions f yield a compact hull $\mathcal{H}(f)$ (see [4, Prop. 3.9]) and thus compact limit sets.

Example A.3 (asymptotically periodic equations) A function f is called *asymptotically periodic*, if there exist $p_+, p_- > 0$ and limit functions $f^\pm : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^d$ satisfying

$$f^\pm(t,x) = f^\pm(t+p_\pm, x), \qquad \lim_{t\to\pm\infty} \sup_{x\in B} \left| f(t,x) - f^\pm(t,x) \right| = 0 \text{ for all } B \subset \Omega \text{ bounded;}$$

if $f^\pm$ do not depend on time, one denotes f as *asymptotically autonomous*. The limit sets

$$\alpha(f) = \left\{ S^t f^- : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^d | \, t \in [0, p_-] \right\}, \qquad \omega(f) = \left\{ S^t f^+ : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^d | \, t \in [0, p_+] \right\},$$

are topological circles (singletons in the asymptotically autonomous case) and hence compact.

Lemma A.4 *The set $\{x \in \Omega \mid \exists g \in G : g(t,x) \equiv 0$ on $\mathbb{R}\}$ is closed for $G \in \{\alpha(f), \omega(f)\}$.*

Proof. Let $\Omega_0 := \{x \in \Omega \mid \exists g \in \omega(f) : g(t,x) \equiv 0 \text{ on } \mathbb{R}\}$ and suppose that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in Ω. To establish that its limit x_0 is contained in Ω_0, we proceed as follows: By definition, there exist $g_n \in \omega(f)$ with

$$0 \equiv g_n(t, x_n) \equiv \lim_{k\to\infty} f(t + s_k^n, x_n) \quad \text{on } \mathbb{R} \quad \text{and for all } n \in \mathbb{N} \tag{A.3}$$

and some real sequence $(s_k^n)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} s_k^n = \infty$. Let $N \in \mathbb{N}$ and set $K := \{x_n : n \in \mathbb{N}_0\}$. The convergence of $(x_n)_{n\in\mathbb{N}}$ to x_0 and the assumed uniform continuity of f implies that there exists an integer $n = n(N) > 0$ such that

$$|f(t, x_0) - f(t, x_n)| < 1/N \text{ for all } t \in \mathbb{R}.$$

On the compact set $[-N, N] \times K$ we derive from (A.3) that there exists a $m = m(N) \in \mathbb{N}$ such that $s_m^n > N$ and

$$|f(t + s_m^n, x_n)| < 1/N \text{ for all } t \in [-N, N]$$

hold. Consequently, this leads to

$$|f(t + s_m^n, x_0)| \leqslant |f(t + s_m^n, x_0) - f(t + s_m^n, x_n)| + |f(t + s_m^n, x_n)| < 2/N$$

for all $t \in [-N, N]$. Then the sequence $s_N := s_{m(N)}^{n(N)}$ satisfies

$$|f(t + s_N, x_0)| < 2/N, \qquad \lim_{N\to\infty} s_N = \infty, \qquad \lim_{N\to\infty} f(t + s_N, x_0) = 0. \tag{A.4}$$

By Lemma A.1 there exists a subsequence $(s_{N_n})_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} S^{s_{N_n}} f = g \text{ for some } g \in \omega(f).$$

From (A.4) we derive $g(t, x_0) \equiv 0$ on $\mathbb{R}$ and hence $x_0 \in \Omega_0$ follows. Thus, the set $\Omega_0 \subseteq \overline{\Omega}$ is closed. The proof for $\alpha(f)$ is analogous. □

A similar skew-product construction as in case of nonlinear functions f is possible for continuous functions $A\colon \mathbb{R} \to L(\mathbb{R}^d)$: One defines the *hull*

$$\mathcal{H}(A) := \overline{\{A(\cdot + s) : \mathbb{R} \to L(\mathbb{R}^d) \mid s \in \mathbb{R}\}},$$

on which the *Bebutov flow* becomes

$$S^s\colon \mathcal{H}(A) \to \mathcal{H}(A), \qquad S^s B := B(\cdot + s) \text{ for all } s \in \mathbb{R}.$$

The closure in this definition of $\mathcal{H}(A)$ is again taken in the compact-open topology, i.e. uniform convergence on bounded sets induced by the metric

$$d(A, \bar{A}) := \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_{t\in[-l,l]} \left|A_t - \bar{A}_t\right| \tag{A.5}$$

and the limit sets now become

$$\omega(A) := \{B \in \mathcal{H}(A) \mid \exists s_n \to \infty : \lim_{n\to\infty} d(A(\cdot + s_n), B) = 0\},$$
$$\alpha(A) := \{B \in \mathcal{H}(A) \mid \exists s_n \to \infty : \lim_{n\to\infty} d(A(\cdot - s_n), B) = 0\}.$$

Lemma A.5 *If $A \in \mathcal{BC}(\mathbb{R}, L(\mathbb{R}^d))$, then $\alpha(A) \neq \emptyset$ and $\omega(A) \neq \emptyset$.*

Proof. The claim follows as in Lemma A.1. □

B The Benevieri-Furi degree

Following [1], let $C(T)$ denote the *correctors* of a linear operator $T \in L(X, Y)$, that is, the set of all $K \in L(X, Y)$ with $\dim R(K) < \infty$ and $T + K \in GL(X, Y)$. We call $K_1, K_2 \in C(T)$ *equivalent*, if $\det((T + K_1)^{-1}(T + K_2)|_{X_0}) > 0$, where X_0 is a finite-dimensional subspace of X containing the range $R((T + K_1)^{-1}(K_1 - K_2))$. Then $C(T)$ contains exactly two equivalence classes and $C(T) \neq \emptyset$ holds if and only if $T \in \Phi_0(X, Y)$. An *orientation* of $T \in \Phi_0(X, Y)$ is an equivalence class of correctors for T according to the above equivalence relation; by the opposite orientation of T we mean the complementary equivalence class in $C(T)$. An *oriented linear Fredholm operator* is a pair (T, σ) consisting of a $T \in \Phi_0(X, Y)$ and an orientation σ.

A closed subspace $Y_0 \subseteq Y$ is called *transversal* to $T \in \Phi_0(X, Y)$, if $R(T) + Y_0 = Y$ and Y_0 is complemented in Y (or $T^{-1}(Y_0)$ is complemented in X). For an oriented Fredholm map (T, σ) and $Y_0 \subseteq Y$ transversal to T one has:

- If $X_0 := T^{-1}(Y_0)$, then the *inherited orientation* of $T_0 := T|_{X_0}$ is

$$\sigma_0 := \{K|_{X_0} \in L(X_0, Y_0) \mid K \in \sigma \text{ and } R(K) \subseteq Y_0\}, \tag{B.1}$$

  $T|_{(X_0,Y_0)} \in \Phi_0(X_0, Y_0)$ and $\dim Y_0 < \infty$ implies $\dim X_0 = \dim Y_0$,

- Y_0 is transversal to T if and only if there are closed subspaces $Y_1 \subseteq Y$ and $X_1 \subseteq X$ satisfying $Y = Y_0 \oplus Y_1$ and $X = X_0 \oplus X_1$ such that $T|_{(X_1,Y_1)} \in GL(X_1, Y_1)$.

This concept can be extended to a continuous map from a metric space Λ to $\Phi_0(X, Y)$. Namely, an orientation of a continuous map $h\colon \Lambda \to \Phi_0(X, Y)$ can be given by assigning a family $\{(U_i, A_i) \mid i \in \mathcal{I}\}$, called an oriented atlas of h, satisfying the following properties:

$$\begin{aligned} &\{U_i \mid i \in \mathcal{I}\} \text{ is an open covering of } \Lambda;\\ &\text{given } i \in \mathcal{I},\ A_i \text{ is a corrector of any } h(\lambda), \text{ for all } \lambda \in U_i;\\ &\text{if } \lambda \in U_i \cap U_j, \text{ then } A_i \text{ is equivalent to } A_j \text{ with respect to } h(\lambda).\end{aligned} \tag{B.2}$$

In particular, one can prove, by using the covering space theory, that if Λ is simply connected and locally path connected, then any continuous map $h\colon \Lambda \to \Phi_0(X,Y)$ is orientable (see [2]). Furthermore, the above concept allows us to define a notion of orientation for nonlinear Fredholm maps of index zero between open subsets of Banach spaces. Let O be an open subset of X and let $f\colon O \to Y$ be Fredholm of index zero. By an orientation of f we shall mean an orientation of $O \ni x \xrightarrow{Df} Df(x) \in \Phi_0(X,Y)$. Thus f is orientable if and only if so is Df according to (B.2).

Let $F\colon O \to Y$ denote an oriented Fredholm map of index 0 on an open subset $O \subseteq X$ and with the compact set $F^{-1}(0)$. An oriented submanifold $Y_0 \subseteq Y$ is called *transversal* to F on $M \subseteq O$, if for each $x \in M \cap F^{-1}(Y_0)$ the subspace $T_{F(x)}Y_0 \subseteq Y$ is transversal to $DF(x)$. What is more, if $F\colon O \to Y$ is oriented, then the restriction $F_0\colon M_0 \to Y_0$ is still an oriented Fredholm map of index zero (see [1, 2, 24] and (B.1)). In [1, 24] it is shown how the orientations of F and Y_0 give an orientation on M_0, that we shall refer to as induced orientation.

Then the *Benevieri-Furi degree* $\deg(F,O,0) \in \mathbb{Z}$ (resp. $\deg(F,O,0) \in \mathbb{Z}_2$ in the nonoriented case) is constructed as follows:

Let $Y_0 \subset Y$ be a finite-dimensional submanifold transversal to F on an open neighborhood $O_0 \subset O$ of $F^{-1}(0)$, F oriented on O_0. The intersection $X_0 := O_0 \cap F^{-1}(Y_0)$ is either empty or an oriented submanifold of the same dimension as Y_0 and of class C^1 (see [24, Thm. 8.55]). If $\dim Y_0 > 0$, then $F_0 := F|_{X_0} \in C^1(X_0,Y_0)$ satisfies the *reduction property*

$$\deg(F,O,0) = \deg_B(F_0,X_0,0), \tag{B.3}$$

where $\deg_B$ is the C^1-Brouwer degree. For $X_0 = \emptyset$ the right-hand side is set to be zero. In the oriented situation, the orientation of F_0 is defined as in (B.1). The sign of an oriented Fredholm operator (T,σ) is $\operatorname{sgn} T = 1$ if $0 \in \sigma$, $\operatorname{sgn} T = -1$ otherwise and $\operatorname{sgn} T = 0$ if $T \notin GL(X,Y)$.

Such a degree is uniquely determined and particularly satisfies (see [1, 24]):

(bf2) **(homotopy invariance)** If $H\colon O \times [0,1] \to Y$ is a generalized (oriented) Fredholm homotopy of index 0 with $H^{-1}(0)$ being compact, then $\deg(H_0,O,0) = \deg(H_1,O,0)$.

Recall that a *generalized Fredholm homotopy* of index 0 is a continuous map $H\colon O \times [0,1] \to Y$ with continuous derivative $(x,t) \mapsto DH_t(x) \in \Phi_0(X,Y)$ for every $t \in [0,1]$. We will say that H is orientable if for any $t \in [0,1]$, the map $O \ni x \mapsto DH_t(x) \in \Phi_0(X,Y)$ is orientable. In fact, one can show that a generalized Fredholm homotopy H is orientable if and only if there exists at least one $t_0 \in [0,1]$ such that the map $H_{t_0}\colon O \to Y$ is orientable (see [1, 2, 24]).

By means of a Lyapunov-Schmidt-like technique we illustrate how the reduction property (B.3) can be used in explicit calculations of the Benevieri-Furi degree. For this endeavor, assume that $O \subsetneq X$ is open, bounded, simply connected and for every $s \in [0,1]$ we suppose

- $L \in \Phi_0(X,Y)$ with $N(L) \neq \{0\}$,
- $R\colon \overline{O} \times [0,1] \to Y$ is continuous and $R(\cdot,s) \in \mathcal{C}^1(O,Y)$,
- $F\colon \overline{O} \times [0,1] \to Y$, $F(x,s) := Lx + sR(x,s)$ is Fredholm of index 0.

Now define $X_1 := N(L)$, $Y_1 := R(L)$ and choose topological complements X_2 and Y_2 with $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Notice that these closed complements exist because X_1 has a finite dimension and Y_1 a finite co-dimensional. There exist projections $P \in L(X)$, $Q \in L(Y)$ such that $R(P) = X_1$, $N(P) = X_2$, $R(Q) = Y_2$, $N(Q) = Y_1$. Thus, the restriction $L_P := L|_{X_2}\colon X_2 \to Y_1$ of L to X_2 is an isomorphism and F admits the decomposition

$$F(x,s) = (Lx + s((\mathrm{id}_Y - Q) \circ R)(x,s), sQR(x,s)).$$

Since $F(\cdot,0) = L$ is linear, it suffices to study the zeros of $F(x,s) = 0$ for $s \in (0,1]$. But then

$$F(x,s) = 0 \quad \Longleftrightarrow \quad \tilde{F}(x,s) = 0 \text{ for all } s \in (0,1],$$

where

$$\tilde{F}\colon \overline{O} \times [0,1] \to Y, \quad \tilde{F}(x,s) = (Lx + s((\mathrm{id}_Y - Q)R)(x,s), QR(x,s)).$$

What is more, with the mapping $F_s\colon O \to Y$ also $\tilde{F}_s\colon O \to Y$, $s \in [0,1]$, are Fredholm maps of index zero, which follows from

$$\tilde{F}(x,s) - F(x,s) = (0, (1-s)QR(x,s)),$$

where the right-hand side of the latter equality is a finite dimensional function. This enables us to study properties of $\tilde{F}$ induced by F.

We close with a result being central in our above application:

Proposition B.1 *If $\tilde{F}\colon \overline{O} \times [0,1] \to Y$ is proper and satisfies*

(i) $\tilde{F}(x,s) \neq 0$ *for all* $0 < s \leqslant 1$ *and* $x \in \partial O$,

(ii) $QR(x,0) \neq 0$ *for all* $x \in X_1 \cap \partial O$,

then $\deg(\tilde{F}_1, O, 0) = \deg_B(QR_0|_{X_1 \cap O}, X_1 \cap O, 0)$.

Proof. Since O is simply connected, $\tilde{F}$ is orientable and the oriented Benevieri-Furi degree applies. The properness of the mapping $\tilde{F}$ implies that the set $\{(x,t) \in O \times [0,1] \mid \tilde{F}(x,t) = 0\}$ is compact. The homotopy invariance (bf2) yields

$$\deg((L, QR_0), O, 0) = \deg(\tilde{F}_0, O, 0) = \deg(\tilde{F}_1, O, 0).$$

Finally, the subspace Y_2 is transversal to $\tilde{F}_0$, $\tilde{F}_0^{-1}(Y_2) = N(L) \cap O = X_1 \cap O$ and consequently (B.3) guarantees that

$$\deg((L, QR_0), O, 0) = \deg_B(QR_0|_{X_1 \cap O}, X_1 \cap O, 0),$$

which concludes the proof. □

C Projector methods

In this section we review a few results about projectors strictly associated with the Fredholm maps considered in this paper. Before this, we must recall the following two facts:

Lemma C.1 *If $\varphi \in N(L_A)$, then $\int_{\mathbb{R}} |\varphi(s)| ds < \infty$ and the same holds for all $\varphi^* \in N(L_A^*)$.*

Proof. The assertion it suffices to prove for $\varphi \in N(L_{\mathcal{A}})$ of the form $\varphi(t) := U(t,0)\xi_i$, where

$$\xi_i \in R(\Pi^+(0)) \cap N(\Pi^-(0)) = \operatorname{span}\{\xi_1, \dots, \xi_m\}.$$

Thus using the assumed dichotomy estimates (see (2.1)), we can estimate $|\varphi(t)|$ as follows

$$|\varphi(t)| = |U(t,0)\xi_i| = \begin{cases} |U(t,0)\Pi^+(0)\xi_i| \leqslant Ke^{-\alpha t}|\xi_i| & \text{if } 0 \leqslant t, \\ |U(t,0)(\mathrm{id}_{\mathbb{R}^d} - \Pi^-(0))\xi_i| \leqslant Ke^{\alpha t}|\xi_i| & \text{if } t \leqslant 0. \end{cases}$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} |\varphi(t)| \mathrm{d}t &= \int_{-\infty}^0 |\varphi(t)| \mathrm{d}t + \int_0^\infty |\varphi(t)| \mathrm{d}t \\ &\leqslant \int_{-\infty}^0 Ke^{\alpha t}|\xi_i| \mathrm{d}t + \int_0^\infty Ke^{-\alpha t}|\xi_i| \mathrm{d}t = \frac{2K}{\alpha}|\xi_i|. \end{aligned}$$

□

Lemma C.2 *Under the above assumptions, the linear functionals $\mu_i\colon \mathcal{C}_0 \to \mathbb{R}$ given by*

$$\mu_i(h) = \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, h(s) \rangle \mathrm{d}s \text{ for all } h \in \mathcal{C}_0 \text{ and } 1 \leqslant i \leqslant m$$

are continuous with $\|\mu_i\| \leqslant 2K\alpha^{-1}|\eta_i^*|$ and $R(L_A) = \bigcap_{i=1}^m N(\mu_i)$.

P r o o f. The corresponding estimation follows directly from Lemma C.1, while the second assertion can be deduced easily from (4.2). □

Consequently, the above lemmas allows us to construct the following projectors.

Lemma C.3 *Under the above assumptions, two mappings $P \in L(\mathcal{C}_0^1)$ and $Q \in L(\mathcal{C}_0)$ given by*

$$
\begin{aligned}
(P\phi)(t) &:= \sum_{i=1}^{m} \langle \xi_i \phi(0) \rangle U(t,0)\xi_i, \\
(Q\phi)(t) &:= \sum_{i=1}^{m} \frac{\omega(t) \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s) \rangle \mathrm{d}s}{|U(t,0)\eta_i^*|} \, U(t,0)\eta_i^*
\end{aligned}
$$

are bounded projections onto $N(L_A)$ and the complement of $R(L_A)$, respectively, where $\omega \in \mathcal{C}_0(\mathbb{R}, (0,\infty))$ is a continuous function satisfying

$$
\int_{\mathbb{R}} \omega(s) |U(s,0)\eta_i|^{-1} ds = 1.
$$

P r o o f. First we will prove that P is a projection onto $N(L_A)$. Observe that (4.4) implies that $P\phi \in N(L_A)$ for all $\phi \in \mathcal{C}_0^1$. Moreover, $P^2 = P$ holds because

$$
\begin{aligned}
P^2\phi = P(P\phi) &= P\left(\sum_{i=1}^{m} \langle \xi_i, \phi(0) \rangle U(\cdot,0)\xi_i \right) \\
&= \sum_{j=1}^{m} \left\langle \xi_j, \left(\sum_{i=1}^{m} \langle \xi_i, \phi(0) \rangle U(0,0)\xi_i \right) \right\rangle U(\cdot,0)\xi_j \\
&= \sum_{i,j=1}^{m} \langle \xi_i, \phi(0) \rangle \langle \xi_j, \xi_i \rangle U(\cdot,0)\xi_j = \sum_{\substack{i,j=1 \\ i=j}}^{m} \langle \xi_i, \phi(0) \rangle \langle \xi_j, \xi_i \rangle U(\cdot,0)\xi_j \\
&= \sum_{i=1}^{m} \langle \xi_i, \phi(0) \rangle U(\cdot,0)\xi_i = P\phi.
\end{aligned}
$$

Furthermore, P is a bounded projector because of the following estimate:

$$
\begin{aligned}
\|P\phi\|_1 &= \left\| \sum_{i=1}^{m} \langle \xi_i, \phi(0) \rangle U(\cdot,0)\xi_i \right\|_1 \leqslant \sum_{i=1}^{m} \| \langle \xi_i, \phi(0) \rangle U(\cdot,0)\xi_i \|_1 \\
&= \sum_{i=1}^{m} |\langle \xi_i, \phi(0) \rangle| \cdot \|U(\cdot,0)\xi_i\|_1 \leqslant \sum_{i=1}^{m} |\xi_i| \cdot \|\phi\|_1 \cdot \|U(\cdot,0)\xi_i\|_1 \\
&= \|\phi\|_1 \left(\sum_{i=1}^{m} |\xi_i| \cdot \|U(\cdot,0)\xi_i\|_1 \right).
\end{aligned}
$$

Now we will prove that Q is a bounded projection onto the complement of $R(L_A)$. It is easy to see that Q is well defined, i.e., $Q\phi \in \mathcal{C}_0$ for any $\phi \in \mathcal{C}_0$. Furthermore, the boundedness of Q follows from the following estimates:

$$
\begin{aligned}
\|Q\phi\|_0 &= \left\| \sum_{i=1}^{m} \frac{\omega(\cdot) \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s) \rangle \mathrm{d}s}{|U(\cdot,0)\eta_i|} \, U(\cdot,0)\eta_i^* \right\|_0 \\
&\leqslant \sum_{i=1}^{m} \left\| \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s) \rangle \mathrm{d}s \right\|_0 \cdot \|\omega\|_0 \\
&\leqslant \left(\sum_{i=1}^{r} 2K\alpha^{-1} \right) \|\phi\|_0 \cdot \|\omega\|_0,
\end{aligned}
$$

where the last inequality follows from Lemma C.2. What is more, Lemma C.2 implies that $N(Q) = R(L_A)$. Finally, it suffices to prove that $Q^2 = Q$. Indeed, let

$$v_j(t) := \frac{\omega(t)U(t,0)\eta_j^*}{|U(t,0)\eta_j^*|} \quad \text{for } 1 \leqslant j \leqslant m.$$

Then

$$\begin{aligned} Q^2\phi = Q(Q\phi) &= Q\left(\sum_{i=1}^m \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s)\rangle \mathrm{d}s \cdot v_i(\cdot)\right) \\ &= \sum_{j=1}^m \int_{\mathbb{R}} \left\langle U(0,s)^T \eta_j^*, \left(\sum_{i=1}^r \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s)\rangle \mathrm{d}s \cdot v_i(\cdot)\right)\right\rangle ds \\ &= \sum_{i=1}^m \sum_{j=1}^m \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s)\rangle \mathrm{d}s \cdot \int_{\mathbb{R}} \left\langle U(0,s)^T \eta_j^*, v_i(s)\right\rangle \mathrm{d}s \cdot v_j(\cdot). \end{aligned}$$

But

$$\begin{aligned} \left\langle U(0,s)^T \eta_j^*, v_i(s)\right\rangle &= \left\langle U(0,s)^T \eta_j^*, \omega(s)|U(s,0)\eta_i|^{-1} U(s,0)\eta_i^*\right\rangle \\ &= \omega(s)|U(s,0)\eta_i|^{-1} \left\langle \eta_j^*, U(0,s)U(s,0)\eta_i^*\right\rangle \\ &= \omega(s)|U(s,0)\eta_i^*|^{-1} \left\langle \eta_j^*, \eta_i^*\right\rangle = \omega(s)|U(s,0)\eta_i^*|^{-1}\delta_{j,i} \end{aligned}$$

where $\delta_{j,i}$ denotes the Kronecker symbol. Continuing the calculations from above, we have

$$\begin{aligned} &\sum_{i,j=1}^m \int_{\mathbb{R}} \langle U(0,s)^T \eta_i^*, \phi(s)\rangle \mathrm{d}s \cdot \int_{\mathbb{R}} \left\langle U(0,s)^T \eta_j^*, v_i(s)\right\rangle \mathrm{d}s \cdot v_j(\cdot) = \\ &\sum_{j=1}^m \int_{\mathbb{R}} \langle U(0,s)^T \eta_j^*, \phi(s)\rangle \mathrm{d}s \cdot \int_{\mathbb{R}} \omega(s)|U(s,0)\eta_i^*|^{-1} ds \cdot v_j(\cdot) = \\ &\sum_{j=1}^m \int_{\mathbb{R}} \langle U(0,s)^T \eta_j^*, \phi(s)\rangle \mathrm{d}s \cdot 1 \cdot v_j(\cdot) = Q\phi, \end{aligned}$$

which completes the proof that $Q^2 = Q$. □

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