A Smoothness Theorem for Invariant Fiber Bundles*

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Abstract

Invariant fiber bundles are the generalization of invariant manifolds from discrete dynamical systems (mappings) to non-autonomous difference equations. In this paper we present a self-contained proof of their existence and smoothness. Our main result generalizes the so-called "Hadamard-Perron-Theorem" for time-dependent families of pseudo-hyperbolic mappings from the finite-dimensional invertible to the infinite-dimensional non-invertible case.

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1 Introduction

While invariant manifolds play a vital role in the theory of dynamical systems, for many important questions the (classical) notion of invariant manifold (for a rest point) is too narrow. In fact, the study of any non-stationary (e.g. non-periodic or chaotic) phenomenon is an intrinsically time-variant matter which involves families of manifolds whose members change in time. In this context, the notion of *invariant fiber bundle* has proved to be

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a proper generalization of *invariant manifold*. Fundamental results concerning invariant fiber bundles can be found in Katok & Hasselblatt [14, pp. 242-243, Theorem 6.2.8] and Aulbach [3]. Some of the applications demonstrating the relevance of the time-dependance inherent in invariant fiber bundles concern invariant manifolds of hyperbolic sets (cf. Katok & Hasselblatt [14, pp. 266-268, Theorem 6.4.9]), invariant foliations (cf. Aulbach & Wanner [7]) and the dynamics near "weakly non-stationary" invariant manifolds (cf. Aulbach & Pötzsche [6]). Also the study of invariant manifolds of continuous-time dynamical systems via discretization methods with variable step-size leads to non-autonomous difference equations with invariant fiber bundles (cf. Aulbach & Garay [4] and Keller [15]).

In this paper we want to overcome some of the restrictions appearing in the fundamental results mentioned above. In Katok & Hasselblatt [14] the systems under consideration are supposed to be finite-dimensional and invertible, while the invariant fiber bundles in Aulbach [3] are only proved to be Lipschitz continuous. In the present paper we generalize those results by proving the existence of smooth invariant fiber bundles for systems of non-autonomous difference equations whose right-hand sides may be non-invertible and whose state spaces may be infinite-dimensional. This opens – at least in principle – the view to applications to partial differential equations.

From a technical point of view the present paper can be seen as a continuation of Aulbach [3] where the existence and global Lipschitz continuity of invariant fiber bundles for a general class of non-autonomous, non-invertible, pseudo-hyperbolic difference equations have been proved. The existence result in [3] is contained in our main theorem (Theorem 4.11), but we additionally prove the differentiability of the fiber bundles. Moreover we use a slightly different construction of the fiber bundles which does not rely on the discrete Gronwall lemma (cf. Aulbach [3, Lemma 2.1]). Related constructions for non-autonomous difference equations can be found in Papaschinopoulos [17], [18] and Schinas [19] and the references therein.

From the theory of ordinary diferential equations it is well-known that the differentiability of invariant manifolds is technically hard to prove. For a modern approach using sophisticated fixed point theorems see Vanderbauwhede & Van Gils [24], Vanderbauwhede [23], Hilger [12] or Siegmund [21]. Vanderbauwhede & Van Gils [24] apply their fixed point theorem to the C^m-smoothness problem for center manifolds of ordinary differential equations in $\mathbb{R}^n$. Another approach to the smoothness of invariant manifolds is essentially based on a lemma by Henry (cf. Chow & Lu [10, Lemma 2.1]) or methods of a more differential topological nature (cf. Hirsch, Pugh & Shub [13] or Shub [20]), namely the C^m-section theorem for fiber contracting maps; see also El-Bialy [11]. In Castañeda & Rosa [8] and Siegmund [22] the problem of higher order smoothness is tackled directly.

In this article we present an accessible "ad hoc" approach to C^1-smoothness of pseudo-hyperbolic invariant fiber bundles which is basically derived from Vanderbauwhede & Van Gils [24]. We also prove higher order smoothness with the classical uniform contraction principle.

2 Preliminaries

Although we are using the same notations as introduced in Aulbach [3] we repeat some of them here to keep the article self-contained. $\mathbb{N}$ denotes the positive and $\mathbb{N}_0$ the non-negative integers. A *discrete interval* I is defined to be the intersection of a real interval with the integers $\mathbb{Z} = \{0, \pm 1, \dots\}$. For an integer $\kappa \in \mathbb{Z}$ we define

$$\mathbb{Z}_\kappa^+ := [\kappa, \infty) \cap \mathbb{Z}, \qquad \mathbb{Z}_\kappa^- := (-\infty, \kappa] \cap \mathbb{Z}.$$

We write

$$x' = f(k, x) \tag{2.1}$$

to denote the difference equation $x(k+1) = f(k, x(k))$, with the right-hand side $f : I \times \mathcal{X} \to \mathcal{X}$, where I is a discrete interval and $\mathcal{X}$ is a Banach space. Let $\lambda(k; \kappa, \xi)$ denote the *general solution* of equation (2.1), i.e. $\lambda(\cdot\,; \kappa, \xi)$ solves (2.1) and satisfies the initial condition $\lambda(\kappa; \kappa, \xi) = \xi$ for $\kappa \in I$, $\xi \in \mathcal{X}$. The general solution can be defined recursively as

$$\lambda(k; \kappa, \xi) := \begin{cases} \xi & \text{for } k = \kappa \\ f(k-1, \lambda(k-1; \kappa, \xi)) & \text{for } k > \kappa \end{cases}$$

and if $f(k, \cdot) : \mathcal{X} \to \mathcal{X}$ has an inverse mapping $f^{-1}(k, \cdot)$ for $k \in \mathbb{Z}_\kappa^-$, we set

$$\lambda(k; \kappa, \xi) := f^{-1}(k+1, \lambda(k+1; \kappa, \xi)) \quad \text{for } k < \kappa.$$

It is easy to see that the so-called *cocycle property*

$$\lambda(k; \kappa, \xi) = \lambda(k; l, \lambda(l; \kappa, \xi)) \quad \text{for } k \geq l \geq \kappa \tag{2.2}$$

holds for the mapping λ. If $f^{-1}(k, \cdot)$ exists for any $k \in I$ then equation (2.2) is true for any $k, l, \kappa \in I$. Given an operator sequence $A : I \to \mathcal{L}(\mathcal{X})$ we define the *evolution operator* $\Phi(k, \kappa) \in \mathcal{L}(\mathcal{X})$ of the linear equation

$$x' = A(k)x$$

as the mapping defined by

$$\Phi(k, \kappa) := \begin{cases} I_\mathcal{X} & \text{for } k = \kappa \\ A(k-1) \cdots A(\kappa) & \text{for } k > \kappa \end{cases}$$

and if $A(k)$ is invertible (in $\mathcal{L}(\mathcal{X})$) for $k \in \mathbb{Z}_\kappa^-$ then

$$\Phi(k, \kappa) := A(k)^{-1} \cdots A(\kappa - 1)^{-1} \quad \text{for } k < \kappa.$$

The Banach spaces $\mathcal{X}, \mathcal{Y}$ are all real or complex throughout this paper and their norm is denoted by $\|\cdot\|_\mathcal{X}$, $\|\cdot\|_\mathcal{Y}$ or simply by $\|\cdot\|$. $\mathcal{L}_n(\mathcal{X}; \mathcal{Y})$ is the Banach space of n-linear continuous operators from $\mathcal{X}^n$ to $\mathcal{Y}$ for $n \in \mathbb{N}$, $\mathcal{L}_0(\mathcal{X}; \mathcal{Y}) := \mathcal{Y}$, $\mathcal{L}_n(\mathcal{X}) := \mathcal{L}_n(\mathcal{X}; \mathcal{X})$, $\mathcal{L}(\mathcal{X}; \mathcal{Y}) := \mathcal{L}_1(\mathcal{X}; \mathcal{Y})$, $\mathcal{L}(\mathcal{X}) := \mathcal{L}_1(\mathcal{X})$, $I_\mathcal{X}$ the identity map on $\mathcal{X}$ and $\mathcal{GL}(\mathcal{X})$ the multiplicative group of bijective mappings in $\mathcal{L}(\mathcal{X})$. On the cartesian product $\mathcal{X} \times \mathcal{Y}$ we always use the norm

$$\|(x, y)\|_{\mathcal{X} \times \mathcal{Y}} := \max \left\{ \|x\|_\mathcal{X}, \|y\|_\mathcal{Y} \right\} \tag{2.3}$$

and write $\Pi_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\Pi_2 : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ for the projections on the first and second component, respectively. We say that a linear subspace $\mathcal{X}_1 \subseteq \mathcal{X}$ is continuously embedded into $\mathcal{X}$ if the embedding operator $J : \mathcal{X}_1 \to \mathcal{X}$, $Jx := x$ is continuous and in this case we write $\mathcal{X}_1 \overset{J}{\hookrightarrow} \mathcal{X}$. The ball in $\mathcal{X}$ with center $x \in \mathcal{X}$ and radius $\varepsilon > 0$ is denoted by $B_\varepsilon(x)$.

3 Quasibounded functions

In this section we introduce the so-called quasiboundedness which is a handy notion describing exponential growth of functions.

Definition 3.1: *For a real constant $\gamma > 0$, an integer $\kappa_0 \in \mathbb{Z}$, a Banach space $\mathcal{X}$, a discrete interval I and a mapping $\lambda : I \to \mathcal{X}$ we say that*

(a) *λ is γ^+-*quasibounded *if $I = \mathbb{Z}^+_{\kappa_0}$ and if $\|\lambda\|^+_{\kappa,\gamma} := \sup_{k\in\mathbb{Z}^+_\kappa} \|\lambda(k)\| \gamma^{\kappa-k} < \infty$ for some $\kappa \in \mathbb{Z}^+_{\kappa_0}$.*

(b) *λ is γ^--*quasibounded *if $I = \mathbb{Z}^-_{\kappa_0}$ and if $\|\lambda\|^-_{\kappa,\gamma} := \sup_{k\in\mathbb{Z}^-_\kappa} \|\lambda(k)\| \gamma^{\kappa-k} < \infty$ for some $\kappa \in \mathbb{Z}^-_{\kappa_0}$.*

(c) *λ is $\gamma^\pm$-*quasibounded *if $I = \mathbb{Z}$ and if $\|\lambda\|^\pm_{\kappa,\gamma} := \sup_{k\in\mathbb{Z}} \|\lambda(k)\| \gamma^{\kappa-k} < \infty$ for some $\kappa \in \mathbb{Z}$.*

By $\ell^+_{\kappa,\gamma}(\mathcal{X})$ and $\ell^-_{\kappa,\gamma}(\mathcal{X})$ we denote the sets of all γ^+- and γ^-–quasibounded functions $\lambda : I \to \mathcal{X}$, respectively.

Obviously $\ell^+_{\kappa,\gamma}(\mathcal{X})$ and $\ell^-_{\kappa,\gamma}(\mathcal{X})$ are non-empty and the following result is immediate:

Lemma 3.2: *For any $\gamma > 0$, $\kappa \in \mathbb{Z}$ the sets $\ell^+_{\kappa,\gamma}(\mathcal{X})$ and $\ell^-_{\kappa,\gamma}(\mathcal{X})$ are Banach spaces with the norms $\|\cdot\|^+_{\kappa,\gamma}$ and $\|\cdot\|^-_{\kappa,\gamma}$, respectively.*

We state the next lemma only in forward time. It will simplify our differential calculus.

Lemma 3.3: *For real constants γ, δ with $0 < \gamma \leq \delta$, $n \in \mathbb{N}$, $\kappa \in \mathbb{Z}$ and Banach spaces $\mathcal{X}, \mathcal{Y}$ the following statements are valid:*

(a) *The Banach spaces $\ell^+_{\kappa,\gamma}(\mathcal{X}) \times \ell^+_{\kappa,\gamma}(\mathcal{Y})$ and $\ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ are isometrically isomorphic, and thus they will be identified.*

(b) *We have $\ell^+_{\kappa,\gamma}(\mathcal{X}) \overset{J^\delta_\gamma}{\hookrightarrow} \ell^+_{\kappa,\delta}(\mathcal{X})$, and the embedding operator $J^\delta_\gamma : \ell^+_{\kappa,\gamma}(\mathcal{X}) \to \ell^+_{\kappa,\delta}(\mathcal{X})$ satisfies*

$$\left\| J^\delta_\gamma \right\|_{\mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))} \leq 1. \tag{3.1}$$

(c) *The Banach spaces* $\ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X}))$ *and* $\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))$ *are isometrically isomorphic by means of the isomorphism* $J_n : \ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X})) \to \mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))$,

$$\big((J_n\Lambda)(\lambda_1,\dots,\lambda_n)\big)(k) := \Lambda(k)\lambda_1(k)\cdots\lambda_n(k) \quad \textit{for } k \in \mathbb{Z}^+_\kappa$$

for any $\Lambda \in \ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X}))$ *and* $\lambda_1,\dots,\lambda_n \in \ell^+_{\kappa,\gamma}(\mathcal{X})$.

Proof. We show only the assertion (c). At first J_n is linear. With arbitrary sequences $\Lambda \in \ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X}))$ and $\lambda_1,\dots,\lambda_n \in \ell^+_{\kappa,\gamma}(\mathcal{X})$ we obtain the estimate

$$\begin{aligned}\|\Lambda(k)\lambda_1(t)\cdots\lambda_n(k)\|\,\delta^{\kappa-k} &\le \|\Lambda(k)\|_{\mathcal{L}_n(\mathcal{X})}\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k}\|\lambda_1(k)\|\,\gamma^{\kappa-k}\cdots\|\lambda_n(k)\|\,\gamma^{\kappa-k} \le \\ &\le \|\Lambda\|^+_{\kappa,\frac{\delta}{\gamma^n}}\|\lambda_1\|^+_{\kappa,\gamma}\cdots\|\lambda_n\|^+_{\kappa,\gamma} \quad \text{for } k \in \mathbb{Z}^+_\kappa.\end{aligned}$$

Thus the continuity of J_n follows from

$$\|J_n\Lambda\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))} = \sup_{\substack{\|\lambda_l\|^+_{\kappa,\gamma}\le 1,\\ l\in\{1,\dots,n\}}}\|(J_n\Lambda)\lambda_1\cdots\lambda_n\|^+_{\kappa,\delta} \le \|\Lambda\|^+_{\kappa,\frac{\delta}{\gamma^n}}. \tag{3.2}$$

On the other hand, the inverse $J_n^{-1} : \mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X})) \to \ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X}))$ of J_n is given by

$$\big(J_n^{-1}\bar\Lambda\big)(k)\lambda_1(k)\cdots\lambda_n(k) := \big(\bar\Lambda\lambda_1\cdots\lambda_n\big)(k) \quad \text{for } k \in \mathbb{Z}^+_\kappa$$

for any $\bar\Lambda \in \mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))$ and $\lambda_1,\dots,\lambda_n \in \ell^+_{\kappa,\gamma}(\mathcal{X})$. By the open mapping theorem (cf. Lang [16, p. 388, Corollary 1.4]) J_n^{-1} is continuous and it remains to show that it is non-expanding. To this purpose we choose n arbitrary points $x_1,\dots,x_n \in \mathcal{X}\setminus\{0\}$ with $\|x_l\| \le 1$ ($l \in \{1,\dots,n\}$) and define sequences $\lambda_l(k) := \gamma^{k-\kappa}x_l$. Obviously $\|\lambda_l\|^+_{\kappa,\gamma} \le 1$ and hence

$$\begin{aligned}\big\|\big(J_n^{-1}\bar\Lambda\big)(k)x_1\cdots x_n\big\|\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k} &= \big\|\big(\bar\Lambda\lambda_1\cdots\lambda_n\big)(k)\big\|\,\delta^{\kappa-k} \le \big\|\bar\Lambda\lambda_1\cdots\lambda_n\big\|^+_{\kappa,\delta} \le \\ &\le \big\|\bar\Lambda\big\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))} \quad \text{for } k \in \mathbb{Z}^+_\kappa.\end{aligned}$$

Now this implies

$$\big\|\big(J_n^{-1}\bar\Lambda\big)(k)\big\|_{\mathcal{L}_n(\mathcal{X})}\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k} \le \big\|\bar\Lambda\big\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))} \quad \text{for } k \in \mathbb{Z}^+_\kappa$$

and hence $\big\|J_n^{-1}\bar\Lambda\big\|^+_{\kappa,\frac{\delta}{\gamma^n}} \le \big\|\bar\Lambda\big\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X});\ell^+_{\kappa,\delta}(\mathcal{X}))}$. Therefore J_n is an isometry. □

We close this section with a result about differentiability.

Lemma 3.4: *Consider constants* $\gamma > 0$, $\kappa \in \mathbb{Z}$, *Banach spaces* $\mathcal{X},\mathcal{Y}$ *and a mapping* $f \in C^m(\mathcal{X},\ell^+_{\kappa,\gamma}(\mathcal{Y}))$ *for some* $m \in \mathbb{N}_0$. *Then* $\big(f(\cdot)\big)(k) \in C^m(\mathcal{X},\mathcal{Y})$ *for every* $k \in \mathbb{Z}^+_\kappa$.

Proof. Let $k \in \mathbb{Z}^+_\kappa$ be fixed. Then the evaluation map $\mathrm{ev}_k : \ell^+_{\kappa,\gamma}(\mathcal{Y}) \to \mathcal{Y}$, $\mathrm{ev}_k(\lambda) := \lambda(k)$ is a continuous homomorphism and hence of class C^∞. It follows from the chain rule that the composition $\big(f(\cdot)\big)(k) = \mathrm{ev}_k \circ f$ has the same smoothness as f. □

4 Construction of invariant fiber bundles

We begin this section by stating our frequently used main assumptions.

Hypothesis 4.1: Let us consider the system of difference equations

$$\begin{cases} x' = A(k)x + F(k,x,y) \\ y' = B(k)y + G(k,x,y) \end{cases} \tag{4.1}$$

where $\mathcal{X}, \mathcal{Y}$ are Banach spaces, the discrete interval I is unbounded to the right, $A : I \to \mathcal{L}(\mathcal{X})$, $B : I \to \mathcal{GL}(\mathcal{Y})$ and the mappings $F : I \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, $G : I \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ are m-times ($m \in \mathbb{N}$) continuously differentiable with respect to (x, y). Moreover we assume:

(i) Hypothesis on linear part: The evolution operators Φ and Ψ of the linear systems $x' = A(k)x$ and $y' = B(k)y$ respectively, satisfy for all $k, l \in I$ the estimates

$$\begin{aligned} \|\Phi(k,l)\| &\leq K_1\alpha^{k-l} \quad \text{for } k \geq l, \\ \|\Psi(k,l)\| &\leq K_2\beta^{k-l} \quad \text{for } l \geq k, \end{aligned} \tag{4.2}$$

with real constants $K_1, K_2 \geq 1$ and α, β with $0 < \alpha < \beta$.

(ii) Hypothesis on perturbation: We have

$$F(k,0,0) \equiv 0, \qquad G(k,0,0) \equiv 0 \quad \text{on } I, \tag{4.3}$$

and the partial derivatives of F and G are globally bounded, i.e. for all $n \in \{1, \dots, m\}$ we have

$$\begin{aligned} |F|_n &:= \sup_{(k,x,y) \in I \times \mathcal{X} \times \mathcal{Y}} \left\| \frac{\partial^n F}{\partial(x,y)^n}(k,x,y) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{X})} < \infty, \\ |G|_n &:= \sup_{(k,x,y) \in I \times \mathcal{X} \times \mathcal{Y}} \left\| \frac{\partial^n G}{\partial(x,y)^n}(k,x,y) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{Y})} < \infty. \end{aligned} \tag{4.4}$$

(iii) Hypothesis on higher order smoothness (if $m \geq 2$): The partial derivatives of F and G are uniformly continuous: For any $\varepsilon > 0$ there exists an $\eta > 0$ such that for all $k \in I$ and $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ we have

$$\left\| \frac{\partial^m (F,G)}{\partial(x,y)^m}(k,x,y) - \frac{\partial^m (F,G)}{\partial(x,y)^m}(k,x_0,y_0) \right\|_{\mathcal{L}_m(\mathcal{X}\times\mathcal{Y})} < \varepsilon \quad \text{for } (x,y) \in B_\eta(x_0,y_0).$$

Remark 4.2: (1) It is an immediate consequence of the mean value theorem (see e.g. Lang [16, p. 341, Theorem 4.2]) and Hypothesis 4.1(ii) that the partial derivatives $\frac{\partial^n F}{\partial(x,y)^n}$, $\frac{\partial^n G}{\partial(x,y)^n}$ are globally Lipschitz continuous (with constants $|F|_n$, $|G|_n$, respectively) for $n \in \{0, \dots, m-1\}$, and hence Hypothesis 4.1(iii) also holds for $\frac{\partial^n F}{\partial(x,y)^n}$, $\frac{\partial^n G}{\partial(x,y)^n}$, with $n \in \{0, \dots, m\}$. Nevertheless Hypothesis 4.1(iii) is of technical nature and only needed for $m \geq 2$.

(2) In a Hilbert space $\mathcal{Z}$ and for a function $R : I \times \mathcal{Z} \to \mathcal{Z}$ with globally bounded derivatives $\frac{\partial^n R}{\partial z^n}$ any system of the form

$$z' = C(k)z + R(k, z)$$

can be transformed into the "decoupled" form (4.1) if the operator $C(k) \in \mathcal{GL}(\mathcal{Z})$ possesses an exponential dichotomy. This can be shown using methods from Aulbach, Van Minh & Zabreiko [5, Theorem 5] via a Lyapunov transformation.

In the sequel we define two linear and two non-linear operators on spaces of quasibounded functions and derive their basic properties concerning continuity and differentiability. This allows to characterize the quasibounded solutions of (4.1) as fixed points of an equation based on these operators.

Lemma 4.3 (the operator $\boldsymbol{\mathcal{S}_\kappa}$)**:** *We assume $I = \mathbb{Z}^+_{\kappa_0}$ ($\kappa_0 \in \mathbb{Z}$) in Hypothesis 4.1 and choose any $\gamma \geq \alpha$. Then for any $\kappa \in \mathbb{Z}^+_{\kappa_0}$ the operator $\mathcal{S}_\kappa : \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$,*

$$\big(\mathcal{S}_\kappa\, \xi\big)(k) := \big(\Phi(k, \kappa)\xi, 0\big) \quad \text{for } k \in \mathbb{Z}^+_\kappa,$$

is linear and continuous with

$$\|\mathcal{S}_\kappa\|_{\mathcal{L}(\mathcal{X};\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}))} \leq K_1. \tag{4.5}$$

Proof. The linearity of $\mathcal{S}_\kappa$ is evident. Furthermore for any $\xi \in \mathcal{X}$ we get the estimate

$$\big\|\big(\mathcal{S}_\kappa\xi\big)(k)\big\|\, \gamma^{\kappa-k} \leq \|\Phi(k, \kappa)\|\, \|\xi\|\, \gamma^{\kappa-k} \overset{(4.2)}{\leq} K_1\, \|\xi\| \quad \text{for } k \in \mathbb{Z}^+_\kappa,$$

hence $\|\mathcal{S}_\kappa\xi\|^+_{\kappa,\gamma} \leq K_1\, \|\xi\|$, and $\mathcal{S}_\kappa\xi$ is well-defined, continuous and (4.5) holds. □

Lemma 4.4 (the operator $\boldsymbol{\mathcal{K}_\kappa}$)**:** *We assume $I = \mathbb{Z}^+_{\kappa_0}$ ($\kappa_0 \in \mathbb{Z}$) in Hypothesis 4.1 and choose an arbitrary $\gamma \in (\alpha, \beta)$. Then for any $\kappa \in \mathbb{Z}^+_{\kappa_0}$ the mapping $\mathcal{K}_\kappa : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$,*

$$\big(\mathcal{K}_\kappa(\mu, \nu)\big)(k) := \left(\sum_{n=\kappa}^{k-1} \Phi(k, n+1)\mu(n),\ -\sum_{n=k}^{\infty} \Psi(k, n+1)\nu(n)\right) \quad \text{for } k \in \mathbb{Z}^+_\kappa,$$

is linear and continuous with

$$\|\mathcal{K}_\kappa\|_{\mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}))} \leq \max\left\{\frac{K_1}{\gamma - \alpha}, \frac{K_2}{\beta - \gamma}\right\}. \tag{4.6}$$

In particular we get

$$\|\Pi_2\mathcal{K}_\kappa(\mu, \nu)\|^+_{\kappa,\gamma} \leq \frac{K_2}{\beta - \gamma}\, \|\nu\|^+_{\kappa,\gamma} \quad \text{for } (\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}). \tag{4.7}$$

Proof. Obviously $\mathcal{K}_\kappa$ is linear. Now choose any pair $(\mu,\nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$. Then by the variation of constants formula the inhomogeneous difference equation $x' = A(k)x + \mu(k)$ has the solution $\tilde{\mu} := \Pi_1\mathcal{K}_\kappa(\mu,\nu) : \mathbb{Z}^+_\kappa \to \mathcal{X}$, satisfying the initial condition $x(\kappa) = 0$. Because of AULBACH [3, Lemma 3.3] the function $\tilde{\mu}$ is γ^+-quasibounded and we get

$$\|\Pi_1\mathcal{K}_\kappa(\mu,\nu)\|^+_{\kappa,\gamma} = \|\tilde{\mu}\|^+_{\kappa,\gamma} \le \frac{K_1}{\gamma-\alpha}\,\|\mu\|^+_{\kappa,\gamma}\,.$$

Similarly AULBACH [3, Lemma 3.4] implies that the linear system $y' = B(k)y + \nu(k)$ has exactly one γ^+-quasibounded solution, namely $\tilde{\nu} := \Pi_2\mathcal{K}_\kappa(\mu,\nu) : \mathbb{Z}^+_\kappa \to \mathcal{Y}$, and this solution satisfies the estimate

$$\|\Pi_2\mathcal{K}_\kappa(\mu,\nu)\|^+_{\kappa,\gamma} = \|\tilde{\nu}\|^+_{\kappa,\gamma} \le \frac{K_2}{\beta-\gamma}\,\|\nu\|^+_{\kappa,\gamma}\,,$$

which is identical with inequality (4.7). Finally we get

$$\begin{aligned}\big\|\big(\mathcal{K}_\kappa(\mu,\nu)\big)(k)\big\|\,\gamma^{\kappa-k} &\overset{(2.3)}{=} \max\left\{\|\tilde{\mu}(k)\|\,\gamma^{\kappa-k}, \|\tilde{\nu}(k)\|\,\gamma^{\kappa-k}\right\} \le \\ &\le \max\left\{\frac{K_1}{\gamma-\alpha}\,\|\mu\|^+_{\kappa,\gamma}\,, \frac{K_2}{\beta-\gamma}\,\|\nu\|^+_{\kappa,\gamma}\right\} \le \\ &\overset{(2.3)}{\le} \max\left\{\frac{K_1}{\gamma-\alpha}, \frac{K_2}{\beta-\gamma}\right\}\|(\mu,\nu)\|^+_{\kappa,\gamma} \quad \text{for } k \in \mathbb{Z}^+_\kappa.\end{aligned}$$

Passing to the least upper bound over $k \in \mathbb{Z}^+_\kappa$ we obtain the estimate

$$\|\mathcal{K}_\kappa(\mu,\nu)\|^+_{\kappa,\gamma} \le \max\left\{\frac{K_1}{\gamma-\alpha}, \frac{K_2}{\beta-\gamma}\right\}\|(\mu,\nu)\|^+_{\kappa,\gamma}$$

and hence the continuity of $\mathcal{K}_\kappa$, as well as the estimate (4.6). □

The two mappings $\mathcal{S}_\kappa$ and $\mathcal{K}_\kappa$ defined in Lemmas 4.3 and 4.4 are continuous homomorphisms and hence continuously differentiable. The non-linear mapping $\mathcal{G}$ which we define next does not have this property in general. This mapping describes the composition of the non-linearities F and G with quasibounded functions and is a special case of a so-called *substitution* or *Nemitskii operator*. Even though there is a vast literature on such operators (see e.g. APPELL & ZABREIKO [2] and the references therein) we have to derive two lemmas in order to meet the particular purposes of this paper.

Lemma 4.5 (the Operator $\boldsymbol{\mathcal{G}}$)**:** *We assume $I = \mathbb{Z}^+_{\kappa_0}$ $(\kappa_0 \in \mathbb{Z})$ in Hypothesis 4.1, choose $\gamma > 0$ and $\kappa \in \mathbb{Z}^+_{\kappa_0}$. Then the non-linear mapping $\mathcal{G} : \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}) \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$,*

$$\big(\mathcal{G}(\mu,\nu)\big)(k) := \big(F(k,\mu(k),\nu(k)), G(k,\mu(k),\nu(k))\big) \quad \textit{for } k \in \mathbb{Z}^+_\kappa,$$

has the following properties:

(a) $\mathcal{G}(0,0) = (0,0) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$,

(b) *$\mathcal{G}$ is globally Lipschitz continuous with*

$$\|\mathcal{G}(\mu,\nu)-\mathcal{G}(\bar{\mu},\bar{\nu})\|_{\kappa,\gamma}^{+} \leq \max\{|F|_1,|G|_1\}\,\|(\mu,\nu)-(\bar{\mu},\bar{\nu})\|_{\kappa,\gamma}^{+} \tag{4.8}$$

for all $(\mu,\nu),(\bar{\mu},\bar{\nu}) \in \ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y})$.

Proof. (a) Because of (4.3) in Hypothesis 4.1(ii) we obtain statement (a).
(b) For arbitrary $(\mu,\nu),(\bar{\mu},\bar{\nu}) \in \ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y})$ it follows from the mean value theorem that

$$\begin{aligned}
&\left\|\big(\mathcal{G}(\mu,\nu)\big)(k)-\big(\mathcal{G}(\bar{\mu},\bar{\nu})\big)(k)\right\|\gamma^{\kappa-k} \leq \\
\stackrel{(4.4)}{\leq}\;& \max\left\{|F|_1\,\|(\mu,\nu)(k)-(\bar{\mu},\bar{\nu})(k)\|\,\gamma^{\kappa-k},|G|_1\,\|(\mu,\nu)(k)-(\bar{\mu},\bar{\nu})(k)\|\,\gamma^{\kappa-k}\right\} \leq \\
\leq\;& \max\{|F|_1,|G|_1\}\,\|(\mu,\nu)-(\bar{\mu},\bar{\nu})\|_{\kappa,\gamma}^{+} \quad \text{for } k \in \mathbb{Z}_{\kappa}^{+}.
\end{aligned}$$

The Lipschitz condition (4.8) is obtained by taking the least upper bound over $k \in \mathbb{Z}_{\kappa}^{+}$ in this estimate. Setting $(\bar{\mu},\bar{\nu}) := (0,0)$ together with statement (a) implies that the operator $\mathcal{G}$ is well-defined, i.e. $\mathcal{G}(\mu,\nu) \in \ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y})$. □

Lemma 4.6 (the operator $\boldsymbol{\mathcal{G}^{(n)}}$)**:** *We assume $I=\mathbb{Z}_{\kappa_0}^{+}$ $(\kappa_0\in\mathbb{Z})$ in Hypothesis 4.1, choose integers $n\in\{1,\ldots,m\}$, $\kappa\in\mathbb{Z}_{\kappa_0}^{+}$ and reals γ,δ such that $0<\gamma\leq\delta$ and $\gamma^n\leq\delta$. Then the mapping $\mathcal{G}^{(n)}:\ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y})\to\ell_{\kappa,\frac{\delta}{\gamma^n}}^{+}(\mathcal{L}_n(\mathcal{X}\times\mathcal{Y}))$,*

$$\big(\mathcal{G}^{(n)}(\mu,\nu)\big)(k) := \frac{\partial^n(F,G)}{\partial(x,y)^n}(k,\mu(k),\nu(k)) \quad \textit{for } k\in\mathbb{Z}_{\kappa}^{+},$$

has the following properties:

(a) *It is well-defined and globally bounded with*

$$\left\|\mathcal{G}^{(n)}(\mu,\nu)\right\|_{\kappa,\frac{\delta}{\gamma^n}}^{+} \leq \max\{|F|_n,|G|_n\} \quad \textit{for } (\mu,\nu)\in\ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y}), \tag{4.9}$$

(b) *for $\gamma<\delta$ and $n=1$ the mapping $\mathcal{G}^{(1)}$ is continuous,*

(c) *for $\gamma\leq 1$ and $m\geq 2$ the mapping $\mathcal{G}^{(n)}$ is continuous as well.*

Proof. (a) For arbitrary functions $(\mu,\nu)\in\ell_{\kappa,\gamma}^{+}(\mathcal{X}\times\mathcal{Y})$ we get

$$\begin{aligned}
\left\|\big(\mathcal{G}^{(n)}(\mu,\nu)\big)(k)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})}\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k} &\leq \left\|\frac{\partial^n(F,G)}{\partial(x,y)^n}(k,\mu(k),\nu(k))\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} \leq \\
&\stackrel{(4.4)}{\leq} \max\{|F|_n,|G|_n\} \quad \text{for } k\in\mathbb{Z}_{\kappa}^{+},
\end{aligned}$$

since $\frac{\delta}{\gamma^n}\geq 1$. Therefore we have $\mathcal{G}^{(n)}(\mu,\nu)\in\ell_{\kappa,\frac{\delta}{\gamma^n}}^{+}(\mathcal{L}_n(\mathcal{X}\times\mathcal{Y}))$ and the estimate (4.9) holds.

(b) To prove the continuity of $\mathcal{G}^{(1)}$ under the assumption $\gamma < \delta$ we choose $\varepsilon > 0$ and $(\mu_0, \nu_0) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ arbitrarily, but fixed. Since $\frac{\gamma}{\delta} < 1$, there exists a $K \in \mathbb{Z}^+_{\kappa+1}$ with $2 \max\{|F|_1, |G|_1\} \left(\frac{\gamma}{\delta}\right)^{K-\kappa} < \frac{\varepsilon}{2}$. Using the triangle inequality we get

$$\begin{aligned} &\left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right)(k) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)(k)\right\| \left(\frac{\delta}{\gamma}\right)^{\kappa-k} \leq \\ \leq\ & \left(\left\|\frac{\partial(F,G)}{\partial(x,y)}(k,\mu(k),\nu(k))\right\| + \left\|\frac{\partial(F,G)}{\partial(x,y)}(k,\mu_0(k),\nu_0(k))\right\|\right) \left(\frac{\delta}{\gamma}\right)^{\kappa-k} \leq \\ \overset{(4.4)}{\leq}\ & 2\max\{|F|_1, |G|_1\} \left(\frac{\delta}{\gamma}\right)^{\kappa-K} < \frac{\varepsilon}{2} \end{aligned}$$

for $k > K$ and all $(\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$. Since the partial derivative $\frac{\partial(F,G)}{\partial(x,y)}$ is continuous, there exists a constant $\delta_1 = \delta_1(\varepsilon) > 0$ such that for $(\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ the estimate

$$\|(\mu,\nu)(k) - (\mu_0,\nu_0)(k)\| < \delta_1 \quad \text{for } k \in \{\kappa, \kappa+1, \ldots, K\}$$

implies

$$\left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right)(k) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)(k)\right\| < \frac{\varepsilon}{2} \quad \text{for } k \in \{\kappa, \kappa+1, \ldots, K\}.$$

Besides one gets

$$\begin{aligned} &\|(\mu,\nu)(k) - (\mu_0,\nu_0)(k)\| \leq \gamma^{k-\kappa} \|(\mu,\nu) - (\mu_0,\nu_0)\|^+_{\kappa,\gamma} \leq \\ \leq\ & \max\left\{1, \gamma^{K-\kappa}\right\} \|(\mu,\nu) - (\mu_0,\nu_0)\|^+_{\kappa,\gamma} < \delta_1 \quad \text{for } k \in \{\kappa, \kappa+1, \ldots, K\}, \end{aligned}$$

for every (μ, ν) with $\|(\mu,\nu) - (\mu_0,\nu_0)\|^+_{\kappa,\gamma} < \delta_2 := \frac{\delta_1}{\max\{1,\gamma^{K-\kappa}\}}$. For such pairs of γ^+-quasibounded functions $(\mu, \nu) \in B_{\delta_2}(\mu_0, \nu_0)$ one has

$$\begin{aligned} &\left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)\right\|^+_{\kappa,\frac{\delta}{\gamma}} = \\ =\ & \sup_{k\in\mathbb{Z}^+_\kappa} \left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right)(k) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)(k)\right\| \left(\frac{\delta}{\gamma}\right)^{\kappa-k} \leq \\ \leq\ & \max\left\{ \sup_{k\in\{\kappa,\ldots,K\}} \left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right)(k) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)(k)\right\|, \right. \\ & \left. \sup_{k>K} \left\|\left(\mathcal{G}^{(1)}(\mu,\nu)\right)(k) - \left(\mathcal{G}^{(1)}(\mu_0,\nu_0)\right)(k)\right\| \left(\frac{\delta}{\gamma}\right)^{\kappa-k} \right\} \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This proves the continuity of the operator $\mathcal{G}^{(1)}$.

(c) Choose $\varepsilon > 0$ and $(\mu_0, \nu_0) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ arbitrarily to prove the continuity of $\mathcal{G}^{(n)}$ in case $\gamma \leq 1$ and $m \geq 2$. By means of Hypothesis 4.1(iii) and the definition of the operator $\mathcal{G}^{(n)}$ there exists an $\eta = \eta(\varepsilon) > 0$ such that the estimate

$$\|(\mu,\nu)(k) - (\mu_0,\nu_0)(k)\| < \eta \quad \text{for } k \in \mathbb{Z}^+_\kappa$$

implies

$$\left\|\big(\mathcal{G}^{(n)}(\mu,\nu)\big)(k)-\big(\mathcal{G}^{(n)}(\mu_0,\nu_0)\big)(k)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})}\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k}\leq$$
$$\leq \left\|\big(\mathcal{G}^{(n)}(\mu,\nu)\big)(k)-\big(\mathcal{G}^{(n)}(\mu_0,\nu_0)\big)(k)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})}<\frac{\varepsilon}{2}\quad\text{for } k\in\mathbb{Z}_\kappa^+$$

for arbitrary $(\mu,\nu)\in\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$, since $\frac{\delta}{\gamma^n}\geq 1$. Moreover we get

$$\begin{aligned}\|(\mu,\nu)(k)-(\mu_0,\nu_0)(k)\| &\leq \gamma^{k-\kappa}\,\|(\mu,\nu)-(\mu_0,\nu_0)\|^+_{\kappa,\gamma}\leq\\ &\leq \|(\mu,\nu)-(\mu_0,\nu_0)\|^+_{\kappa,\gamma}\quad\text{for } k\in\mathbb{Z}^+_\kappa.\end{aligned}$$

Taking $(\mu,\nu)\in B_\eta(\mu_0,\nu_0)\subseteq\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ our assertion follows because we have

$$\left\|\mathcal{G}^{(n)}(\mu,\nu)-\mathcal{G}^{(n)}(\mu_0,\nu_0)\right\|^+_{\kappa,\frac{\delta}{\gamma^n}}=$$
$$=\sup_{k\in\mathbb{Z}^+_\kappa}\left\|\big(\mathcal{G}^{(n)}(\mu,\nu)\big)(k)-\big(\mathcal{G}^{(n)}(\mu_0,\nu_0)\big)(k)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})}\left(\frac{\delta}{\gamma^n}\right)^{\kappa-k}\leq\frac{\varepsilon}{2}<\varepsilon.$$

Therefore Lemma 4.6 is proved. □

Now the question arises whether the assumption $\gamma<\delta$ in statement (b) of Lemma 4.6 is of a purely technical nature. In fact, one cannot get rid of it because γ^+-quasibounded functions can be unbounded. We now give an example which demonstrates that the non-linear operator $\mathcal{G}^{(1)}$ may not be continuous for $\gamma=\delta>1$.

Example 4.7: We consider the continuously differentiable function $\varphi:\mathbb{R}\to\mathbb{R}$ defined as $\varphi(t):=t-\arctan t$. Obviously the derivative $\dot\varphi(t)=\frac{t^2}{1+t^2}$ is globally bounded on $\mathbb{R}$ by the constant 1. Then the non-linearities

$$F(k,x,y):=\varphi(x),\qquad\qquad G(k,x,y):=\varphi(y)$$

satisfy Hypothesis 4.1(ii) for $m=1$. Now we choose $\gamma=\delta>1$ and $\kappa\in\mathbb{Z}$. For the continuity of $\mathcal{G}^{(1)}:\ell^+_{\kappa,\gamma}(\mathbb{R}^2)\to\ell^+_{\kappa,1}(\mathbb{R}^{2\times 2})$ it is necessary and sufficient that the “scalar” operator $\tilde{\mathcal{G}}^{(1)}:\ell^+_{\kappa,\gamma}(\mathbb{R})\to\ell^+_{\kappa,1}(\mathbb{R})$, $\big(\tilde{\mathcal{G}}^{(1)}(\mu)\big)(k):=\dot\varphi(\mu(k))$ is continuous. Now $\mu(k):=\gamma^{k-\kappa}$ defines a γ^+-quasibounded sequence with the γ^+-norm $\|\mu\|^+_{\kappa,\gamma}=1$. Hence $\mu_n:=\gamma^{\kappa-n}\mu$ $(n\in\mathbb{Z}^+_\kappa)$ converges to zero in the linear space $\ell^+_{\kappa,\gamma}(\mathbb{R})$, because of $\gamma>1$. But

$$\left\|\tilde{\mathcal{G}}^{(1)}(\mu_n)\right\|^+_{\kappa,1}=\sup_{k\in\mathbb{Z}^+_\kappa}\left|\dot\varphi(\gamma^{k-n})\right|=\sup_{k\in\mathbb{Z}^+_\kappa}\frac{\gamma^{2(k-n)}}{1+\gamma^{2(k-n)}}\geq\frac{1}{2}\quad\text{for } n\in\mathbb{Z}^+_\kappa.$$

Consequently $(\tilde{\mathcal{G}}^{(1)}(\mu_n))_{n\in\mathbb{Z}^+_\kappa}$ cannot converge to $0=\tilde{\mathcal{G}}^{(1)}(0)\in\ell^+_{\kappa,1}(\mathbb{R})$ and therefore $\mathcal{G}^{(1)}$ is not continuous at the point 0.

Next we investigate the differentiability of $\mathcal{G}$. It will turn out that not only the smoothness of the functions F and G is essential but also the particular choice of the spaces of quasibounded sequences as domain and range of $\mathcal{G}$. As candidates for the derivatives of the substitution operator $\mathcal{G}$ the mappings $\mathcal{G}^{(n)}$ seem to be a good choice since they are defined with the aid of the derivatives of the mappings (F,G).

Lemma 4.8 (continuous differentiability of $\mathcal{G}$)**:** *We assume $I = \mathbb{Z}^+_{\kappa_0}$ ($\kappa_0 \in \mathbb{Z}$) in Hypothesis 4.1, choose reals γ, δ with $0 < \gamma \leq \delta$ and an integer $\kappa \in \mathbb{Z}^+_{\kappa_0}$. Then the operator $\mathcal{G}^{(0)} := J^\delta_\gamma \mathcal{G} : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \ell^+_{\kappa,\delta}(\mathcal{X} \times \mathcal{Y})$ has the following properties:*

(a) For $\gamma < \delta$ it is continuously differentiable,

(b) for $\gamma \leq 1$ and $m \geq 2$ it is m-times continuously differentiable.

In any case and for any $n \in \{1, \dots, m\}$ the derivatives are given by

$$D^n \mathcal{G}^{(0)} = J_n \mathcal{G}^{(n)} : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}); \ell^+_{\kappa,\delta}(\mathcal{X} \times \mathcal{Y})), \tag{4.10}$$

and they are globally bounded with

$$\left\| \left(D^n \mathcal{G}^{(0)}\right)(\mu, \nu) \right\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}); \ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y}))} \leq \max\left\{ |F|_n, |G|_n \right\} \quad \text{for } (\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}).$$

Proof. We start with some preparations. To this end consider two arbitrary sequences $(\mu, \nu), (\mu_0, \nu_0) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$. Now we keep the pair (μ_0, ν_0) fixed and define the mappings $r_n : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}^+_0$ for $n \in \{1, \dots, m-1\}$ by

$$r_n(\mu, \nu) := \sup_{t \in [0,1]} \left\| \mathcal{G}^{(n)}(\mu_0 + t\mu, \nu_0 + t\nu) - \mathcal{G}^{(n)}(\mu_0, \nu_0) \right\|^+_{\kappa, \frac{\delta}{\gamma^n}}.$$

By Lemma 3.3(c), $\mathcal{G}^{(n)}(\mu_0, \nu_0)$ can be considered as a sequence in $\ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}))$ as well as a n-linear mapping in $\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}); \ell^+_{\kappa,\delta}(\mathcal{X} \times \mathcal{Y}))$. The mean value theorem implies for $n \in \{0, \dots, m-1\}$ the estimate

$$\begin{aligned}
&\left\| \mathcal{G}^{(n)}(\mu_0 + \mu, \nu_0 + \nu) - \mathcal{G}^{(n)}(\mu_0, \nu_0) - \mathcal{G}^{(n+1)}(\mu_0, \nu_0)(\mu, \nu) \right\|^+_{\kappa, \frac{\delta}{\gamma^n}} = \\
&= \sup_{k \in \mathbb{Z}^+_\kappa} \left\| \frac{\partial^n (F,G)}{\partial(x,y)^n}(k, \mu_0(k) + \mu(k), \nu_0(k) + \nu(k)) - \frac{\partial^n (F,G)}{\partial(x,y)^n}(k, \mu_0(k), \nu_0(k)) - \right. \\
&\quad \left. - \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k, \mu_0(k), \nu_0(k)) \begin{pmatrix} \mu(k) \\ \nu(k) \end{pmatrix} \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} \left(\frac{\delta}{\gamma^n} \right)^{\kappa - k} = \\
&= \sup_{k \in \mathbb{Z}^+_\kappa} \left\| \left(\int_0^1 \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k, \mu_0(k) + t\mu(k), \nu_0(k) + t\nu(k))\, dt \right) \begin{pmatrix} \mu(k) \\ \nu(k) \end{pmatrix} - \right. \\
&\quad \left. - \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k, \mu_0(k), \nu_0(k)) \begin{pmatrix} \mu(k) \\ \nu(k) \end{pmatrix} \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} \left(\frac{\delta}{\gamma^n} \right)^{\kappa - k} \leq \\
&\leq \sup_{k \in \mathbb{Z}^+_\kappa} \int_0^1 \left\| \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k, \mu_0(k) + t\mu(k), \nu_0(k) + t\nu(k)) - \right. \\
&\quad \left. - \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k, \mu_0(k), \nu_0(k)) \right\|_{\mathcal{L}_{n+1}(\mathcal{X}\times\mathcal{Y})} dt \left(\frac{\delta}{\gamma^{n+1}} \right)^{\kappa - k} \|(\mu(k), \nu(k))\| \gamma^{\kappa - k}.
\end{aligned}$$

Estimating the integral we get

$$\left\| \mathcal{G}^{(n)}(\mu_0 + \mu, \nu_0 + \nu) - \mathcal{G}^{(n)}(\mu_0, \nu_0) - \mathcal{G}^{(n+1)}(\mu_0, \nu_0)(\mu, \nu) \right\|^+_{\kappa, \frac{\delta}{\gamma^n}} \leq$$

$$
\begin{aligned}
&\leq \sup_{k\in\mathbb{Z}_\kappa^+}\sup_{t\in[0,1]}\Bigg\| \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k,\mu_0(k)+t\mu(k),\nu_0(k)+t\nu(k)) - \\
&\quad - \frac{\partial^{n+1}(F,G)}{\partial(x,y)^{n+1}}(k,\mu_0(k),\nu_0(k))\Bigg\|_{\mathcal{L}_{n+1}(\mathcal{X}\times\mathcal{Y})}\left(\frac{\delta}{\gamma^{n+1}}\right)^{\kappa-k}\|(\mu(k),\nu(k))\|\,\gamma^{\kappa-k} \leq \\
&\leq \sup_{t\in[0,1]}\left\|\mathcal{G}^{(n+1)}(\mu_0+t\mu,\nu_0+t\nu)-\mathcal{G}^{(n+1)}(\mu_0,\nu_0)\right\|^+_{\kappa,\frac{\delta}{\gamma^{n+1}}}\|(\mu,\nu)\|^+_{\kappa,\gamma} = \\
&= r_{n+1}(\mu,\nu)\,\|(\mu,\nu)\|^+_{\kappa,\gamma}\,.
\end{aligned}
$$

(a) In case $\gamma<\delta$ the operator $\mathcal{G}^{(1)}:\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})\to\mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y});\ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y}))$ is continuous by Lemma 4.6(b), hence the mapping r_1 is well-defined and we get $\lim_{(\mu,\nu)\to(0,0)} r_1(\mu,\nu)=0$. Now the above estimate for $n=0$ shows the differentiability of $\mathcal{G}^{(0)}=J^\delta_\gamma\mathcal{G}$ in any point $(\mu_0,\nu_0)\in\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ with continuous derivative $\mathcal{G}^{(1)}$.

(b) For reals $\gamma\leq 1$ and integers $m\geq 2$, $n\in\{0,\ldots,m-1\}$ obviously $\gamma^{1+n}\leq\delta$ holds. Therefore the operators $\mathcal{G}^{(n+1)}:\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})\to\mathcal{L}_{n+1}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y});\ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y}))$ are continuous by Lemma 4.6(c) and the mappings r_{n+1} are well-defined and fulfill $\lim_{(\mu,\nu)\to(0,0)} r_{n+1}(\mu,\nu)=0$. Furthermore the above estimate shows again that each mapping $\mathcal{G}^{(n)}:\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})\to\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y});\ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y}))$ is differentiable and has the derivative $\mathcal{G}^{(n+1)}$. Now the assertion follows by mathematical induction. Finally we get the estimate

$$
\begin{aligned}
&\left\|\left(D^n\mathcal{G}^{(0)}\right)(\mu,\nu)\right\|_{\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y});\ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y}))}\leq \\
&\overset{(4.10)}{\leq} \|J_n\|_{\mathcal{L}(\ell^+_{\kappa,\frac{\delta}{\gamma^n}}(\mathcal{L}_n(\mathcal{X}\times\mathcal{Y}));\mathcal{L}_n(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y});\ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y})))}\left\|\mathcal{G}^{(n)}(\mu,\nu)\right\|^+_{\kappa,\frac{\delta}{\gamma^n}}\leq \\
&\overset{(3.2)}{\leq} \left\|\mathcal{G}^{(n)}(\mu,\nu)\right\|^+_{\kappa,\frac{\delta}{\gamma^n}}\overset{(4.9)}{\leq}\max\left\{|F|_n,|G|_n\right\}\quad\text{for } n\in\{1,\ldots,m\},
\end{aligned}
$$

and the proof of Lemma 4.8 is complete. □

We already pointed out in Example 4.7 that for $\gamma=\delta>1$ the continuity of the mapping $\mathcal{G}^{(1)}$ may get lost. In fact, one cannot even expect the differentiability of the operator $\mathcal{G}=\mathcal{G}^{(0)}$. Nonlinearities demonstrating this can be obtained by modifying an example from Siegmund [21, pp. 35–38] to the case of difference equations.

Example 4.9: The real-valued functions

$$
F(k,x,y):=\sin(x),\qquad\qquad G(k,x,y):=\sin(y)
$$

satisfy the Hypothesis 4.1(ii) for $m\in\mathbb{N}$. We consider the situation $\gamma=\delta>1$. For a fixed integer $\kappa\in\mathbb{Z}$ the differentiability of $\mathcal{G}:\ell^+_{\kappa,\gamma}(\mathbb{R}^2)\to\ell^+_{\kappa,\gamma}(\mathbb{R}^2)$ is equivalent to the existence of the derivative of $\tilde{\mathcal{G}}:\ell^+_{\kappa,\gamma}(\mathbb{R})\to\ell^+_{\kappa,\gamma}(\mathbb{R})$, $\big(\tilde{\mathcal{G}}(\mu)\big)(k):=\sin\mu(k)$ in each point $\mu\in\ell^+_{\kappa,\gamma}(\mathbb{R})$. We now assume that this operator $\tilde{\mathcal{G}}$ is differentiable. Then there exists a function $r:\ell^+_{\kappa,\gamma}(\mathbb{R})\to\mathbb{R}^+_0$ with $\lim_{\mu\to 0} r(\mu)=0$ such that for any $\vartheta>\gamma$ and $\mu,\mu_0\in\ell^+_{\kappa,\gamma}(\mathbb{R})$ we have

$$
\left\|\tilde{\mathcal{G}}(\mu_0+\mu)-\tilde{\mathcal{G}}(\mu_0)-(D\tilde{\mathcal{G}})(\mu_0)\mu\right\|^+_{\kappa,\vartheta}\overset{(3.1)}{\leq}\left\|\tilde{\mathcal{G}}(\mu_0+\mu)-\tilde{\mathcal{G}}(\mu_0)-(D\tilde{\mathcal{G}})(\mu_0)\mu\right\|^+_{\kappa,\gamma}\leq
$$

$$\leq \; r(\mu) \left\| \mu \right\|^+_{\kappa,\gamma} .$$

Now a similar argument as in Lemma 4.8 yields

$$\big((D\tilde{\mathcal{G}})(\mu_0)\big)(k) = \cos \mu_0(k);$$

here we have identified $\mathbb{R}$ with $\mathcal{L}(\mathbb{R})$. Obviously the sequence $\mu_0(k) := 2\pi \left[\gamma^{k-\kappa}\right]$ is γ^+-quasibounded with $\|\mu_0\|^+_{\kappa,\gamma} \leq 2\pi$, where $[\cdot]$ denotes the greatest integer function. Additionally $\mu_n := \frac{1}{n}\mu_0$ $(n \in \mathbb{N})$ converges to zero in $\ell^+_{\kappa,\gamma}(\mathbb{R})$. Using $\gamma > 1$, for every $n \in \mathbb{N}$ one can choose integers $k(n) := \min\left\{k \in \mathbb{Z}^+_\kappa : \gamma^{k-\kappa} \geq 2 + \frac{n}{\pi}\right\}$. Consequently with elementary properties of the trigonometric functions we get

$$\begin{aligned}
&\left\| \tilde{\mathcal{G}}(\mu_0 + \mu_n) - \tilde{\mathcal{G}}(\mu_0) - (D\tilde{\mathcal{G}})(\mu_0)\mu_n \right\|^+_{\kappa,\gamma} = \\
&= \sup_{k \in \mathbb{Z}^+_\kappa} \left| \sin\left(2\pi\left[\gamma^{k-\kappa}\right](1+\frac{1}{n})\right) - \sin(2\pi\left[\gamma^{k-\kappa}\right]) - 2\pi\cos(2\pi\left[\gamma^{k-\kappa}\right])\frac{\left[\gamma^{k-\kappa}\right]}{n} \right| \gamma^{\kappa-k} = \\
&= \sup_{k \in \mathbb{Z}^+_\kappa} \left| \sin(\frac{2\pi\left[\gamma^{k-\kappa}\right]}{n}) - \frac{2\pi\left[\gamma^{k-\kappa}\right]}{n} \right| \gamma^{\kappa-k} \geq \\
&\geq \frac{2\pi\left[\gamma^{k(n)-\kappa}\right]}{n}\gamma^{\kappa-k(n)} - \sin(\frac{2\pi\left[\gamma^{k(n)-\kappa}\right]}{n})\gamma^{\kappa-k(n)} \geq \\
&\geq \frac{2\pi}{n}\left[\gamma^{k(n)-\kappa}\right]\gamma^{\kappa-k(n)} - \gamma^{\kappa-k(n)} \geq \frac{2\pi}{n}(1-\gamma^{\kappa-k(n)}) - \gamma^{\kappa-k(n)} \geq \frac{\pi}{n}.
\end{aligned}$$

But this yields the contradiction

$$\pi \leq n \left\| \tilde{\mathcal{G}}(\mu_0 + \mu_n) - \tilde{\mathcal{G}}(\mu_0) - (D\tilde{\mathcal{G}})(\mu_0)\mu_n \right\|^+_{\kappa,\gamma} \leq r(\mu_n) \left\|\mu_0\right\|^+_{\kappa,\gamma},$$

since the sequence $(r(\mu_n))_{n\in\mathbb{N}}$ converges to zero.

In the previous last lemmas we investigated the linear mappings $\mathcal{S}_\kappa$, $\mathcal{K}_\kappa$ and the more subtle substitution operator $\mathcal{G}$. With the help of these mappings we can now characterize the quasibounded solutions of the difference equation (4.1) quite easily as fixed points.

Lemma 4.10 (solutions in $\boldsymbol{\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})}$ as fixed points)**:** *Let us assume $I = \mathbb{Z}^+_{\kappa_0}$ $(\kappa_0 \in \mathbb{Z})$ in Hypothesis 4.1, choose constants $\gamma \in (\alpha, \beta)$, $\kappa \in \mathbb{Z}^+_{\kappa_0}$ and a vector $\xi \in \mathcal{X}$. Then for the mapping $\mathcal{T}_\kappa : \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}) \times \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$,*

$$\mathcal{T}_\kappa(\mu,\nu;\xi) := \mathcal{S}_\kappa\xi + \mathcal{K}_\kappa\mathcal{G}(\mu,\nu), \tag{4.11}$$

the following two statements are equivalent:

(a) $(\mu^*,\nu^*) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ is a solution of the difference equation (4.1) *with $\mu^*(\kappa) = \xi$,*

(b) $(\mu^*,\nu^*) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ is a solution of the equation

$$(\mu^*,\nu^*) = \mathcal{T}_\kappa(\mu^*,\nu^*;\xi). \tag{4.12}$$

Proof. Lemmas 4.3, 4.4 and 4.5 imply that $\mathcal{T}_\kappa$ is well-defined and may be written explicitly as

$$\big(\mathcal{T}_\kappa(\mu^*,\nu^*;\xi)\big)(k) = \Bigg(\Phi(k,\kappa)\xi + \sum_{n=\kappa}^{k-1}\Phi(k,n+1)F(n,\mu^*(n),\nu^*(n)), \\ -\sum_{n=k}^{\infty}\Psi(k,n+1)G(n,\mu^*(n),\nu^*(n))\Bigg).$$

(a) $\Rightarrow$ (b) The sequence μ^* is also a solution of the linear inhomogeneous equation

$$x' = A(k)x + F(k,\mu^*(k),\nu^*(k)) \tag{4.13}$$

with the initial condition $x(\kappa) = \xi$. By the variation of constant formula it is given by $\Pi_1\mathcal{T}_\kappa(\mu^*,\nu^*;\xi)$. Additionally using the mean value theorem we get

$$\|G(k,\mu^*(k),\nu^*(k))\|\,\gamma^{\kappa-k} \overset{(4.3)}{=} \|G(k,\mu^*(k),\nu^*(k)) - G(k,0,0)\|\,\gamma^{\kappa-k} \leq \\ \overset{(4.4)}{\leq} |G|_1\,\|(\mu^*,\nu^*)\|^+_{\kappa,\gamma} \quad \text{for } k\in\mathbb{Z}^+_\kappa,$$

and hence the inhomogeneous part of equation

$$y' = B(k)y + G(k,\mu^*(k),\nu^*(k)) \tag{4.14}$$

is γ^+-quasibounded. With the aid of Aulbach [3, Lemma 3.4] one can show that the mapping ν^* is the only γ^+-quasibounded solution of (4.14) and has the claimed form.

(b) $\Rightarrow$ (a) If $(\mu^*,\nu^*) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ is a solution of the fixed point problem (4.12), the mapping μ^* has to be the (unique) solution of the difference equation (4.13) with $\mu^*(\kappa) = \xi$ by the variation of constant formula. Furthermore, again Aulbach [3, Lemma 3.4(a)] implies that the sequence ν^* is a solution of the linear system (4.14) in $\ell^+_{\kappa,\gamma}(\mathcal{Y})$. □

Having all preparatory results at hand we may now head for our main theorem. As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to non-autonomous equations. In order to be more precise we call a subset S of the extended state space $I\times\mathcal{X}\times\mathcal{Y}$ an *invariant fiber bundle* if for any triple $(\kappa,\xi,\eta)\in S$ one has $(k,\lambda(k;\kappa,\xi,\eta))\in S$ for all $k\geq\kappa$, $k\in I$, where λ denotes the general solution of (4.1).

Theorem 4.11 (existence of invariant fiber bundles)**:** *We assume Hypothesis 4.1 and let the global Lipschitz constants* $|F|_1$ *and* $|G|_1$ *satisfy the estimate*

$$0 \leq \max\{|F|_1, |G|_1\} < \frac{\beta-\alpha}{2\max\{K_1,K_2\}}. \tag{4.15}$$

Moreover we choose a fixed real number $\sigma \in \big(\max\{K_1,K_2\}\max\{|F|_1,|G|_1\}, \frac{\beta-\alpha}{2}\big]$ *and let* λ *denote the general solution of* (4.1)*. Then the following statements are true:*

(a) *In case* $I = \mathbb{Z}_{\kappa_0}^+$ $(\kappa_0 \in \mathbb{Z})$ *there exists a uniquely determined mapping* $s : I \times \mathcal{X} \to \mathcal{Y}$ *whose graph* $S := \{(\kappa, \xi, s(\kappa, \xi)) : \kappa \in I, \xi \in \mathcal{X}\}$ *can be characterized dynamically for any constant* $\gamma \in [\alpha + \sigma, \beta - \sigma]$ *as*

$$S = \left\{(\kappa, \xi, \eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\cdot\,; \kappa, \xi, \eta) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})\right\}. \tag{4.16}$$

Furthermore we have

(a_1) $s(\kappa, 0) \equiv 0$ *on* I,

(a_2) *the graph* S *is a global invariant fiber bundle of* (4.1). *Additionally* s *is a solution of the* invariance equation

$$s\left(\kappa + 1, A(\kappa)\xi + F(\kappa, \xi, s(\kappa, \xi))\right) = B(\kappa)s(\kappa, \xi) + G(\kappa, \xi, s(\kappa, \xi)) \tag{4.17}$$

for all $(\kappa, \xi) \in I \times \mathcal{X}$,

(a_3) $s : I \times \mathcal{X} \to \mathcal{Y}$ *is continuous and* $s(\kappa, \cdot) : \mathcal{X} \to \mathcal{Y}$ *is continuously differentiable for any* $\kappa \in I$ *with globally bounded derivative*

$$\left\| \frac{\partial s}{\partial \xi}(\kappa, \xi) \right\| \leq \frac{K_1 K_2 \max\left\{|F|_1, |G|_1\right\}}{\sigma - \max\left\{K_1, K_2\right\} \max\left\{|F|_1, |G|_1\right\}} \tag{4.18}$$

for all $(\kappa, \xi) \in I \times \mathcal{X}$,

(a_4) *provided that* $\alpha + \sigma \leq 1$, *then* $s(\kappa, \cdot) : \mathcal{X} \to \mathcal{Y}$ *is* m*-times continuously differentiable for any* $\kappa \in I$ *with globally bounded derivatives.*

S *is called the* pseudo-stable fiber bundle *of* (4.1).

(b) *In case* $I = \mathbb{Z}$ *there exists a uniquely determined mapping* $r : I \times \mathcal{Y} \to \mathcal{X}$ *whose graph* $R := \{(\kappa, r(\kappa, \eta), \eta) : \kappa \in I, \eta \in \mathcal{Y}\}$ *can be characterized dynamically for any constant* $\gamma \in [\alpha + \sigma, \beta - \sigma]$ *as*

$$R = \left\{(\kappa, \xi, \eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\cdot\,; \kappa, \xi, \eta) \in \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})\right\}. \tag{4.19}$$

Furthermore we have

(b_1) $r(\kappa, 0) \equiv 0$ *on* I,

(b_2) *the graph* R *is a global invariant fiber bundle of* (4.1). *Additionally* r *is a solution of the* invariance equation

$$r\left(\kappa + 1, B(\kappa)\eta + G(\kappa, r(\kappa, \eta), \eta)\right) = A(\kappa)r(\kappa, \eta) + F(\kappa, r(\kappa, \eta), \eta)$$

for all $(\kappa, \eta) \in I \times \mathcal{Y}$,

(b_3) $r : I \times \mathcal{Y} \to \mathcal{X}$ *is continuous and* $r(\kappa, \cdot) : \mathcal{Y} \to \mathcal{X}$ *is continuously differentiable for any* $\kappa \in I$ *with globally bounded derivative*

$$\left\| \frac{\partial r}{\partial \eta}(\kappa, \eta) \right\| \leq \frac{K_1 K_2 \max\left\{|F|_1, |G|_1\right\}}{\sigma - \max\left\{K_1, K_2\right\} \max\left\{|F|_1, |G|_1\right\}}$$

for all $(\kappa, \eta) \in I \times \mathcal{Y}$,

(b_4) *provided that* $1 \leq \beta - \sigma$, *then* $r(\kappa, \cdot) : \mathcal{Y} \to \mathcal{X}$ *is* m*-times continuously differentiable for any* $\kappa \in I$ *with globally bounded derivatives.*

R *is called the* pseudo-unstable fiber bundle *of* (4.1).

(c) In case $I = \mathbb{Z}$ *only the zero solution of equation* (4.1) *is contained both in* S *and* R, *i.e.* $S \cap R = \mathbb{Z} \times \{0\} \times \{0\}$, *and hence the zero solution is the only* $\gamma^{\pm}$*-quasibounded solution of* (4.1).

Remark 4.12: (1) It is easy to see that the existence of suitable values for σ follows from assumption (4.15). Because of $\sigma \leq \frac{\beta-\alpha}{2}$ also the interval $[\alpha + \sigma, \beta - \sigma]$ containing γ is non-empty.

(2) Although our construction of the invariant fiber bundles S and R is different from the one in Aulbach [3] the fiber bundles are the same. This can be seen using their dynamical characterization in (4.16) and (4.19). Hence all the additional properties of S and R proved in Aulbach [3, Theorem 4.1] remain valid. The same applies for the result Aulbach [3, Corollary 4.2] concerning autonomous and periodic equations. In fact, if the mappings A, B, F and G are periodic in k with period $\Theta \in \mathbb{N}$, then the mappings s and r are also Θ-periodic in k. Nevertheless, the assumption on the Lipschitz constants of F and G made in (4.15) is weaker then the one in Aulbach [3, Theorem 4.1].

(3) The assumption of global boundedness of the derivatives in (4.4) can be replaced by the global Lipschitz continuity of F and G. If this is done the functions s and r defining the invariant fiber bundles S and R are also globally Lipschitz continuous with respect to $\xi \in \mathcal{X}$ and $\eta \in \mathcal{Y}$ (independent of $\kappa \in \mathbb{Z}^+_{\kappa_0}$), respectively. This result can be easily derived by a slight modification of the subsequent proof of Theorem 4.11.

(4) If the functions F and G are m-times continuously differentiable with respect to (x, y) with globally bounded derivatives $\frac{\partial^n F}{\partial(x,y)^n}$ and $\frac{\partial^n G}{\partial(x,y)^n}$ for $n \in \{1, \dots, m\}$ then one would expect the same degree of smoothness for the invariant fiber bundles S and R. For this to be true the constants $0 < \alpha < \beta$ have to satisfy a so-called *Gap-Condition* $\alpha^m < \beta$ or $\alpha < \beta^m$, respectively. Such a generalization of Theorem 4.11 can be found for ordinary differential equations in Siegmund [21, p. 73, Satz 8.1]. For difference equations it will be shown in a forthcoming paper. Our Theorem 4.11 at least provides higher order smoothness under the following conditions:

- For $\alpha < \beta \leq 1$ the pseudo-stable fiber bundle S is of class C^m.
- For $1 \leq \alpha < \beta$ the pseudo-unstable fiber bundle R is of class C^m.
- In the hyperbolic case $\alpha < 1 < \beta$ and under the additional assumption

$$0 \leq \max\{|F|_1, |G|_1\} < \frac{\min\{1-\alpha, \beta-1\}}{\max\{K_1, K_2\}}$$

one can always choose a real number $\sigma \in \left(\max\{K_1, K_2\} \max\{|F|_1, |G|_1\}, \frac{\beta-\alpha}{2}\right]$ such that $\alpha + \sigma \leq 1 \leq \beta - \sigma$. In this case S and R are as smooth as the right-hand side of the difference equation (4.1) and they are called the *stable fiber bundle* and the *unstable fiber bundle* of (4.1), respectively.

(5) By means of cut-off-functions we can deduce a theorem on locally invariant C^1-fiber bundles for equation (4.1) from the above Theorem 4.11. The essential fact hereby is that one can replace the strong assumption of the existence of $|F|_1, |G|_1 < \infty$ and (4.15) by

$$\lim_{(x,y)\to(0,0)} \frac{\partial(F,G)}{\partial(x,y)}(k,x,y) = 0 \quad \text{uniformly in } k \in I.$$

The construction can be found in many references (cf. e.g. Vanderbauwhede & Van Gils [24]) for autonomous equations and it is easily lifted to our non-autonomous setting. Nevertheless it is worth mentioning that although C^∞-cut-off-functions always exist in Hilbert spaces, in general Banach spaces even C^1-cut-off-functions may fail to exist (cf. Abraham, Marsden & Ratiu [1, p. 273, Lemma 4.2.13]).

The following example shows that the invariant fiber bundles S and R from Theorem 4.11 are not C^2 in general, even if the non-linearities F and G are C^∞-functions.

Example 4.13: The two-dimensional autonomous difference equation

$$\begin{cases} x' = e^{-2}x + e^{-2}y^2\Theta_\rho(x^2+y^2) \\ y' = e^{-1}y \end{cases} \tag{4.20}$$

satisfies Hypothesis 4.1 with $\alpha = e^{-2}$, $\beta = e^{-1}$ and $K_1 = K_2 = 1$, where $\Theta_\rho : [0,\infty) \to [0,1]$ is a C^∞-cut-off-function with

$$\Theta_\rho(t) = \begin{cases} 1 & \text{for } t \in [0,\rho] \\ 0 & \text{for } t \in [2\rho,\infty) \end{cases}.$$

Here we choose the real constant $\rho > 0$ so small that condition (4.15) is satisfied. Now for every $c \in \mathbb{R}$ the sets $R_c := \left\{(\xi,\eta) \in B_\rho(0,0) \setminus \{(0,0)\} : \xi = c\eta^2 - \frac{\eta^2}{2}\ln\eta^2\right\} \cup \{(0,0)\}$ contain the origin and are positively invariant with respect to (4.20), i.e. $\mathbb{Z} \times R_c$ is an invariant fiber bundle. Additionally, each point $(\xi,\eta) \in B_\rho(0,0)$, $\eta \neq 0$ is contained in exactly one of the sets R_c, namely for $c = \frac{\xi}{\eta^2} + \frac{\ln\eta^2}{2}$. Hence the restriction of the pseudo-unstable fiber bundle R from Theorem 4.11 to $\mathbb{Z} \times B_\rho(0,0)$ has the form $\mathbb{Z} \times R_c$ for some $c \in \mathbb{R}$. On the other hand, each R_c is a graph of a C^1-function $r_c(\eta) = \xi$, but r_c fails to be two times continuously differentiable. Note that in the present example the gap-condition $\alpha < \beta^2$ is violated.

Proof (of Theorem 4.11): (*a*) By $\lambda = (\lambda_1, \lambda_2)$ we denote the general solution of the difference equation (4.1). We show first that for any pair $(\kappa,\xi) \in \mathbb{Z}^+_{\kappa_0} \times \mathcal{X}$ there exists exactly one $s(\kappa,\xi) \in \mathcal{Y}$ such that $\lambda(\cdot\,;\kappa,\xi,s(\kappa,\xi)) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ for every $\gamma \in [\alpha+\sigma, \beta-\sigma]$. Then the function $s : \mathbb{Z}^+_{\kappa_0} \times \mathcal{X} \to \mathcal{Y}$ defines the invariant fiber bundle S. Now for $\gamma \in [\alpha+\sigma,\beta-\sigma]$ we get the estimate

$$\max\left\{\frac{K_1}{\gamma-\alpha}, \frac{K_2}{\beta-\gamma}\right\} \max\{|F|_1, |G|_1\} \leq \frac{\max\{K_1,K_2\}}{\sigma} \max\{|F|_1, |G|_1\} =: L. \tag{4.21}$$

Because of assumption (4.15) we get $L \in [0,1)$ and hence the mapping $\mathcal{T}_\kappa : \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}) \times \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ defined in Lemma 4.10 is a uniform contraction in $\xi \in \mathcal{X}$, since for any

$(\mu,\nu),(\bar\mu,\bar\nu)\in\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ and $\xi\in\mathcal{X}$ we have the estimate

$$\begin{aligned}&\|\mathcal{T}_\kappa(\mu,\nu;\xi)-\mathcal{T}_\kappa(\bar\mu,\bar\nu;\xi)\|^+_{\kappa,\gamma}\overset{(4.11)}{=}\left\|\mathcal{K}_\kappa\big(\mathcal{G}(\mu,\nu)-\mathcal{G}(\bar\mu,\bar\nu)\big)\right\|^+_{\kappa,\gamma}\le\\ \overset{(4.6)}{\le}\;&\max\left\{\frac{K_1}{\gamma-\alpha},\frac{K_2}{\beta-\gamma}\right\}\|\mathcal{G}(\mu,\nu)-\mathcal{G}(\bar\mu,\bar\nu)\|^+_{\kappa,\gamma}\le\\ \overset{(4.8)}{\le}\;&\max\left\{\frac{K_1}{\gamma-\alpha},\frac{K_2}{\beta-\gamma}\right\}\max\{|F|_1,|G|_1\}\|(\mu,\nu)-(\bar\mu,\bar\nu)\|^+_{\kappa,\gamma}\le\\ \overset{(4.21)}{\le}\;&L\|(\mu,\nu)-(\bar\mu,\bar\nu)\|^+_{\kappa,\gamma}.\end{aligned}\tag{4.22}$$

Consequently Banach's fixed point theorem guarantees the unique existence of a fixed point $(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))\in\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ of $\mathcal{T}_\kappa(\cdot\,;\xi)$. This fixed point is independent of the growth constant $\gamma\in[\alpha+\sigma,\beta-\sigma]$ because with Lemma 3.3(b) we have the inclusion $\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})\subseteq\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ and every mapping $\mathcal{T}_\kappa(\cdot\,;\xi):\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})\to\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ has the same fixed point as the restriction $\mathcal{T}_\kappa(\cdot\,;\xi)\big|_{\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})}$. Formally we can write

$$J^\gamma_{\alpha+\sigma}(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))\equiv(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))\equiv\mathcal{T}_\kappa(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi);\xi)\quad\text{on }\mathcal{X}.\tag{4.23}$$

Because of Lemma 4.10 the fixed point $(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))$ is a solution of equation (4.1) with $\big(\mu^*_\kappa(\xi)\big)(\kappa)=\xi$. Now we define $s(\kappa,\xi):=\big(\nu^*_\kappa(\xi)\big)(\kappa)$ and have to prove (4.16).
($\subseteq$) Because of the uniqueness of solutions we get $\lambda(\cdot\,;\kappa,\xi,s(\kappa,\xi))=(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))$ and therefore $\lambda(\cdot\,;\kappa,\xi,s(\kappa,\xi))\in\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$.
($\supseteq$) On the other hand the sequence $\lambda(\cdot\,;\kappa,\xi,\eta)$ is a γ^+-quasibounded solution of the difference equation (4.1) with $\lambda_1(\kappa;\kappa,\xi,\eta)=\xi$, and with Lemma 4.10 it is the unique solution of the fixed point problem (4.12). So we obtain $(\lambda_1,\lambda_2)(\cdot\,;\kappa,\xi,\eta)=(\mu^*_\kappa(\xi),\nu^*_\kappa(\xi))$ and finally get $\eta=\big(\nu^*_\kappa(\xi)\big)(\kappa)=s(\kappa,\xi)$.

(a_1) Using Hypothesis 4.1(ii) we have $\lambda(k;\kappa,0,0)\equiv(0,0)$ on $\mathbb{Z}^+_\kappa$ and since this zero solution is obviously γ^+-quasibounded, the identity

$$s(\kappa,0)\equiv\big(\nu^*_\kappa(0)\big)(\kappa)\equiv\lambda_2(\kappa;\kappa,0,0)\overset{(4.3)}{\equiv}0\quad\text{on }\mathbb{Z}^+_{\kappa_0}$$

follows from the uniqueness statement proved before.

(a_2) So far the proof provides the fact that the function $\lambda(\cdot\,;\kappa,\xi,\eta)$ is γ^+-quasibounded for arbitrary pairs of initial values $(\kappa,\xi,\eta)\in S$. The cocycle property (2.2) now implies for any integer $k_0\in\mathbb{Z}^+_\kappa$ that

$$\lambda(k;k_0,\lambda(k_0;\kappa,\xi,\eta))\overset{(2.2)}{\equiv}\lambda(k;\kappa,\xi,\eta)\quad\text{on }\mathbb{Z}^+_{k_0}.$$

Hence also $\lambda(\cdot\,;k_0,\lambda(k_0;\kappa,\xi,\eta))$ is a γ^+-quasibounded function and additionally this yields $(k_0,\lambda(k_0;\kappa,\xi,\eta))\in S$ for any $k_0\in\mathbb{Z}^+_\kappa$. To verify the invariance equation (4.17) for all pairs $(\kappa,\xi)\in\mathbb{Z}^+_{\kappa_0}\times\mathcal{X}$ we use the inclusion $(\kappa+1,(\lambda_1,\lambda_2)(\kappa+1;\kappa,\xi,s(\kappa,\xi)))\in S$ and the solution property of λ to get

$$B(\kappa)s(\kappa,\xi)+G(\kappa,\xi,s(\kappa,\xi))\overset{(4.1)}{\equiv}\lambda_2(\kappa+1;\kappa,\xi,s(\kappa,\xi))\equiv$$

$$\begin{aligned} &\equiv s(\kappa+1, \lambda_1(\kappa+1;\kappa,\xi,s(\kappa,\xi))) \equiv \\ &\overset{(4.1)}{\equiv} s(\kappa+1, A(\kappa)\xi + F(\kappa,\xi,s(\kappa,\xi))). \end{aligned}$$

(a_3) In this step we examine the continuous differentiability of the function $s(\kappa,\cdot):\mathcal{X}\to\mathcal{Y}$ defining the invariant fiber bundle S. The primary tool in this endeavour is Theorem 5.1 from the appendix whose assumptions we check now. To obtain the notation from Theorem 5.1 we declare for any $\gamma \in (\alpha+\sigma, \beta-\sigma]$ the Banach spaces $\mathcal{X} := \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})$, $\mathcal{X}_1 := \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$, $\mathcal{A} := \mathcal{X}$ and consider the mapping $\mathcal{T}_\kappa$. Due to Lemma 3.3(b) we have the continuous embedding $\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}) \overset{J^\gamma_{\alpha+\sigma}}{\hookrightarrow} \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$. Because of relation (4.22) the function $\mathcal{T}_\kappa : \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})\times\mathcal{X}\to\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})$ satisfies a uniform Lipschitz condition

$$\left\|\mathcal{T}_\kappa(\mu,\nu;\xi) - \mathcal{T}_\kappa(\bar\mu,\bar\nu;\xi)\right\|^+_{\kappa,\alpha+\sigma} \le L\left\|(\mu,\nu)-(\bar\mu,\bar\nu)\right\|^+_{\kappa,\alpha+\sigma}$$

for arbitrary pairs $(\mu,\nu),(\bar\mu,\bar\nu) \in \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})$ and $\xi\in\mathcal{X}$. We define the operator $\mathcal{G}^{(1)}_1 : \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \ell^+_{\kappa,1}(\mathcal{L}(\mathcal{X}\times\mathcal{Y}))$ by $\mathcal{G}^{(1)}_1(\mu,\nu) := \mathcal{G}^{(1)}(\mu,\nu)$; one should keep in mind the range of $\mathcal{G}^{(1)}$ in Lemma 4.6. Now the embedded mapping $J^\gamma_{\alpha+\sigma}\mathcal{T}_\kappa$ is continuously differentiable with respect to (μ,ν). This follows from the identity

$$\begin{aligned} \frac{\partial(J^\gamma_{\alpha+\sigma}\mathcal{T}_\kappa)}{\partial(\mu,\nu)}(\mu,\nu;\xi) &\overset{(4.11)}{\equiv} D\big(J^\gamma_{\alpha+\sigma}\mathcal{K}_\kappa\mathcal{G}\big)(\mu,\nu) \equiv D\big(\mathcal{K}_\kappa J^\gamma_{\alpha+\sigma}\mathcal{G}\big)(\mu,\nu) \equiv \\ &\equiv \mathcal{K}_\kappa D\big(J^\gamma_{\alpha+\sigma}\mathcal{G}\big)(\mu,\nu) \equiv \\ &\overset{(4.10)}{\equiv} \mathcal{K}_\kappa J_1 \mathcal{G}^{(1)}(\mu,\nu) \in \mathcal{L}(\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}); \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})) \end{aligned}$$

and Lemma 4.8. It is obvious that the two linear operators $\mathcal{K}_\kappa \in \mathcal{L}(\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}))$ and $\mathcal{K}_\kappa \in \mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}))$, respectively, and $J^\gamma_{\alpha+\sigma}$ commute; the continuous homomorphism $\mathcal{K}_\kappa$ and the differential operator D commute because of Lang [16, p. 339, Corollary 3.2]. Furthermore we have

$$\frac{\partial(J^\gamma_{\alpha+\sigma}\mathcal{T}_\kappa)}{\partial(\mu,\nu)}(\mu,\nu;\xi) \equiv J^\gamma_{\alpha+\sigma}\mathcal{K}_\kappa J_1\mathcal{G}^{(1)}_1(\mu,\nu) \equiv \mathcal{K}_\kappa J_1 \mathcal{G}^{(1)}(\mu,\nu) J^\gamma_{\alpha+\sigma},$$

and hence relation (5.1) is verified for the (not necessarily continuous) functions

$$\begin{aligned} \mathcal{T}^{(1)}_1 &:= \mathcal{K}_\kappa J_1 \mathcal{G}^{(1)}_1 : \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{L}(\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})), \\ \mathcal{T}^{(1)} &:= \mathcal{K}_\kappa J_1 \mathcal{G}^{(1)} : \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})). \end{aligned}$$

Since $\mathcal{T}_\kappa$ is linear in ξ it is differentiable and the derivative is given by

$$\frac{\partial\mathcal{T}_\kappa}{\partial\xi}(\mu,\nu;\xi) \overset{(4.11)}{\equiv} \mathcal{S}_\kappa \in \mathcal{L}(\mathcal{X}; \ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})),$$

obviously $\frac{\partial\mathcal{T}_\kappa}{\partial\xi}$ is continuous and hence $\mathcal{T}_\kappa$ is continuously differentiable with respect to the parameter $\xi\in\mathcal{X}$. After all, for any pair $(\mu,\nu)\in\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y})$ we get the estimate

$$\left\|\mathcal{T}^{(1)}_1(\mu,\nu)\right\|_{\mathcal{L}(\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}))} \overset{(4.6)}{\le} \max\left\{\frac{K_1}{\sigma}, \frac{K_2}{\beta-\alpha-\sigma}\right\} \left\|J_1\mathcal{G}^{(1)}_1(\mu,\nu)\right\|_{\mathcal{L}(\ell^+_{\kappa,\alpha+\sigma}(\mathcal{X}\times\mathcal{Y}))} \le$$

$$\stackrel{(3.2)}{\leq} \max\left\{\frac{K_1}{\sigma}, \frac{K_2}{\beta-\alpha-\sigma}\right\} \left\|\mathcal{G}_1^{(1)}(\mu,\nu)\right\|_{\kappa,1}^{+} \leq$$
$$\stackrel{(4.9)}{\leq} \max\left\{\frac{K_1}{\sigma}, \frac{K_2}{\beta-\alpha-\sigma}\right\} \max\left\{|F|_1, |G|_1\right\} \stackrel{(4.21)}{\leq} L$$

as well as $\left\|\mathcal{T}^{(1)}(\mu,\nu)\right\|_{\mathcal{L}(\ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}))} \leq L$. Because of Theorem 5.1 the function $J^\gamma_{\alpha+\sigma}\nu^*_\kappa = \nu^*_\kappa : \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{Y})$ has to be of class C^1. Using Lemma 3.4 we also know that $s(\kappa,\cdot) = \big(\nu^*_\kappa(\cdot)\big)(\kappa) : \mathcal{X} \to \mathcal{Y}$ is, for any fixed $\kappa \in \mathbb{Z}^+_{\kappa_0}$, a continuously differentiable mapping. It remains to prove the estimate (4.18), since $s : \mathbb{Z}^+_{\kappa_0} \times \mathcal{X} \to \mathcal{Y}$ is continuous. To this end we consider $\xi, \bar{\xi} \in \mathcal{X}$ and the corresponding fixed points $(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)), (\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi})) \in \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ of $\mathcal{T}_\kappa(\cdot\,;\xi)$ and $\mathcal{T}_\kappa(\cdot\,;\bar{\xi})$. For constants $\gamma \in [\alpha+\sigma, \beta-\sigma]$ we have

$$\begin{aligned}
&\left\|(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - (\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\right\|^+_{\kappa,\gamma} \leq \\
\stackrel{(4.23)}{\leq}\; &\left\|\mathcal{T}_\kappa(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi);\xi) - \mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\xi)\right\|^+_{\kappa,\gamma} + \\
&+ \left\|\mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\xi) - \mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\bar{\xi})\right\|^+_{\kappa,\gamma} \leq \\
\stackrel{(4.22)}{\leq}\; &\frac{\max\{K_1,K_2\}}{\sigma} \max\{|F|_1, |G|_1\} \left\|(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - (\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\right\|^+_{\kappa,\gamma} + \\
&+ \left\|\mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\xi) - \mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\bar{\xi})\right\|^+_{\kappa,\gamma}.
\end{aligned}$$

Consequently we get

$$\begin{aligned}
&\left\|(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - (\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\right\|^+_{\kappa,\gamma} \leq \\
\leq\; &\frac{\sigma}{\sigma - \max\{K_1,K_2\}\max\{|F|_1, |G|_1\}} \cdot \\
&\cdot \left\|\mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\xi) - \mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\bar{\xi})\right\|^+_{\kappa,\gamma} = \\
\stackrel{(4.11)}{=}\; &\frac{\sigma}{\sigma - \max\{K_1,K_2\}\max\{|F|_1, |G|_1\}} \left\|\mathcal{S}_\kappa(\xi-\bar{\xi})\right\|^+_{\kappa,\gamma} \leq \\
\stackrel{(4.5)}{\leq}\; &\frac{\sigma K_1}{\sigma - \max\{K_1,K_2\}\max\{|F|_1, |G|_1\}} \left\|\xi-\bar{\xi}\right\|.
\end{aligned} \tag{4.24}$$

Finally the function $s(\kappa,\cdot) : \mathcal{X} \to \mathcal{Y}$ is globally Lipschitz continuous uniformly in $\kappa \in \mathbb{Z}^+_{\kappa_0}$ because of

$$\begin{aligned}
\left\|s(\kappa,\xi) - s(\kappa,\bar{\xi})\right\| &= \left\|\big(\nu^*_\kappa(\xi)\big)(\kappa) - \big(\nu^*_\kappa(\bar{\xi})\big)(\kappa)\right\| \leq \left\|\nu^*_\kappa(\xi) - \nu^*_\kappa(\bar{\xi})\right\|^+_{\kappa,\gamma} = \\
&= \left\|\Pi_2\left(\mathcal{T}_\kappa(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi);\xi) - \mathcal{T}_\kappa(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi});\bar{\xi})\right)\right\|^+_{\kappa,\gamma} = \\
&\stackrel{(4.11)}{=} \left\|\Pi_2\left(\mathcal{K}_\kappa\big(\mathcal{G}(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - \mathcal{G}(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\big)\right)\right\|^+_{\kappa,\gamma} \leq \\
&\stackrel{(4.7)}{\leq} \frac{K_2}{\beta-\gamma} \left\|\mathcal{G}(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - \mathcal{G}(\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\right\|^+_{\kappa,\gamma} \leq \\
&\stackrel{(4.8)}{\leq} \frac{K_2\max\{|F|_1, |G|_1\}}{\sigma} \left\|(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) - (\mu^*_\kappa(\bar{\xi}), \nu^*_\kappa(\bar{\xi}))\right\|^+_{\kappa,\gamma} \leq
\end{aligned}$$

$$\stackrel{(4.24)}{\leq} \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\} \max\{|F|_1, |G|_1\}} \left\|\xi - \bar{\xi}\right\|.$$

Since differentiable and globally Lipschitz continuous mappings (with Lipschitz constant L, say) have a derivative which is bounded by L the estimate (4.18) follows.

(a_4) We are going to show now that $s(\kappa, \cdot) : \mathcal{X} \to \mathcal{Y}$ is m-times continuously differentiable under the assumption $\alpha + \sigma \leq 1$. Therefore we do not have to use the whole embedding procedure but rather may use the well-known uniform contraction principle (cf. Chow & Hale [9, p. 25, Theorem 2.2]), applied to the uniform contraction $\mathcal{T}_\kappa : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \times \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$. We may choose $\gamma \leq 1$ and $m \geq 2$ here. Because of the chain rule and by setting $\gamma = \delta$ in Lemma 4.6(b) we see that $\mathcal{T}_\kappa(\cdot\,; \xi) : \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ $(\xi \in \mathcal{X})$ is m-times continuously differentiable with the derivative

$$\frac{\partial^m \mathcal{T}_\kappa}{\partial(\mu, \nu)^m}(\mu, \nu; \xi) = \mathcal{K}_\kappa J_m \mathcal{G}^{(m)}(\mu, \nu).$$

On the other hand, $\mathcal{T}_\kappa(\mu, \nu; \cdot) : \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ $((\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}))$ is a linear continuous mapping and consequently C^∞ with identically vanishing derivatives of order $m \geq 2$. For this reason $\mathcal{T}_\kappa$ is m-times continuously differentiable and with the aid of the uniform contraction principle the fixed-point mapping $(\mu^*_\kappa, \nu^*_\kappa) : \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ is of the class C^m as well. Using Lemma 3.4 again, for arbitrarily fixed integers $\kappa \in \mathbb{Z}^+_{\kappa_0}$, the function $s(\kappa, \cdot) = \big(\nu^*_\kappa(\cdot)\big)(\kappa) : \mathcal{X} \to \mathcal{Y}$ is m-times continuously differentiable. To show the global boundedness of the derivatives we differentiate the identity

$$(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) = \mathcal{S}_\kappa \xi + \mathcal{K}_\kappa \mathcal{G}(\mu^*_\kappa(\xi), \nu^*_\kappa(\xi)) \quad \text{on } \mathcal{X}$$

with respect to $\xi \in \mathcal{X}$ by using the higher order chain rule (see Abraham, Marsden & Ratiu [1, pp. 96–97]). Then mathematical induction and Lemma 4.6 lead to the assertion, since the derivatives $D^n \mathcal{G}$ $(n \in \{1, \ldots, m\})$ are globally bounded and $\mathcal{K}_\kappa$ is also continuous.

(b) Since part (b) of the theorem can be proved along the same lines as part (a) we present only a rough sketch of the proof. Analogously to Lemma 4.10, for initial values $\eta \in \mathcal{Y}$, the γ^--quasibounded solutions of (4.1) may be characterized as the fixed points of a mapping $\bar{\mathcal{T}}_\kappa : \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \times \mathcal{Y} \to \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$,

$$\begin{aligned}\big(\bar{\mathcal{T}}_\kappa(\mu^*, \nu^*; \eta)\big)(k) \;:=\; \Bigg(&\sum_{n=-\infty}^{k-1} \Phi(k, n+1) F(n, \mu^*(n), \nu^*(n)),\\ &\Psi(k, \kappa)\eta - \sum_{n=k+1}^{\kappa} \Psi(k, n) G(n-1, \mu^*(n-1), \nu^*(n-1))\Bigg).\end{aligned}$$

Therefore the variation of constant formula in backward time and Aulbach [3, Lemma 3.2(a)] is needed. Furthermore, $\bar{\mathcal{T}}_\kappa$ may be decomposed into two linear mappings and a substitution operator, as we have done it for $\mathcal{T}_\kappa$ in relation (4.11). Now counterparts to our preparatory Lemmas 4.3, 4.4, 4.5, 4.6 and 4.8 hold true in the Banach spaces $\ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$. Note that in order to prove the counterpart of Lemma 4.4 (on the linear operator $\mathcal{K}_\kappa$) the two results Aulbach [3, Lemma 3.3, Lemma 3.4] have to be replaced by Aulbach [3, Lemma 3.2, Lemma 3.5]. It follows from the assumption (4.15) that also $\bar{\mathcal{T}}_\kappa$ is a contraction on $\ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ and if $(\mu^*_\kappa(\eta), \nu^*_\kappa(\eta)) \in \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ denotes its unique fixed point we define the function $r : I \times \mathcal{Y} \to \mathcal{X}$ by $r(\kappa, \eta) := \big(\mu^*_\kappa(\eta)\big)(\kappa)$. The claimed properties of r can be proved using the same arguments as in step (a).

(*c*) The proof of part (c) can be done just as in Aulbach [3, Theorem 4.1(c)] and hence the proof of Theorem 4.11 is complete. □

5 Appendix

Since the substitution operators under consideration (see Lemma 4.5) become differentiable only after composition with certain embeddings we present here an appropriate fixed point theorem. It goes back to Vanderbauwhede & Van Gils [24].

Theorem 5.1 (contractions between embedded Banach spaces)**:** *Let us consider three Banach spaces $\mathcal{X}$, $\mathcal{X}_1$ and $\mathcal{A}$ with a continuous embedding*

$$\mathcal{X} \overset{J}{\hookrightarrow} \mathcal{X}_1.$$

Furthermore the mapping $\mathcal{T} : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ may satisfy the following assumptions:

(i) There exists a constant $L \in [0, 1)$ with

$$\|\mathcal{T}(x;\alpha) - \mathcal{T}(\bar{x};\alpha)\| \le L\,\|x - \bar{x}\| \quad \text{for } x, \bar{x} \in \mathcal{X},\ \alpha \in \mathcal{A},$$

(ii) $J\mathcal{T} : \mathcal{X} \times \mathcal{A} \to \mathcal{X}_1$ has a continuous partial derivative $\frac{\partial(J\mathcal{T})}{\partial x} : \mathcal{X} \times \mathcal{A} \to \mathcal{L}(\mathcal{X};\mathcal{X}_1)$, where

$$\frac{\partial(J\mathcal{T})}{\partial x}(x;\alpha) \equiv J\mathcal{T}_1^{(1)}(x;\alpha) \equiv \mathcal{T}^{(1)}(x;\alpha)J \quad \text{on } \mathcal{X} \times \mathcal{A} \tag{5.1}$$

for certain functions $\mathcal{T}_1^{(1)} : \mathcal{X} \times \mathcal{A} \to \mathcal{L}(\mathcal{X})$ and $\mathcal{T}^{(1)} : \mathcal{X} \times \mathcal{A} \to \mathcal{L}(\mathcal{X}_1)$,

(iii) $\mathcal{T}$ has a continuous partial derivative $\frac{\partial \mathcal{T}}{\partial \alpha} : \mathcal{X} \times \mathcal{A} \to \mathcal{L}(\mathcal{A};\mathcal{X})$,

(iv) $\mathcal{T}_1^{(1)}$ and $\mathcal{T}^{(1)}$ are bounded by the constant L defined above, i.e. we have

$$\left\|\mathcal{T}_1^{(1)}(x;\alpha)\right\|_{\mathcal{L}(\mathcal{X})} \le L, \qquad \left\|\mathcal{T}^{(1)}(x;\alpha)\right\|_{\mathcal{L}(\mathcal{X}_1)} \le L \quad \text{for } (x,\alpha) \in \mathcal{X} \times \mathcal{A}.$$

Then for any parameter value $\alpha \in \mathcal{A}$ the mapping $\mathcal{T}(\cdot\,;\alpha)$ has exactly one fixed point $x^*(\alpha) \in \mathcal{X}$, i.e. there exists a function $x^* : \mathcal{A} \to \mathcal{X}$ with the property $\mathcal{T}(x^*(\alpha);\alpha) \equiv x^*(\alpha)$ on $\mathcal{A}$. Additionally x^* is Lipschitz continuous and $Jx^* : \mathcal{A} \to \mathcal{X}_1$ is continuously differentiable with derivative

$$D\big(Jx^*\big)(\alpha) = JT(\alpha),$$

where $T(\alpha) \in \mathcal{L}(\mathcal{A};\mathcal{X})$ is the unique fixed point of the linear operator equation

$$T = \mathcal{T}_1^{(1)}(x^*(\alpha);\alpha)T + \frac{\partial \mathcal{T}}{\partial \alpha}(x^*(\alpha);\alpha).$$

Remark 5.2: For $\mathcal{X} = \mathcal{X}_1$ this is the classical fixed point theorem on C^1-dependence of the fixed point of a uniform contraction $\mathcal{T} : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$.

Proof. Theorem 5.1 is a special case of Vanderbauwhede & Van Gils [24, Theorem 3] as well as of Hilger [12, Theorem 6.1]. □

References

[1] R. H. Abraham, J. E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Applied Mathematical Sciences, 75, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[2] J. Appell and P. P. Zabreijko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.

[3] B. Aulbach, *The fundamental existence theorem on invariant fiber bundles*, Journal of Difference Equations and Applications, 3 (1998), 501–537.

[4] B. Aulbach and B. M. Garay, *Discretization of semilinear differential equations with an exponential dichotomy*, Computers and Mathematics with Applications, 28 (1994), 23–35.

[5] B. Aulbach, Nguyen Van Minh and P. P. Zabreiko, *The concept of spectral dichotomy for linear difference equations*, Journal of Mathematical Analysis and Applications, 185 (1994), 275–287.

[6] B. Aulbach and C. Pötzsche, *Invariant Manifolds with Asymptotic Phase for Nonautonomous Difference Equations*, Computers and Mathematics with Applications, to appear.

[7] B. Aulbach and T. Wanner, *Invariant Foliations and Decoupling of Nonautonomous Difference Equations*, Journal of Difference Equations and Applications, to appear.

[8] N. Castañeda and R. Rosa, *Optimal estimates for the uncoupling of differential equations*, Journal of Dynamics and Differential Equations, 8(1) (1996), 103–139.

[9] S.-N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Grundlehren der mathematischen Wissenschaften, 251, Springer-Verlag, Berlin-Heidelberg-New York, 1996.

[10] S.-N. Chow and K. Lu, *C^k centre unstable manifolds*, Proceedings of the Royal Society Edinburgh, Section A 108, No. 3/4 (1988), 303–320.

[11] M. S. ElBialy, *On pseudo-stable and pseudo-unstable manifolds for maps*, Journal of Mathematical Analysis and Applications, 232 (1999), 229–258.

[12] S. Hilger, *Smoothness of invariant manifolds*, Journal of Functional Analysis, 106(1) (1992), 95–129.

[13] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, 583, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[14] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, 1995.

[15] S. Keller, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*, Dissertation, Universität Augsburg, 1999.

[16] S. Lang, *Real and Functional Analysis*, Graduate Texts in Mathematics, 142, Springer-Verlag, Berlin-Heidelberg-New York, 1993.

[17] G. Papaschinopoulos, *On the summable manifold for discrete systems*, Math. Japonica 33, No. 3 (1988), 457–468.

[18] G. Papaschinopoulos, *Linearization near the summable manifold for discrete systems*, Stud. Scient. Math. Hungarica 25 (1990), 275–289.

[19] J. Schinas, *Stability and conditional stability of time-dependent difference equations in Banach spaces*, J. Inst. Maths Applics 14 (1974), 335–346.

[20] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[21] S. Siegmund, *On the differentiability of integral manifolds*, (in german) Diploma thesis, Universität Augsburg, 1996.

[22] ——, *Spectral theory, smooth foliations and normal forms for differential equations of Carathéodory-type*, (in german) Ph. D. Dissertation, Universität Augsburg, 1999.

[23] A. Vanderbauwhede, *Center manifolds, normal forms and elementary bifurcations*, in Dynamics Reported, Vol. 2, U. Kirchgraber and H.-O. Walther, eds., B. G. Teubner, Stuttgart, 1989, 89–169.

[24] A. Vanderbauwhede and S. A. Van Gils, *Center manifolds and contractions on a scale of Banach spaces*, Journal of Functional Analysis, 72 (1987), 209–224.