NUMERICAL DYNAMICS OF INTEGRODIFFERENCE EQUATIONS: GLOBAL ATTRACTIVITY IN A C^0-SETTING*

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Abstract. Integrodifference equations are successful and popular models in theoretical ecology to describe spatial dispersal and temporal growth of populations with nonoverlapping generations. In relevant situations, such infinite-dimensional discrete dynamical systems have a globally attractive periodic solution. We show that this property persists under sufficiently accurate spatial (semi-) discretizations of collocation- and degenerate kernel-type using linear splines. Moreover, convergence preserving the order of the method is established. This justifies theoretically that simulations capture the behavior of the original problem. Several numerical illustrations confirm our results.

Key words. Integrodifference equation, collocation method, degenerate kernel method, piecewise linear approximation, global attractivity, Urysohn operator, Hammerstein operator

AMS subject classifications. 45G15; 65R20; 65P40; 37C55

1. Introduction.

Integrodifference equations (short IDEs) are a recursions

$$\text{(I}_0\text{)} \qquad \boxed{u_{t+1} = \mathcal{F}_t(u_t),}$$

whose right-hand side is a nonlinear integral operator

$$\text{(1.1)} \qquad \mathcal{F}_t(u)(x) := G_t\left(x, \int_\Omega f_t(x,y,u(y))\,\mathrm{d}y\right) \quad \text{for all } t \in \mathbb{Z},\, x \in \Omega$$

acting on an ambient *state space* of functions u over a domain Ω. Such infinite-dimensional discrete dynamical systems arise in various contexts: In the life sciences they originate from population genetics [12], but gained a remarkable popularity in theoretical ecology [7] over the last decades. Here, they model the growth and spatial dispersal of populations with non-overlapping generations. At the same token, they might serve in epidemiology. In applied mathematics, IDEs occur as time-1-maps of evolutionary differential equations or as iterative schemes to solve (nonlinear) boundary value problems.

When simulating the dynamical behavior of IDEs (I_0), appropriate discretizations are due in order to arrive at finite-dimensional state spaces and to replace (I_0) by a corresponding recursion. For this purpose, we apply standard techniques in the numerical analysis of integral eqns. [1] to (1.1), namely collocation and degenerate kernel methods. This triggers the question whether such numerical approximations actually reflect the dynamics of the original problem (I_0)?

Since the resulting discretization error typically grows exponentially in time [10, Thm. 4.1], corresponding estimates are of little use when questions on the asymptotic behavior are of interest. Indeed, while the global error only yields convergence on finite intervals, we investigate the long-term dynamics of IDEs versus their discretizations. More detailed, it is shown that global convergence of a sequence $(u_t)_{t\geq 0}$ generated by (I_0) to a fixed point or a periodic solution, independent of the initial function u_0, persists under discretization. In addition, we prove that the original and and the limit of the discretized equation are close to each other respecting the error

*Submitted to the editors January 31, 2019.

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order of the approximation method. This can be seen as a first contribution to the numerical dynamics of IDEs, i.e. the field in theoretical numerical analysis investigating the question, which qualitative properties of a dynamical system persist under discretization? A survey of such results addressing time-discretizations of ODEs is given in [14], while we tackle a corresponding theory for spatial discretizations of IDEs.

In applications the existence of globally attractive solutions to (I$_0$) is of eminent importance and holds in various representative models. Indeed, conditions for global attractivity of periodic solutions to IDEs were given in [2]. We study the robustness of this property using a quantitative version of a result by Smith and Waltman [13].

The content and framework of this paper are as follows: We consider IDEs (I$_0$) being periodic in t; this assumption is well-motivated from applications in the life sciences to describe seasonality. As state space for (I$_0$) serve the continuous functions over a compact domain and technical preliminaries were given in [10]. For conceptional clarity we restrict to discretizations based on piecewise linear functions, although our perturbation results apparently allow higher-order approximations. Moreover, the given analysis covers semi-discretization methods only.

After summarizing the essential assumptions on and properties of (I$_0$) in Sect. 2, we present our crucial perturbation result given by Thm. 2.1. It is applied to spatial discretizations of (1.1) based on collocation with piecewise linear functions. The corresponding interpolation estimates yield quadratic convergence (cf. Prop. 2.3), which is numerically confirmed by two examples. Hammerstein IDEs frequently arise in applications (see [7]), where (1.1) simplifies to a Hammerstein operator. This relevant special case particularly allows degenerate kernel approximations. In Sect. 3 we provide an adequate discretization and convergence theory. Since Hammerstein operators have a simpler structure than (1.1), the associate Prop. 3.1 is more accessible than the general Prop. 2.3. For illustrative purposes, we numerically study 4-periodic solutions to a Beverton-Holt-type IDE, which affirms our theoretical results. An appendix contains a quantitative version of [13, Thm. 2.1] in terms of Thm. A.1.

Notation. Let $\mathbb{R}_+ := [0, \infty)$, denote the norm on linear spaces X, Y by $\|\cdot\|$ and V° is the interior of a (nonempty) subset $V \subseteq X$. If a function $f : V \to Y$ satisfies a Lipschitz condition, then $\operatorname{lip} f$ is its smallest Lipschitz constant and

$$\omega(\delta, f) := \sup_{\|x-\bar{x}\|<\delta} \|f(x) - f(\bar{x})\| \quad \text{for all } \delta > 0$$

the *modulus of continuity* of f. The limit relation $\lim_{\delta \searrow 0} \omega(\delta, f) = 0$ holds if and only if f is uniformly continuous. The classes $\mathfrak{N} := \{\Gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \lim_{\rho \searrow 0} \Gamma(\rho) = 0\}$ and $\mathfrak{N}^* := \{\Gamma \in \mathfrak{N} \mid \Gamma \text{ is nondecreasing}\}$ of limit 0 functions are convenient.

Throughout this text, let $\Omega \subset \mathbb{R}^\kappa$ denote a nonempty, compact set without isolated points. If $U \subseteq \mathbb{R}^d$, then we write

$$C(\Omega, U) := \{u : \Omega \to U \mid u \text{ is continuous}\}, \qquad C_d := C(\Omega, \mathbb{R}^d)$$

and the maximum norm $\|u\| := \max_{x \in \Omega} |u(x)|$ makes $C(\Omega, \mathbb{R}^d)$ a Banach space. The set of $u : \Omega \to \mathbb{R}^d$, whose derivatives $D^j u$ up to order $j \le m$ have a continuous extension from the interior $\Omega^\circ \neq \emptyset$ to Ω is $C^m(\Omega, \mathbb{R}^d)$, $m \in \mathbb{N}_0$.

2. Urysohn integrodifference equations and perturbation. The right-hand sides of (I$_0$) are mappings $\mathcal{F}_t : U_t \subseteq C_d \to C_d$, $t \in \mathbb{Z}$, defined on the space of $\mathbb{R}^d$-valued continuous functions. For $d = 1$ we speak of *scalar* eqns. (I$_0$).

A *solution* of (I$_0$) is a sequence $\phi = (\phi_t)_{t \in \mathbb{Z}}$ satisfying $\phi_{t+1} = \mathcal{F}_t(\phi_t)$ and $\phi_t \in U_t$ for every $t \in \mathbb{Z}$. If there exists a $\theta \in \mathbb{N}$ such that $\phi_{t+\theta} = \phi_t$ holds for all $t \in \mathbb{Z}$, then

ϕ is called *θ-periodic*. Given an initial time $\tau \in \mathbb{Z}$ and an initial state $u_\tau \in U_\tau$, then the *general solution* of (I$_0$) is

$$\varphi_0(t;\tau,u_\tau) := \begin{cases} u_\tau, & t = \tau, \\ \mathcal{F}_{t-1} \circ \ldots \circ \mathcal{F}_\tau, & t > \tau; \end{cases} \tag{2.1}$$

it is defined for times $t > \tau$ as long as the compositions stay in the domains U_t.

We are dealing with IDEs (I$_0$) being periodic in time, i.e. there exists a *period* $\theta \in \mathbb{N}$ such that $f_t = f_{t+\theta}$ and $G_t = G_{t+\theta}$ hold for all $t \in \mathbb{Z}$. Then (1.1) implies $\mathcal{F}_t = \mathcal{F}_{t+\theta}$, $t \in \mathbb{Z}$, and (I$_0$) becomes a *θ-periodic difference equation*. In case $\theta = 1$, i.e. the right-hand sides $\mathcal{F}_t$ are independent of t, one speaks of an *autonomous* equation. The following standing assumptions are supposed to hold for all $s \in \mathbb{Z}$: Let $m \in \mathbb{N}$,

- $f_s : \Omega^2 \times U_s^1 \to \mathbb{R}^p$ is continuous on an open, convex, nonempty $U_s^1 \subseteq \mathbb{R}^d$ and the derivatives $D_1^j f_s : \Omega^2 \times U_s^1 \to \mathbb{R}^p$ for $1 \le j \le m$, $D_3 f_s : \Omega \times U_s^1 \to \mathbb{R}^{p\times d}$ may exist as continuous functions. Furthermore, for every $\varepsilon > 0$ and $x, y \in \Omega$ there may exist a $\delta > 0$ such that
$$|z_1 - z_2| < \delta \quad \Rightarrow \quad |D_3 f_s(x,y,z_1) - D_3 f_s(x,y,z_2)| < \delta \quad \text{for all } z_1, z_2 \in U_s^1.$$
- $G_s : \Omega \times U_s^2 \to \mathbb{R}^d$ is a C^m-function on an open, convex, nonempty $U_s^2 \subseteq \mathbb{R}^p$. Moreover, for every $\varepsilon > 0$, $x \in \Omega$, there may exist a $\delta > 0$ such that
$$|z_1 - z_2| < \delta \quad \Rightarrow \quad |D_2 G_s(x,z_1) - D_2 G_s(x,z_2)| < \delta \quad \text{for all } z_1, z_2 \in U_s^2$$
and the following domain is assumed to be convex:
$$U_s := \left\{ u \in C(\Omega, U_s^1) \,\middle|\, \int_\Omega f_s(x,y,u(y))\,\mathrm{d}y \in U_s^2 \text{ for all } x \in \Omega \right\}.$$

Then the *Urysohn operator*

$$\mathcal{U}_s : C(\Omega, U_s^1) \to C_p, \qquad \mathcal{U}_s(u) := \int_\Omega f_s(\cdot, y, u(y))\,\mathrm{d}y \tag{2.2}$$

is completely continuous and of class C^1 on the interior $C(\Omega, U_s^1)^\circ$. Referring to [10][1] this guarantees that the general solution of (I$_0$) fulfills:

(P_1) $\varphi_0(t;\tau,\cdot) : U_\tau \to C_d$ is completely continuous for all $\tau < t$ (see [10, Cor. 2.2]),
(P_2) $\varphi_0(t;\tau,u) \in C^m(\Omega^\circ, \mathbb{R}^d)$ for all $\tau < t$, $u \in C_d$ (see [10, Cor. 2.6]),
(P_3) $\varphi_0(t;\tau,\cdot) \in C^1(U_\tau, C_d)$ for all $\tau \le t$ (see [10, Prop. 2.7]).

Along with (I$_0$) we consider difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \tag{I$_n$}$$

depending on a discretization parameter $n \in \mathbb{N}$. Defining the *local discretization error*

$$\varepsilon_t(u) := \mathcal{F}_t(u) - \mathcal{F}_t^n(u) \quad \text{for all } u \in U_t,$$

we denote $(\text{I}_n)_{n\in\mathbb{N}}$ as *bounded convergent*, if $\lim_{n\to\infty} \sup_{u\in B} \|\varepsilon_t^n(u)\| = 0$ holds for all $t \in \mathbb{Z}$ and every bounded $B \subset U_t$. One says (I$_n$) has *convergence rate* $\gamma > 0$, if for every bounded $B \subseteq U_t$ there exists a $K(B) \ge 0$ such that

$$\|e_t^n(u)\| \le \frac{K(B)}{n^\gamma} \quad \text{for all } t \in \mathbb{Z}, u \in B.$$

Now, under appropriate assumptions we arrive at the crucial perturbation result:

[1] This reference assumes a globally defined operator $\mathcal{F}_s$, i.e. $U_s = C_d$. Yet, the reader might verify that the corresponding proofs merely require the domains U_s^1, U_s^2 to be convex (as assumed above).

THEOREM 2.1. *Suppose there exists a θ-periodic solution ϕ^* of* (I₀) *with $\phi_t^* \in U_t^\circ$ for all $t \in \mathbb{Z}$ and the following properties:*

(i) ϕ^* is globally attractive, i.e. the limit $\lim_{t\to\infty} \|\varphi_0(t;\tau,u_\tau) - \phi_t^*\| = 0$ holds for all $\tau \in \mathbb{Z}$, $u_\tau \in U_\tau$,

(ii) $\sigma(D\mathcal{F}_\theta(\phi_\theta^*) \cdots D\mathcal{F}_1(\phi_1^*)) \subset B_{q_0}(0)$ for some $q_0 \in (0,1)$.

If a bounded convergent discretization $(\mathrm{I}_n)_{n\in\mathbb{N}}$ is θ-periodic and satisfies

(iii) $\mathcal{F}_s^n : U_s \to C_d$ is completely continuous, of class C^1, $D\mathcal{F}_s^n : U_s \to L(C_d)$ are bounded[2] (uniformly in $n \in \mathbb{N}$) and

$$\lim_{n\to\infty} \|D\varepsilon_s^n(u)\| = 0 \quad \textit{for all } u \in U_s, \tag{2.3}$$

(iv) there exist $\rho_0 > 0$ and functions $\Gamma_0^0, \Gamma_0^1, \gamma^1 \in \mathfrak{N}$ so that for all $n \in \mathbb{N}$ one has

$$\left\|D^j \varepsilon_s^n(\phi_s^*)\right\| \le \Gamma_0^j(\tfrac{1}{n}) \quad \textit{for all } j = 0, 1, \tag{2.4}$$

$$\|D\mathcal{F}_s^n(u) - D\mathcal{F}_s^n(\phi_s^*)\| \le \gamma^1(\|u - \phi_s^*\|) \quad \textit{for all } u \in B_{\rho_0}(\phi_s^*) \cap U_s, \tag{2.5}$$

(v) for every $n \in \mathbb{N}_0$ there is a bounded set $B_n \subset U_s$ such that $\bigcup_{n\in\mathbb{N}_0} B_n$ is bounded and for every $u \in C_d$ there is a $T \in \mathbb{N}$ with $\varphi_n(s + T\theta; s, u) \in B_n$

for each $1 \le s \le \theta$, then there exists a $N \in \mathbb{N}$ such that the following holds: Every discretization $(\mathrm{I}_n)_{n\ge N}$ possesses a globally attractive θ-periodic solution ϕ^n and there exist $q \in (q_0, 1)$, $K \ge 1$ such that

$$\sup_{t\in\mathbb{Z}} \|\phi_t^n - \phi_t^*\| \le \frac{K}{1-q} \Gamma_0^0(\tfrac{1}{n}) \quad \textit{for all } n \ge N. \tag{2.6}$$

Remark 2.2. A careful study of the subsequent proof shows:

(1) If ϕ^* is a globally attractive fixed-point of an autonomous eqn. (I₀), then the assumption of bounded derivatives $D\mathcal{F}_s^n$ in (*iii*) is redundant.

(2) The constant $K \ge 1$ in (2.6) essentially depends on Lipschitz constants of $\mathcal{F}_t$ in a vicinity of the solution ϕ^* (cf. (2.8)). Similarly, the larger these Lipschitz constants are, and the closer one has to choose q_0 to 1 in (ii), the larger N becomes.

Proof. Let $\tau \in \mathbb{Z}$, $u \in U_\tau$ be fixed. In order to match the setting of Thm. A.1, consider the parameter set $\Lambda := \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ as metric subspace of $\mathbb{R}$ and define $\lambda_0 := 0$, $u_0 := \phi_\tau^*$, $U := U_\tau$. If φ_n denote the general solutions of (I_n), $n \in \mathbb{N}_0$, then

$$\Pi_\lambda(u) := \begin{cases} \varphi_0(\tau + \theta; \tau, u), & \lambda = 0, \\ \varphi_n(\tau + \theta; \tau, u), & \lambda = \frac{1}{n} \end{cases} \tag{2.7}$$

are the corresponding time-θ-maps. It follows from (P_3) that $\Pi_{\lambda_0} : U_\tau \to C_d$ is continuously differentiable. Moreover, each $\Pi_\lambda : U_\tau \to C_d$ is a composition of the C^1-mappings $\mathcal{F}_\tau^n, \ldots, \mathcal{F}_{\tau+\theta-1}^n$ (due to (iii)) and therefore also continuously differentiable for all $\lambda > 0$. We gradually verify the assumptions (i'–v') of Thm. A.1 next:

ad (i'): Combining global attractivity (i) and periodicity of ϕ^* implies

$$\left\|\Pi_{\lambda_0}^s(u) - \phi_\tau^*\right\| \overset{(2.7)}{=} \left\|\varphi_0(\tau + s\theta; \tau, u) - \phi_{\tau+s\theta}^*\right\| \xrightarrow[s\to\infty]{(i)} 0.$$

ad (ii'): Using mathematical induction one easily derives from (2.1) that

$$D_3\varphi_0(t;\tau,u) = D\mathcal{F}_{t-1}(\varphi_0(t-1;\tau,u)) \cdots D\mathcal{F}_\tau(\varphi_0(\tau;\tau,u)) \quad \text{for all } \tau < t$$

[2] Understood as mapping bounded sets into bounded sets.

and hence $D\Pi_{\lambda_0}(\phi_\tau^*) = D\mathcal{F}_{\tau+\theta-1}(\phi_{\tau+\theta-1}^*)\cdots D\mathcal{F}_\tau(\phi_\tau^*)$ holds. Because the spectrum $\sigma(D\mathcal{F}_\theta(\phi_{\tau+\theta-1}^*)\cdots D\mathcal{F}_1(\phi_\tau^*))\setminus\{0\}$ is independent of τ, our assumption (ii) implies the inclusion $\sigma(D\Pi_{\lambda_0}(\phi_\tau^*)) \subset B_{q_0}(0)$. If we choose $q \in (q_0, 1)$, then referring to [5, p. 6, Technical lemma] there exists an equivalent norm $\|\cdot\|$ on X with $\|D\Pi_{\lambda_0}(\phi_\tau^*)\| \le q$ and we use this norm from now on (without changing notation). The still owing continuity of $D\Pi_\lambda(u)$ in (u, λ) will be shown below.

ad (iii'): The main argument is based on error estimates having been prepared in [10, Prop. 4.5], whose notation we adopt from now on. Due to assumption (iii), the sets $D\mathcal{F}_t^n(B_{\rho_0}(\phi_t^*)) \subset L(C_d)$ are bounded uniformly in n and consequently there exists a θ-periodic sequence $(L_t)_{t\in\mathbb{Z}}$ in $\mathbb{R}_+$ such that

$$\|\mathcal{F}_t^n(u) - \mathcal{F}_t^n(\bar{u})\| \le L_t \|u - \bar{u}\| \quad \text{for all } u, \bar{u} \in B_{\rho_0}(\phi_t^*) \cap U_t \tag{2.8}$$

holds, yielding the required Lipschitz condition [10, (4.6)]. In [10, Prop. 4.5(a)] we verified that there exists a $N_0 \in \mathbb{N}$ such that $n \ge N_0$ implies the error estimate

$$\|\varphi_n(t; \tau, u_\tau) - \phi_t^*\| \le \left(\prod_{r=\tau}^{t-1} L_r\right) \|u_\tau - \phi_\tau^*\| + \Gamma_0^0(\tfrac{1}{n}) \sum_{s=\tau}^{t-1} \prod_{r=s+1}^{t-1} L_r.$$

Supposing $n \ge N_0$ (or equivalently $\lambda < \frac{1}{N_0}$) from now on, this leads to

$$\|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \overset{(2.7)}{=} \|\varphi_n(\tau+\theta; \tau, \phi_\tau^*) - \varphi_0(\tau+\theta; \tau, \phi_\tau^*)\| \le \Gamma_0(\tfrac{1}{n}),$$

where we define $\Gamma_0(\delta) := \Gamma_0^0(\delta) \sum_{s=\tau}^{\tau+\theta-1} \prod_{r=s+1}^{\tau+\theta-1} L_r$. Thanks to $\Gamma_0 \in \mathfrak{N}$, the assumption (A.1) is satisfied. In order to also establish (A.2), we furthermore deduce from the inequality derived in [10, Prop. 4.5(b)] that

$$\begin{aligned} \|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u_0)\| &= \|D_3\varphi_n(\tau+\theta; \tau, u) - D_3\varphi_0(\tau+\theta; \tau, \phi_\tau^*)\| \\ &\le \gamma_0(\|u - \phi_\tau^*\|, \tfrac{1}{n}) \end{aligned}$$

with the function

$$\gamma_0(\rho, \delta) := \sum_{s=\tau}^{\tau+\theta-1} \ell_s \left[\gamma^1(\tilde{\gamma}_s(\rho, \delta)) + \Gamma_0^1(\delta)\right] \prod_{r=s+1}^{\tau+\theta-1} L_r,$$

where $\tilde{\gamma}_t(\rho, \delta) := \rho \prod_{r=\tau}^{t-1} L_r + \delta \sum_{s=\tau}^{t-1} \prod_{r=s+1}^{t-1} L_r$ and $\ell_t := \prod_{s=\tau}^{t-1} \|D\mathcal{F}_s(\phi_s^*)\|$ for every $\tau \le t < \tau+\theta$. Due to $\gamma_0(\rho, \delta) \to 0$ in the limit $\rho, \delta \searrow 0$, the assumption (A.2) is verified. This eventually brings us into the position to establish (ii') completely, i.e. to show that $(u, \lambda) \mapsto D\Pi_\lambda(u)$ is continuous:

- In pairs $(\tilde{u}_0, \lambda) \in C_d \times \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ this results by the continuity of every derivative $D\mathcal{F}_s^n$, which was required in (iii).
- In the remaining points $(\tilde{u}_0, 0)$ we obtain

$$\|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(\tilde{u}_0)\| \le \|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u)\| + \|D\Pi_{\lambda_0}(u) - D\Pi_{\lambda_0}(\tilde{u}_0)\|.$$

  The first summand tends to 0 as $\lambda \to \lambda_0$, since assumption (iii) implies convergence of the derivatives $D\mathcal{F}_s^n$, the assumed bounded convergence of the family $(\mathrm{I}_n)_{n\in\mathbb{N}}$ guarantees convergence of the solutions, and thus due to the convergence of every factor in the product,

$$D\Pi_\lambda(u) = \prod_{s=\tau}^{\tau+\theta-1} D\mathcal{F}_s^n(\varphi_n(s; \tau, u)) \xrightarrow[\lambda\to\lambda_0]{} \prod_{s=\tau}^{\tau+\theta-1} D\mathcal{F}_s(\varphi_0(s; \tau, u)) = D\Pi_{\lambda_0}(u).$$

The second term in the sum has limit 0 as $u \to \tilde{u}_0$ because of the continuity of $D\mathcal{F}_s$ ensured by (P_3).

ad (iv'): Thanks to (v), the bounded sets $\tilde{B}_\lambda := B_n$ (with $\lambda = \frac{1}{n}$), $\tilde{B}_{\lambda_0} := B_0$ satisfy the assumption that for all $u \in U_\tau$ there is a $T \in \mathbb{N}$ with $\Pi_\lambda^T(u) \in \tilde{B}_\lambda$.

ad (v'): Property (P_1) and assumption (iii) imply that each $\Pi_\lambda(\tilde{B}_\lambda) \subseteq C_d$, $\lambda \in \Lambda$, is relatively compact. Due to the Arzelà-Ascoli theorem [4, p. 44, Thm. 3.3] it remains to show that $\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda)$ is bounded and equicontinuous:

ad boundedness: The set $B := \bigcup_{\lambda\in\Lambda} \tilde{B}_\lambda$ is bounded due to (v). First, as completely continuous mapping, $\Pi_{\lambda_0} : U_\tau \to C_d$ is bounded and there exists a $R_1 > 0$ satisfying the inclusion $\Pi_{\lambda_0}(B) \subset B_{R_1}(0)$. Second, because $(I_n)_{n\in\mathbb{N}}$ is bounded convergent, we obtain a $R_2 > 0$ with $\|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| \le R_2$ for all $u \in B$ and

$$\|\Pi_\lambda(u)\| \le \|\Pi_{\lambda_0}(u)\| + \|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| \le R_1 + R_2 \quad \text{for all } u \in B,\ \lambda > 0$$

readily implies $\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda) \subseteq B_{R_1+R_2}(0)$.

ad equicontinuity: Let $\varepsilon > 0$. The assumed bounded convergence of $(I_n)_{n\in\mathbb{N}}$ guarantees that there exists a $\lambda_* \in \Lambda$ such that

$$\|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| < \tfrac{\varepsilon}{4} \quad \text{for all } u \in B,\ \lambda < \lambda_*. \tag{2.9}$$

Because $\Pi_{\lambda_0}(B)$ is relatively compact, the Arzelà-Ascoli theorem [4, p. 44, Thm. 3.3] ensures that $\Pi_{\lambda_0}(B)$ is equicontinuous and by [4, p. 43, Prop. 3.1] in turn uniformly equicontinuous. That is, there exists a $\delta > 0$ such that the implication

$$|x - y| < \delta \quad \Rightarrow \quad |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| < \tfrac{\varepsilon}{4} \tag{2.10}$$

holds for all $x, y \in \Omega$. Hence, for $\lambda < \lambda_*$ and $|x - y| < \delta$ the triangle inequality yields

$$\begin{aligned} & |\Pi_\lambda(u)(x) - \Pi_\lambda(u)(y)| \\ \le\ & |\Pi_\lambda(u)(x) - \Pi_{\lambda_0}(u)(x)| + |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| + |\Pi_{\lambda_0}(u)(y) - \Pi_\lambda(u)(y)| \\ \overset{(2.9)}{\le}\ & \tfrac{\varepsilon}{2} + |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| \overset{(2.10)}{\le} \tfrac{3\varepsilon}{4} < \varepsilon \quad \text{for all } u \in B. \end{aligned}$$

Therefore, the union $\bigcup_{\lambda<\lambda_*} \Pi_\lambda(B)$ is equicontinuous, and as subset of this equicontinuous set, also $\bigcup_{\lambda<\lambda_*} \Pi_\lambda(\tilde{B}_\lambda)$. Finally, because equicontinuity is preserved under finite unions, the desired set $\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda)$ is equicontinuous.

In conclusion Thm. A.1 applies, if we choose $\rho > 0$ so small and $N \ge N_0$ so large that $\Gamma_0(\frac{1}{n}) \le \frac{1-q}{2n}$, $\gamma_0(\rho, \frac{1}{n}) \le \frac{1-q}{2}$ for all $n \ge N$. Hence, there exists a globally attractive fixed point $u^*(\lambda)$ of Π_λ (where $\lambda = \frac{1}{n}$). Since the fixed points of Π_λ correspond to the θ-periodic solutions of (I_n), we define $\phi_t^n := \varphi_n(t; \tau, u^*(\frac{1}{n}))$. This is the desired θ-periodic solution of (I_n). In particular, it is not difficult to see that ϕ^n is globally attractive w.r.t. $(I_n)_{n\ge N}$, where Thm. A.1(b) implies (2.6). ∎

Next we concretize Thm. 2.1 to collocation and degenerate kernel discretizations of (I_0). In doing so, let us for simplicity restrict to piecewise linear approximation.

2.1. Piecewise linear collocation. Given $n \in \mathbb{N}$, for reals $a_i < b_i$, $1 \le i \le \kappa$, we introduce the nodes $\xi_j^i := a_i + j\frac{b_i - a_i}{n}$. Let us define the *hat functions*

$$e_j^i : [a, b] \to [0, 1], \qquad e_j^i(x) := \max\left\{0, 1 - \tfrac{n}{b_i - a_i}\left|x - \xi_j^i\right|\right\} \quad \text{for all } 0 \le j \le n$$

and assume that the domain of integration for (I_0) (the habitat) is the κ-dimensional rectangle $\Omega = [a_1, b_1] \times \cdots \times [a_\kappa, b_\kappa]$ having Lebesgue measure $\lambda_\kappa(\Omega) = \prod_{i=1}^{\kappa}(b_i - a_i)$. With the set of multiindices $I_n^\kappa := \{0, \dots, n\}^\kappa$ we define the projections

$$P_n u := \sum_{\iota \in I_n^\kappa} e_\iota u(\xi_{\iota_1}^1, \dots, \xi_{\iota_\kappa}^\kappa), \qquad e_\iota(x) := \prod_{i=1}^{\kappa} e_{\iota_i}^i(x_{\iota_i}) \quad \text{for all } \iota \in I_n^\kappa$$

from C_d into the continuous $\mathbb{R}^d$-valued functions over Ω having piecewise linear components. These projections satisfy

$$\|P_n\| \le 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.11}$$

Introducing the *partial moduli of continuity*

$$\omega_i(\rho, u) := \sup_{x \in \Omega} \{|u(x_1, \dots, \bar{x}_i, \dots, x_\kappa) - u(x_1, \dots, x_i, \dots, x_\kappa)| : |\bar{x}_i - x_i| < \rho\}$$

over the coordinates $1 \le i \le \kappa$, we obtain from [11, Thm. 5.2(ii) and (iii)] (combined with (2.11)) the interpolation estimate

$$\|u - P_n u\| \le \sum_{i=1}^{\kappa} \left(\tfrac{b_i - a_i}{n}\right)^j \omega_i\left(\tfrac{b_i - a_i}{n}, D_i^j u\right) \quad \text{for all } n \in \mathbb{N}, \tag{2.12}$$

if $u \in C^j(\Omega, \mathbb{R}^d)$ and $j \in \{0, 1\}$. In case $u \in C^2(\Omega, \mathbb{R}^d)$ one even has (cf. [3, p. 227])

$$\|u - P_n u\| \le \tfrac{1}{8} \sum_{i=1}^{\kappa} \left(\tfrac{b_i - a_i}{n}\right)^2 \max_{x \in \Omega} \left|D_i^2 u(x)\right| \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

The semi-discretizations (I_n) may have the right-hand sides

$$\mathcal{F}_t^n(u) := P_n \mathcal{F}_t(u) = \sum_{\iota \in I_n^\kappa} e_\iota G_t\left(\xi_{\iota_1}^1, \dots, \xi_{\iota_\kappa}^\kappa, \int_\Omega f_t(\xi_{\iota_1}^1, \dots, \xi_{\iota_\kappa}^\kappa, y, u(y))\, \mathrm{d}y)\right). \tag{2.14}$$

This allows the following persistence and convergence result for globally attractive periodic solutions to general IDEs (I_0):

PROPOSITION 2.3 (piecewise linear collocation). *Suppose that a θ-periodic solution ϕ^* of an Urysohn IDE (I_0) with right-hand side (1.1) satisfies the assumptions (i–ii) of Thm. 2.1 and choose $q \in (q_0, 1)$. If there exist a*

(i_c) $\rho_0 > 0$, functions $\tilde{\gamma}_0 \in \mathfrak{N}$, $\tilde{\gamma}, \tilde{\gamma}_1, \tilde{\Gamma} \in \mathfrak{N}^*$, and for bounded $B_1 \subset U_s^1$, $B_2 \subset U_s^2$ there exist $\gamma_{B_1}^*, \Gamma_{B_2}^1 \in \mathfrak{N}$, $\Gamma_{B_1}^2 \in \mathfrak{N}^*$ so that for $x, \bar{x}, y \in \Omega$ one has

$$|f_s(x, y, z) - f_s(\bar{x}, y, z)| \le \tilde{\gamma}(|x - \bar{x}|) \quad \textit{for all } z \in B_1,$$
$$\left|D_3^j f_s(x, y, z) - D_3^j f_s(x, y, \bar{z})\right| \le \tilde{\gamma}_j(|z - \bar{z}|) \quad \textit{for all } z, \bar{z} \in B_{\rho_0}(\phi_s^*(y)),$$
$$|D_3 f_s(x, y, z) - D_3 f_s(\bar{x}, y, z)| \le \gamma_{B_1}^*(|x - \bar{x}|) \quad \textit{for all } z \in B_1$$

and

$$|G_s(x, z) - G_s(\bar{x}, z)| \le \Gamma_{B_2}^1(|x - \bar{x}|) \quad \textit{for all } z \in B_2,$$
$$|G_s(x, z) - G_s(x, \bar{z})| \le \Gamma_{B_2}^2(|z - \bar{z}|) \quad \textit{for all } z, \bar{z} \in B_2,$$
$$|D_2 G_s(x, z) - D_2 G_s(x, \bar{z})| \le \tilde{\Gamma}(|z - \bar{z}|) \quad \textit{for all } z, \bar{z} \in U_s^2,$$

(ii$_c$) $C \geq 0$ such that $|f_s(x,y,z)| \leq C$ for all $x,y \in \Omega$, $z \in U_s^1$ for each $1 \leq s \leq \theta$, then there exists a $N \in \mathbb{N}$ so that every collocation discretization (I$_n$) with right-hand side (2.14) and $n \geq N$ possesses a globally attractive θ-periodic solution ϕ^n. Furthermore, there is a $\tilde{K} \geq 1$ such that for all $n \geq N$ the following holds:

(a) $\|\phi_t^n - \phi_t^*\| \leq \frac{\tilde{K}}{1-q} \sum_{i=1}^{\kappa} \max_{s=1}^{\theta} \omega_i\big(\frac{b_i-a_i}{n}, \mathcal{F}_s(\phi_s^*)\big)$ *for all* $t \in \mathbb{Z}$,

(b) if $m = 1$, then

$$\|\phi_t^n - \phi_t^*\| \leq \frac{\tilde{K}}{(1-q)n} \sum_{i=1}^{\kappa} (b_i - a_i) \max_{s=1}^{\theta} \omega_i\big((b_i-a_i)\rho, D_i(\mathcal{F}_s(\phi_s^*))\big) \text{ for all } t \in \mathbb{Z},$$

(c) if $m = 2$, then

$$\|\phi_t^n - \phi_t^*\| \leq \frac{\tilde{K}}{8(1-q)n^2} \sum_{i=1}^{\kappa} (b_i - a_i)^2 \max_{s=1}^{\theta} \big\|D_i^2(\mathcal{F}_s(\phi_s^*))\big\| \quad \text{for all } t \in \mathbb{Z}.$$

The quadratic error decay in (c) also holds on non-rectangular $\Omega \subset \mathbb{R}^\kappa$. For e.g. polygonal Ω a corresponding interpolation inequality is mentioned in [10, Sect. 3.1.3].

Remark 2.4 (functions in (i_c)). In concrete applications, the functions $\tilde{\gamma}, \tilde{\gamma}_j, \gamma_{B_1}^*$ and $\Gamma_{B_2}^1, \Gamma_{B_2}^2, \tilde{\Gamma}$ are realized by means of (local) Lipschitz or Hölder conditions on f_s resp. G_s. Although they do not appear in the assertion of Prop. 2.3, the interested reader might use them, combined with estimates in the subsequent proof, to obtain a more quantitative version of Prop. 2.3.

Remark 2.5 (dependence of $\tilde{K}, N$). In addition to Rem. 2.2(2) concerning the dependence of $\tilde{K}$ and N on the properties of (I$_0$), the following proof shows that these constants also grow with the measure $\lambda_\kappa(\Omega)$ of the domain Ω.

Remark 2.6 (dissipativity). The global boundedness assumption (ii$_c$) appears to be rather restrictive, but is valid in various applications (see [7]), since growth functions in population dynamical models are typically bounded. Yet, a weaker condition ensuring dissipativity is given in [9, pp. 190–191, Prop. 4.1.5].

Proof. Let $t \in \mathbb{Z}$, $u \in U_t$ be fixed and choose $v \in C_d$, $\|v\| = 1$. Suppose $B_1 \subseteq U_t^1$ is a bounded set containing $u(\Omega)$. We begin with preliminaries and notation: If $\mathcal{U}_t$ denotes the Urysohn integral operator (2.2), then we briefly write $V_t(x) := \mathcal{U}_t(u)(x)$, $V_t^*(x) := \mathcal{U}_t(\phi_t^*)(x)$ and choose $B_2 \subseteq U_t^2$ so that $V_t(\Omega) \subseteq B_2$. Hence, (ii$_c$) implies

$$|V_t(x)| \leq \int_\Omega |f_t(x,y,u(y))| \,\mathrm{d}y \leq \lambda_\kappa(\Omega) C \quad \text{for all } x \in \Omega. \tag{2.15}$$

Furthermore, the Fréchet derivative

$$[D\mathcal{F}_t(u)v](x) = D_2 G_t(x, V_t(x)) \int_\Omega D_3 f_t(x,y,u(y)) v(y) \,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{2.16}$$

exists due to (P_3). Note that θ-periodicity of G_t, f_t readily extends to $\mathcal{F}_t$ and $\mathcal{F}_t^n$. Let us now check the remaining assumptions of Thm. 2.1.

ad (iii): With [10, Thm. 3.1], $\mathcal{F}_t^n$ are completely continuous and of class C^1 with

$$\|D\mathcal{F}_t^n(u)\| \overset{(2.14)}{=} \|P_n D\mathcal{F}_t(u)\| \overset{(2.11)}{\leq} \|D\mathcal{F}_t(u)\|$$

$$\stackrel{(2.16)}{\le} \max_{\xi\in\Omega} |D_2G_t(\xi, V_t(\xi))| \left\| \int_\Omega |D_3f_t(\cdot, y, u(y))| \, \mathrm{d}y \right\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the derivatives $D\mathcal{F}_t^n$ are bounded maps (uniformly in $n \in \mathbb{N}$). The functions $F_t : \Omega \to L(\mathbb{R}^p, \mathbb{R}^d)$, $F_t(x) := D_2G_t(x, V_t(x))$ are continuous, hence uniformly continuous on the compact set Ω and their modulus $\omega(\cdot, F_t)$ of continuity satisfy the limit relation $\lim_{\rho\searrow 0} \omega(\rho, F_t) = 0$. Then

$$\begin{aligned}
&|[D\mathcal{F}_t(u)v](x) - [D\mathcal{F}_t(u)v](\bar{x})| \\
&\qquad \stackrel{(2.16)}{\le} |F_t(x) - F_t(\bar{x})| \int_\Omega |D_3f_t(x, y, u(y))v(y)| \, \mathrm{d}y \\
&\qquad + |F_t(\bar{x})| \int_\Omega |D_3f_t(x, y, u(y))v(y) - D_3f_t(\bar{x}, y, u(y))v(y)| \, \mathrm{d}y \\
&\qquad \le \max_{s=1}^{\theta} \left\| \int_\Omega |D_3f_s(\cdot, y, u(y))| \, \mathrm{d}y \right\| \omega(|x - \bar{x}|, F_s) \\
&\qquad\qquad + \lambda_\kappa(\Omega) \max_{s=1}^{\theta} \max_{\xi\in\Omega} |F_s(\xi)| \, \gamma_{B_1}^*(|x - \bar{x}|) \quad \text{for all } x, \bar{x} \in \Omega
\end{aligned}$$

results from the triangle inequality. Thus, the continuous function $D\mathcal{F}_t(u)v : \Omega \to \mathbb{R}^d$ has a modulus of continuity being uniform in v (with $\|v\| = 1$), which implies

$$\|D\varepsilon_t^n(u)\| = \sup_{\|v\|=1} \|[I - P_n]D\mathcal{F}_t(u)v\| \stackrel{(2.12)}{\le} \sup_{\|v\|=1} \sum_{i=1}^{\kappa} \omega_i\big(\tfrac{b_i - a_i}{n}, D\mathcal{F}_t(u)v\big) \xrightarrow[n\to\infty]{} 0$$

and therefore (2.3) holds. In addition, we also verified (2.4) (for $j = 1$) with

$$\begin{aligned}
\Gamma_0^1(\rho) &:= \max_{s=1}^{\theta} \left\| \int_\Omega |D_3f_s(\cdot, y, \phi_s^*(y))| \, \mathrm{d}y \right\| \omega(\rho, F_s) \\
&\qquad + \lambda_\kappa(\Omega) \max_{s=1}^{\theta} \max_{\xi\in\Omega} \left| D_2G_s\left(\xi, \int_\Omega f_s(\xi, y, \phi_t^*(y)) \, \mathrm{d}y\right) \right| \gamma_{B_1}^*(\rho);
\end{aligned}$$

note here that $\Gamma_0^1 \in \mathfrak{N}$. Moreover, for arbitrary $x, \bar{x} \in \Omega$ we obtain

$$|V_t(x) - V_t(\bar{x})| \stackrel{(2.2)}{\le} \int_\Omega |f_t(x, y, u(y)) - f_t(\bar{x}, y, u(y))| \, \mathrm{d}y \le \lambda_\kappa(\Omega)\tilde{\gamma}(|x - \bar{x}|)$$

and consequently by the triangle inequality

$$\begin{aligned}
&|\mathcal{F}_t(u)(x) - \mathcal{F}_t(u)(\bar{x})| \\
&\qquad \le |G_t(x, V_t(x)) - G_t(\bar{x}, V_t(x))| + |G_t(\bar{x}, V_t(x)) - G_t(\bar{x}, V_t(\bar{x}))| \\
&\qquad \stackrel{(2.15)}{\le} \Gamma_{B_2}^1(|x - \bar{x}|) + \Gamma_{B_2}^2(|V_t(x) - V_t(\bar{x})|) \le \bar{\omega}(|x - \bar{x}|, \mathcal{F}_t(u)).
\end{aligned}$$

Here, the function $\bar{\omega}(\rho, \mathcal{F}_t(u)) := \Gamma_{B_2}^1(\rho) + \Gamma_{B_2}^2(\lambda_\kappa(\Omega)\tilde{\gamma}(\rho))$ clearly majorizes the partial moduli of continuity for $\mathcal{F}_t(u)$ and (2.12) implies for each $n \in \mathbb{N}$ that

$$\|\varepsilon_t^n(u)\| \le \sum_{i=1}^{\kappa} \omega_i\big(\tfrac{b_i - a_i}{n}, \mathcal{F}_t(u)\big) \le \sum_{i=1}^{\kappa} \big(\Gamma_{B_2}^1(\tfrac{b_i - a_i}{n}) + \Gamma_{B_2}^2(\lambda_\kappa(\Omega)\tilde{\gamma}(\tfrac{b_i - a_i}{n}))\big). \tag{2.17}$$

This leads to the bounded convergence of $(\mathrm{I}_n)_{n\in\mathbb{N}}$. If $u \in B_{\rho_0}(\phi_t^*)$ holds, then

$$|V_t(x) - V_t^*(x)| \le \int_\Omega |f_t(x,y,u(y)) - f_t(x,y,\phi_t^*(y))|\,\mathrm{d}y \le \lambda_\kappa(\Omega)\tilde\gamma_0(\|u-\phi_t^*\|) \tag{2.18}$$

and furthermore for every $n \in \mathbb{N}$ one has

$$\begin{aligned}
|[D\mathcal{F}_t^n(u)v - D\mathcal{F}_t^n(\phi_t^*)v](x)| &\overset{(2.14)}{=} |P_n[D\mathcal{F}_t(u)v - D\mathcal{F}_t(\phi_t^*)v](x)| \\
&\overset{(2.11)}{\le} |[D\mathcal{F}_t(u)v - D\mathcal{F}_t(\phi_t^*)v](x)| \\
&\overset{(2.16)}{\le} \Big|F_t(x)\int_\Omega D_3f_t(x,y,u(y))v(y)\,\mathrm{d}y \\
&\quad - D_2G_t(x,V_t^*(x))\int_\Omega D_3f_t(x,y,\phi_t^*(y))v(y)\,\mathrm{d}y\Big| \\
&\le \Big|F_t(x)\int_\Omega \big(D_3f_t(x,y,u(y)) - D_3f_t(x,y,\phi_t^*(y))\big)v(y)\,\mathrm{d}y\Big| \\
&\quad + \Big|(F_t(x) - D_2G_t(x,V_t^*(x)))\int_\Omega D_3f_t(x,y,\phi_t^*(y))v(y)\,\mathrm{d}y\Big| \\
&\le \max_{\xi\in\Omega}|F_t(\xi)|\int_\Omega |D_3f_t(x,y,u(y)) - D_3f_t(x,y,\phi_t^*(y))|\,\mathrm{d}y \\
&\quad + \Big\|\int_\Omega |D_3f_t(\cdot,y,\phi_t^*(y))|\,\mathrm{d}y\Big\|\,|F_t(x) - D_2G_t(x,V_t^*(x))| \\
&\le \lambda_\kappa(\Omega)\max_{\xi\in\Omega}|F_t(\xi)|\,\tilde\gamma_1(\|u-\phi_t^*\|) \\
&\quad + \Big\|\int_\Omega |D_3f_t(\cdot,y,\phi_t^*(y))|\,\mathrm{d}y\Big\|\,\tilde\Gamma(|V_t(x) - V_t^*(x)|) \\
&\overset{(2.18)}{\le} \lambda_\kappa(\Omega)\max_{\xi\in\Omega}|F_t(\xi)|\,\tilde\gamma_1(\|u-\phi_t^*\|) \\
&\quad + \Big\|\int_\Omega |D_3f_t(\cdot,y,\phi_t^*(y))|\,\mathrm{d}y\Big\|\,\tilde\Gamma(\lambda_\kappa(\Omega)\tilde\gamma_0(\|u-\phi_t^*\|)) \quad \text{for all } x\in\Omega.
\end{aligned}$$

After passing to the supremum over $x \in \Omega$, the inequality (2.5) is valid with

$$\begin{aligned}
\gamma^1(\rho) := {}& \lambda_\kappa(\Omega)\max_{s=1}^{\theta}\max_{\xi\in\Omega}|F_s(\xi)|\,\tilde\gamma_1(\rho) \\
&+ \max_{s=1}^{\theta}\Big\|\int_\Omega |D_3f_s(\cdot,y,\phi_s^*(y))|\,\mathrm{d}y\Big\|\,\tilde\Gamma(\lambda_\kappa(\Omega)\tilde\gamma_0(\rho));
\end{aligned}$$

note again that $\gamma^1 \in \mathfrak{N}$.

It remains to determine a function Γ_0^0 yielding the convergence rates in (2.6), which depend on the respective smoothness properties of $\mathcal{F}_t(u)$.

(a) The estimate (2.17) allows us to define the function

$$\Gamma_0^0(\rho) := \max_{s=1}^{\theta}\sum_{i=1}^{\kappa}\omega_i\big((b_i - a_i)\rho, \mathcal{F}_s(\phi_s^*)\big)$$

in order to fulfill (2.4), when $\mathcal{F}_t(\phi_t^*)$ is merely continuous.

(b) For $m = 1$ we derive from (P_2) that $\mathcal{F}_t(\phi_t^*) \in C^1(\Omega, \mathbb{R}^d)$ holds. Hence, applying the interpolation estimate (2.12) for $j = 1$ leads to

$$\|\varepsilon_t^n(\phi_t^*)\| \le \sum_{i=1}^{\kappa} \tfrac{b_i - a_i}{n} \omega_i\big(\tfrac{b_i - a_i}{n}, D_i(\mathcal{F}_t(\phi_t^*))\big).$$

Thus, the inequality (2.4) will be satisfied, if we choose

$$\Gamma_0^0(\rho) := \rho \max_{s=1}^{\theta} \sum_{i=1}^{\kappa} (b_i - a_i)\omega_i\big((b_i - a_i)\rho, D_i(\mathcal{F}_s(\phi_s^*))\big).$$

(c) For $m = 2$ we obtain from (P_2) that $\mathcal{F}_t(\phi_t^*)$ is twice continuously differentiable. We deduce the error $\|\varepsilon_t^n(\phi_t^*)\| \le \frac{1}{8n^2} \sum_{i=1}^{\kappa} (b_i - a_i)^2 \left\|D_i^2(\mathcal{F}_t(\phi_t^*))\right\|$ for all $n \in \mathbb{N}$ from (2.13), and therefore (2.4) holds for the function

$$\Gamma_0^0(\rho) := \frac{\rho^2}{8} \sum_{i=1}^{\kappa} (b_i - a_i)^2 \max_{s=1}^{\theta} \left\|D_i^2(\mathcal{F}_s(\phi_s^*))\right\|.$$

ad (v): Because of (2.15) the Urysohn operator $\mathcal{U}_t$ is globally bounded. Since $\mathcal{G}_t$ is bounded due to [10, Thm. B.1], we obtain that $\mathcal{F}_t = \mathcal{G}_t \circ \mathcal{U}_t$ is globally bounded. Referring to (2.11) it follows that $\mathcal{F}_t^n = P_n \mathcal{F}_t$ is globally bounded uniformly in $n \in \mathbb{N}$. This carries over to the general solutions φ_n for all $n \in \mathbb{N}_0$.

Whence, the proof is concluded. ∎

2.2. Simulations. For convenience, let us restrict to interval domains $\Omega = [a, b]$ with reals $a < b$, i.e. $\kappa = 1$, and scalar IDEs

$$u_{t+1}(x) = G_t\left(x, \int_a^b f_t(x, y, u_t(y))\,\mathrm{d}y\right) \quad \text{for all } x \in [a, b]. \tag{2.19}$$

We apply piecewise linear collocation based on the hat functions $e_0, \dots, e_n : [a, b] \to \mathbb{R}$ (from above) with uniformly distributed nodes $\eta_j^n := a + j\frac{b-a}{n}$, $0 \le j \le n$ and $n \in \mathbb{N}$. This yields a semi-discretization (2.14). In order to arrive at full discretizations, the remaining integrals are approximated by the trapezoidal rule

$$\int_a^b u(y)\,\mathrm{d}y = \tfrac{b-a}{2n}\big(u(a) + 2\sum_{j=1}^{n-1} u(\eta_j^n) + u(b)\big) - \tfrac{(b-a)^3}{12n^2} u''(\xi) \tag{2.20}$$

with some intermediate $\xi \in [a, b]$. This leads to an explicit recursion

$$\boxed{\upsilon_{t+1} = \hat{\mathcal{F}}_t^n(\upsilon_t)} \tag{2.21}$$

in $\mathbb{R}^{n+1}$, with general solution $\hat{\varphi}_n$ and whose right-hand side reads as

$$\hat{\mathcal{F}}_t^n(\upsilon) := \left(G_t\left(\eta_i, \tfrac{b-a}{2n}\left(f_t(\eta_i, a, \upsilon(0)) + 2\sum_{j=1}^{n-1} f_t\left(\eta_i, \eta_j^n, \upsilon(j)\right) + f_t(\eta_i, b, \upsilon(n))\right)\right)\right)_{i=0}^{n}.$$

Then the coordinates $\upsilon_t(i)$ approximate the solution values $u_t(\eta_i)$. As error between the (globally attractive) θ-periodic solutions ϕ^* of (2.19) and υ^n to (2.21) we consider

$$\operatorname{err}(n) := \frac{1}{n} \sum_{t=0}^{\theta-1} \sum_{j=0}^{n} \left|\phi_t^*(\eta_j^n) - \upsilon_t^n(j)\right|.$$

The θ-periodic solutions of (2.21) are computed from the system of θ equations

$$v_0 = \hat{\mathfrak{F}}^n_{\theta-1}(v_{\theta-1}),\ v_1 = \hat{\mathfrak{F}}^n_0(v_0),\ v_2 = \hat{\mathfrak{F}}^n_1(v_1),\ \dots,\ v_{\theta-1} = \hat{\mathfrak{F}}^n_{\theta-2}(v_{\theta-2})$$

using inexact Newton-Armijo iteration implemented in the solver `nsoli` from [6].

Example 2.7. Let $\Omega = [0,1]$ and $\alpha \in \mathbb{R}$, $c \in \mathbb{R}_+$. We consider an autonomous IDE (2.19) (that is $\theta = 1$) with $U_t^1 = U_t^2 = \mathbb{R}$,

$$f_t(x,y,z) := \frac{\alpha}{1+x+z^2},$$

$$G_t(x,z) := z + \frac{1}{c+x} + \frac{\alpha}{1+x}\left(\frac{\arctan((1+c)\sqrt{1+x}) - \arctan(c\sqrt{1+x})}{\sqrt{1+x}} - 1\right)$$

and the constant solution $\phi^*(x) = \frac{1}{c+x}$. The mean value theorem leads to the Lipschitz estimate $\operatorname{lip}\mathcal{F}_t \leq \frac{3\sqrt{3}}{8}|\alpha|$. For $\alpha = \frac{3}{2}$, $c = \frac{1}{5}$ the right-hand side of (2.19) is contractive and the fixed-point u_n^* of (I_n) can be approximated by iteration. Choosing the initial function $u_0(x) := x$ the temporal evolution of the error

$$\operatorname{err}_n(t) := \frac{1}{n}\sum_{j=0}^{n}\left|\hat{\varphi}_n(t;0,u_0)(j) - \phi^*(\eta_j^n)\right|$$

is shown in Fig. 1 (left) for $n \in \{10^1, 10^2, 10^3\}$; it becomes stationary after a modest number of iterations. The limit is denoted by ϕ^n and is a fixed-point of (I_n). From Fig. 1 (left) we deduce that 20 iterates yield a good approximation. The error $\operatorname{err}(n)$ between v^n and ϕ^* as function of the discretization parameter n is illustrated in Fig. 1 (right). The slope of the curve in this diagram has the value -2.001, which confirms the quadratic convergence of piecewise linear collocation stated in (2.13).

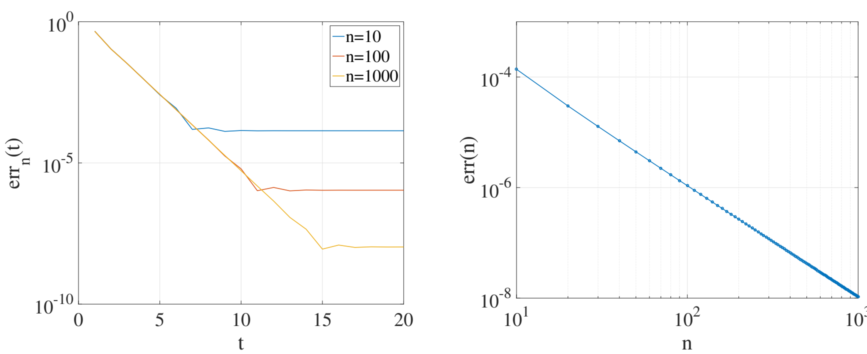

Fig. 1. *Quadratically decaying errors in Exam. 2.7*

While the right-hand side in Exam. 2.7 was arbitrarily smooth, we next discuss a less smooth example, being only Hölder (with exponent $\frac{1}{2}$) in x:

Example 2.8. Let $\Omega = [0,1]$, $\alpha \in \mathbb{R}$. We anew study an autonomous IDE (2.19) with $U_t^1 = U_t^2 = \mathbb{R}$,

$$f_t(x,y,z) := \alpha\frac{\sqrt{x}+y}{1+x+z^2}, \quad G_t(x,z) := z + \sqrt{x} - \alpha\left(1 + (1+x-\sqrt{x})\ln\frac{1+x}{2+x}\right)$$

and the constant solution $\phi^*(x) \equiv \sqrt{x}$. In order to derive a Lipschitz estimate for the right-hand side of (2.19) we obtain from the mean value theorem

$$\left| \tfrac{\sqrt{x}+y}{1+x+z^2} - \tfrac{\sqrt{x}+y}{1+x+\bar{z}^2} \right| \leq \tfrac{3\sqrt{3}(\sqrt{x}+y)}{8\sqrt{1+x}^3} \left|z - \bar{z}\right| \quad \text{for all } z, \bar{z} \in \mathbb{R},$$

consequently for every $u, \bar{u} \in C[0,1]$ it results

$$\begin{aligned} \left|\mathcal{F}(u)(x) - \mathcal{F}(\bar{u})(x)\right| &\leq |\alpha| \int_0^1 \tfrac{3\sqrt{3}(\sqrt{x}+y)}{8\sqrt{1+x}^3} \,\mathrm{d}y \,\|u - \bar{u}\| \\ &\leq |\alpha| \tfrac{3\sqrt{3}}{16} \max_{x\in[0,1]} \tfrac{2\sqrt{x}+1}{\sqrt{1+x}^3} \|u - \bar{u}\| = \tfrac{2\sqrt{2}(4+\sqrt{2(25-3\sqrt{41})})}{\sqrt{19-\sqrt{41}}^3} |\alpha| \,\|u - \bar{u}\| \end{aligned}$$

and thus $\operatorname{lip}\mathcal{F} \leq 0.47\,|\alpha|$. For $\alpha = 2$ the IDE (2.19) is contractive and the fixed-point u_n^* of (I_n) can be approximated by iteration. Using $u_0(x) := x$ as initial function, the temporal evolution of the error $\operatorname{err}_n(t)$ is shown in Fig. 2 (left) for $n \in \{10^1, 10^2, 10^3\}$ and becomes stationary after 80 iterations, while the dependence of $\operatorname{err}(n)$ is illustrated in Fig. 2 (right). The slope of the curve in this diagram has the value -2.003 yielding quadratic convergence, although the right-hand side is not of class C^2 in x anymore.

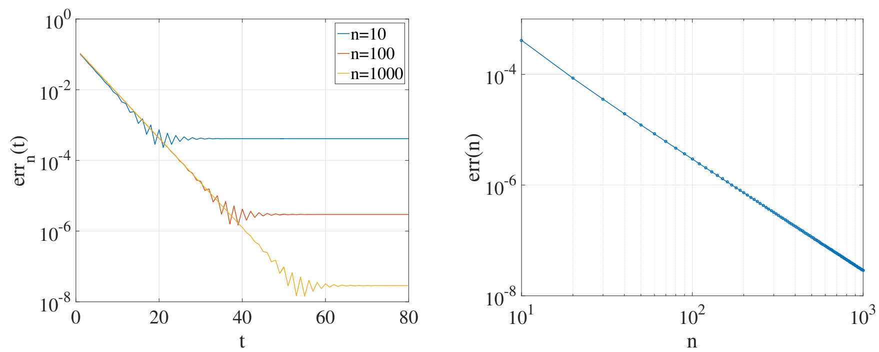

FIG. 2. *Quadratically decaying errors in Exam. 2.8*

Comparing Exam. 2.7 and 2.8 it is apparent that, although the same convergence rate is reached, iteration in the less smooth Exam. 2.8 needs longer to become stationary.

The following example is less academic and mimics biological models for species, which first disperse spatially and then grow. Here, explicit solutions are not known and in order to determine the convergence rate γ, we use an asymptotic formula

$$\frac{\left\|\phi^n - \phi^{2n}\right\|}{\left\|\phi^{2n} - \phi^{4n}\right\|} = \frac{\frac{K}{n^\gamma} - \frac{K}{(2n)^\gamma} + O(n^{-(\gamma+1)})}{\frac{K}{(2n)^\gamma} - \frac{K}{(4n)^\gamma} + O(n^{-(\gamma+1)})} = \frac{1 - 2^{-\gamma} + O(\frac{1}{n})}{2^{-\gamma} - 2^{-2\gamma} + O(\frac{1}{n})} = 2^\gamma + O(\tfrac{1}{n})$$

(as $n \to \infty$), relating the globally attractive θ-periodic solutions ϕ^n to (I_n). After a full discretization, the corresponding solutions υ^n and υ^{2n} are provided on different grids. To handle this, we compute the piecewise linear approximation $\hat{\phi}^n : [a,b] \to \mathbb{R}$ obtained from the values υ^n and work with the approximation

$$\left\|\phi^n - \phi^{2n}\right\| \approx \frac{1}{2n} \sum_{i=0}^{\theta-1} \sum_{j=0}^{2n-1} \left|\upsilon_t^{2n}(j) - \hat{\phi}_t^n(\eta_j^{2n})\right|$$

kernel	$k_\alpha(x)$	r_1^*	r_1
Gauß	$\frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}$	1.32	1.31
Laplace	$\frac{\alpha}{2} e^{-\alpha\lvert x\rvert}$	1.43	1.42

TABLE 1
Typical convolution kernels and critical parameter values in Exam. 2.9 and 3.2

in order to obtain convergence rates. In conclusion, our indicator for convergence rates is the limit of $c(n) := \log_2 \frac{\|\phi^n - \phi^{2n}\|}{\|\phi^{2n} - \phi^{4n}\|}$ for large values of n.

Example 2.9 (periodic Beverton-Holt equation). Let $\Omega = [-2, 2]$ and consider the 4-periodic sequence $\alpha_t := 5+4\sin\frac{\pi t}{2}$. We study the spatial Beverton-Holt equation

$$u_{t+1}(x) = r\frac{(2 - \frac{3}{2}\cos\frac{x}{2})\int_{-2}^{2} k_{\alpha_t}(x-y)u_t(y)\,\mathrm{d}y}{1 + \left|\int_{-2}^{2} k_{\alpha_t}(x-y)u_t(y)\,\mathrm{d}y\right|} \quad \text{for all } x \in [-2,2], \tag{2.22}$$

which is of the form (1.1) with $G_t(x, z) := r\frac{(2-\frac{3}{2}\cos\frac{x}{2})z}{1+|z|}$, $f_t(x, y, z) := k_{\alpha_t}(x-y)z$ and $U_t^1 = U_t^2 = \mathbb{R}$, where $k_\alpha : \mathbb{R} \to \mathbb{R}$ is a dispersal kernel from Tab. 1. The growth rate $r > 0$ is interpreted as bifurcation parameter and the trivial solution of (2.22) exhibits a transcritical bifurcation for some critical $r_1^* > 0$. If we choose $r = 4$, then Fig. 3 shows the 4-periodic orbits $\{\phi_0^*, \phi_1^*, \phi_2^*, \phi_3^*\}$ for the Gauß- (left) and Laplace-kernel (right). The table in Fig. 4 (left) indicates quadratic convergence of the scheme and

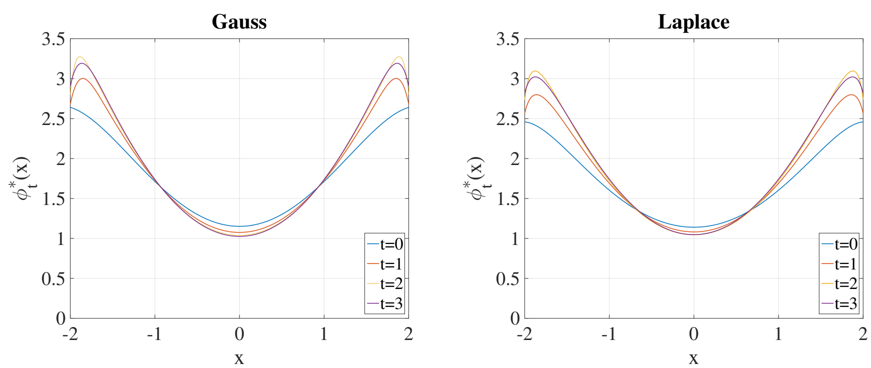

FIG. 3. *For Exam. 2.9 with $r = 4$: Attractive 4-periodic solutions of the Beverton-Holt IDE (3.8) with 4-periodic dispersal rates $(\alpha_t)_{t\in\mathbb{Z}}$: Gauß kernel (left) and Laplace kernel (right)*

thus confirms our theoretical result from Prop. 2.3(c). Moreover, the smooth Gauß kernel yields more accurate results than the Laplace kernel (see Fig. 4 (right)), which is not differentiable along the diagonal.

3. Hammerstein integrodifference equations. This section deals with systems of d Hammerstein IDEs, which often arise in applications [7]. Their right-hand side reads as

$$\mathcal{F}_t(u) := \int_a^b K_t(\cdot, y)g_t(y, u(y))\,\mathrm{d}y + h_t, \tag{3.1}$$

n	Gauß	Laplace
16	3.401293516	1.697576232
32	2.010916523	1.945062175
64	2.019632435	2.000945171
128	2.013192291	2.006543793
256	2.007446186	2.005257811
512	2.003910442	2.003008231
1024	2.002006939	2.001654344
2048	2.001024625	2.000882723

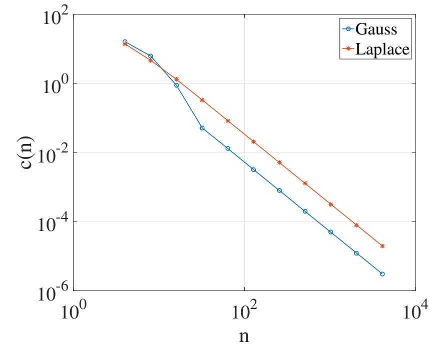

FIG. 4. *For Exam. 2.9 with $r = 4$: Approximations to the convergence rates $c(n)$ (left) and development of the error $\|\phi^{2n} - \phi^n\|$ (right) for $n \in \{2^2, \dots, 2^{11}\}$*

where we restrict to domains $\Omega = [a, b]$ for simplicity. Higher-dimensional domains Ω can be investigated like the rectangle Ω in Sect. 2.

For *kernels* $K_t : [a, b]^2 \to \mathbb{R}^{d\times p}$, *growth functions* $g_t : [a, b] \times U_t^1 \to \mathbb{R}^p$ and *inhomogeneities* $h_t : [a, b] \to \mathbb{R}^d$ we assume that there exists a period $\theta \in \mathbb{N}$ such that $K_t = K_{t+\theta}$, $g_t = g_{t+\theta}$ and $h_t = h_{t+\theta}$, $t \in \mathbb{Z}$.

Furthermore, let us impose the following standing assumptions for all $s \in \mathbb{Z}$:

- K_s is of class C^2 and $h_s \in C^2[a, b]^d$,
- $U_s^1 \subseteq \mathbb{R}^d$ is open, convex and nonempty, $g_s : [a, b] \times U_s^1 \to \mathbb{R}^p$ is a continuous function, the derivative $D_2 g_s : [a, b] \times U_s^1 \to \mathbb{R}^{p\times d}$ exists as continuous function and for all $\varepsilon > 0$, $x \in [a, b]$ there exists a $\delta > 0$ such that

$$|z_1 - z_2| < \delta \quad \Rightarrow \quad |D_2 g_s(x, z_1) - D_2 g_s(x, z_2)| < \varepsilon \text{ for all } z_1, z_2 \in U_s^1.$$

Since Hammerstein eqns. (I_0) are a special case of the IDEs studied in Sect. 2 with

$$U_s^2 = \mathbb{R}^d, \qquad G_s(x, z) := z + h_s(x), \qquad f_s(x, y, z) := K_s(x, y) g_s(y, z)$$

and convex domains $U_s := C([a, b], U_s^1)$, $s \in \mathbb{Z}$, this guarantees the properties (P_1–P_3) of their general solution φ_0 (cf. [10, Sect. 3.2]). In particular, the compact Fréchet derivative of $\mathcal{F}_s$ is

$$D\mathcal{F}_s(u)v = \int_a^b K_s(\cdot, y) D_2 g_s(y, u(y)) v(y)\,\mathrm{d}y \quad \text{for all } u \in U_s,\ v \in C_d.$$

Formally, a degenerate kernel discretization of (3.1) is given as

$$\mathcal{F}_t^n(u) := \int_a^b K_t^n(\cdot, y) g_t(y, u(y))\,\mathrm{d}y + h_t, \tag{3.2}$$

where $K_t^n : [a, b]^2 \to \mathbb{R}^{d\times p}$ serves as approximation of the original kernel K_t. In the following we discuss two possibilities, in which $e_j := e_j^1 : [a, b] \to [0, 1]$ denote the hat functions introduced in Sect. 2.1 with notes $\xi_j := a + \frac{j}{n}(b - a)$ for $0 \le j \le n$.

3.1. Linear degenerate kernels. A piecewise linear approximation of $K_t(\cdot, y)$, $y \in [a, b]$ fixed, yields the degenerate kernels

$$K_t^n(x, y) := \sum_{i=0}^{n} K_t(\xi_i, y) e_j(x) \quad \text{for all } n \in \mathbb{N},\ x, y \in [a, b].$$

The resulting discretization (3.2) essentially coincides with the collocation method discussed in Sect. 2.1. In fact, applying the projection operator $P_n \in L(C_d)$ onto $\operatorname{span}\{e_0, \dots, e_n\}$ to the right-hand side (3.1) yields $\mathcal{F}_t^n(u) = P_n\mathcal{F}_t(u) + h_t - P_n h_t$. Thus, apart from an occurrence of the term $h_t - P_n h_t$, the convergence analysis is covered by Prop. 2.3.

3.2. Bilinear degenerate kernels. In order to obtain an alternative semi-discretization ($\mathrm{I_n}$) of the Hammerstein IDE ($\mathrm{I_0}$), we apply the degenerate kernels

$$K_t^n(x,y) := \sum_{j_1=0}^n \sum_{j_2=0}^n e_{j_2}(y) K_t(\xi_{j_1}, \xi_{j_2}) e_{j_1}(x) \quad \text{for all } n \in \mathbb{N},\ x,y \in [a,b];$$

this yields a piecewise linear approximation of K_t. Since the kernels were assumed to be of class C^2, the interpolation estimate [3, p. 267] applies to each matrix entry and using the matrix norm induced by the maximum vector norm, leads to

$$\begin{aligned} |K_t^n(x,y) - K_t(x,y)| &= \max_{j_1=1}^d \sum_{j_2=1}^p |K_t^n(x,y)_{j_1j_2} - K_t(x,y)_{j_1j_2}| \\ &\le \frac{(b-a)^2}{8n^2} \max_{j_1=1}^d \sum_{j_2=1}^p \sum_{l=1}^2 \left\| D_l^2 K_t(\cdot)_{j_1j_2} \right\| \quad \text{for all } x,y \in [a,b]. \end{aligned} \tag{3.3}$$

We arrive at the semi-discretization ($\mathrm{I_n}$) with right-hand sides

$$\mathcal{F}_t^n(u) := \sum_{i_1=0}^n \left(\sum_{i_2=0}^n \int_a^b e_{i_2}(y) K_t(x_{i_1}, x_{i_2}) g_t(y, u_t(y))\, \mathrm{d}y \right) e_{i_1} + h_t \tag{3.4}$$

and the subsequent persistence and convergence result:

Proposition 3.1 (bilinear degenerate kernel). *Suppose that a θ-periodic solution ϕ^* of a Hammerstein IDE ($\mathrm{I_0}$) with right-hand side (3.1) satisfies the assumptions (i–ii) of Thm. 2.1 and choose $q \in (q_0, 1)$. If there exists a*

(i_{dg}) $\rho_0 > 0$ and a function $\tilde\gamma_1 \in \mathfrak{N}^*$ such that for all $y \in [a,b]$ holds

$$|D_2 g_s(y,z) - D_2 g_s(y,\bar z)| \le \tilde\gamma_1(|z - \bar z|) \quad \textit{for all } z, \bar z \in B_{\rho_0}(\phi_s^*(y)), \tag{3.5}$$

(ii_{dg}) $C \ge 0$ such that $|g_s(y,z)| \le C$ for all $y \in [a,b]$, $z \in U_s^1$

and each $1 \le s \le \theta$, then there exists a $N \in \mathbb{N}$ so that every degenerate kernel discretization ($\mathrm{I_n}$) with right-hand side (3.4) and $n \ge N$ possesses a globally attractive θ-periodic solution ϕ^n. Moreover, there is a $\tilde K \ge 1$ such that for all $n \ge N$ the following holds:

$$\|\phi_t^n - \phi_t^*\| \le \frac{\tilde K}{(1-q)n^2} \quad \textit{for all } t \in \mathbb{Z}.$$

We point out that Rem. 2.5 and 2.6 also apply in the present situation.

Proof. Let $n \in \mathbb{N}$. Before gradually verifying the assumptions of Thm. 2.1 applied to the right-hand sides (3.1) and (3.4), we begin with a convenient abbreviation

$$e_t := \frac{(b-a)^2}{8} \max_{j_1=1}^d \sum_{j_2=1}^p \sum_{l=1}^2 \left\| D_l^2 K_t(\cdot)_{j_1j_2} \right\| \quad \text{for all } t \in \mathbb{Z}$$

and an elementary estimate

$$|K_t^n(x,y)| \le |K_t(x,y)| + |K_t^n(x,y) - K_t(x,y)| \overset{(3.3)}{\le} \|K_t\| + \frac{e_t}{n^2} =: C_t(n) \tag{3.6}$$

for all $t \in \mathbb{Z}$ and $x, y \in [a,b]$. Clearly, the constants $C_t(n)$ are nonincreasing in $n \in \mathbb{N}$.

First, θ-periodicity of K_t, g_t and h_t extends to $\mathcal{F}_t^n$. For $t \in \mathbb{Z}$, $u \in U_t$ fixed and $v \in C_d$ with $\|v\| = 1$, we obtain the local discretization error

$$\begin{aligned}
|\varepsilon_t^n(u)(x)| &\overset{(3.4)}{\le} \int_a^b |K_t(x,y) - K_t^n(x,y)|\, |g_t(y,u(y))|\, \mathrm{d}y \\
&\overset{(3.3)}{\le} \frac{e_t}{n^2} \int_a^b |g_t(y,u(y))|\, \mathrm{d}y \quad \text{for all } x \in [a,b].
\end{aligned}$$

Second, from [10, Thm. 3.5(b)] we see that every $\mathcal{F}_t^n$ is continuously differentiable and

$$\begin{aligned}
|[D\varepsilon_t^n(u)v](x)| &\le \int_a^b |K_t(x,y) - K_t^n(x,y)|\, |D_2 g_t(y,u(y))v(y)|\, \mathrm{d}y \\
&\overset{(3.3)}{\le} \frac{e_t}{n^2} \int_a^b |D_2 g_t(y,u(y))|\, \mathrm{d}y \quad \text{for all } x \in [a,b].
\end{aligned}$$

Passing to the supremum over $x \in [a,b]$ in the previous two estimates leads to

$$\left\|D^j \varepsilon_t^n(u)\right\| \le \frac{e_t}{n^2} \int_a^b \left|D_2^j g_t(y,u(y))\right| \mathrm{d}y \quad \text{for all } j \in \{0,1\}\,. \tag{3.7}$$

Among the several consequences of this error estimate (3.7), we initially note that, because the substitution operator induced by the continuous function g_t is bounded, it follows from [10, Thm. B.1] that $(I_n)_{n\in\mathbb{N}}$ is bounded convergent.

ad (iii): It results using [10, Thm. 3.5] that all semi-discretizations $\mathcal{F}_t^n$ are completely continuous. The estimate (3.7) for $j = 1$ readily yields (2.3). Thanks to

$$D\mathcal{F}_t^n(u)v = \int_a^b K_t^n(\cdot,y) D_2 g_t(y,u(y))v(y)\, \mathrm{d}y$$

it results

$$\|D\mathcal{F}_t^n(u)\| \overset{(3.6)}{\le} C_t(n) \int_a^b |D_2 g_t(y,u(y))|\, \mathrm{d}y,$$

from which we furthermore observe that $D\mathcal{F}_t^n$ are bounded uniformly in $n \in \mathbb{N}$, because of $C_t(n) \le C_1(1)$. Moreover, (3.7) for $j = 0$ implies $\lim_{n\to\infty} \|\varepsilon_t^n(u)\| = 0$.

ad (iv): Again keeping an eye on the estimate (3.7), one can define

$$\Gamma_0^j(\rho) := \rho^2 \max_{s=1}^{\theta} e_s \int_a^b \left|D_2^j g_s(y,\phi_s^*(y))\right| \mathrm{d}y \quad \text{for all } j \in \{0,1\}$$

and consequently (2.4) holds. Moreover, given $u \in B_{\rho_0}(\phi_t^*)$, the estimate

$$\begin{aligned}
&|[D\mathcal{F}_t^n(u)v - D\mathcal{F}_t^n(\phi_t^*)v](x)| \\
&\quad \le \int_a^b |K_t^n(x,y)|\, |D_2 g_t(y,u(y)) - D_2 g_t(y,\phi_t^*(y))|\, |v(y)|\, \mathrm{d}y
\end{aligned}$$

$$\stackrel{(3.6)}{\leq} C_t(n) \int_a^b |D_2 g_t(y, u(y)) - D_2 g_t(y, \phi_t^*(y))| \, \mathrm{d}y$$

$$\stackrel{(3.5)}{\leq} (b-a) C_t(n) \tilde{\gamma}_1(\|u - \phi_t^*\|) \quad \text{for all } x \in [a,b],$$

after passing to the supremum over $x \in [a,b]$, allows us to choose

$$\gamma^1(\rho) := (b-a)\tilde{\gamma}_1(\rho) \max_{s=1}^{\theta} C_s(1)$$

in the final required inequality (2.5).

ad (v): The boundedness assumption (ii_{dg}) implies that both $\mathcal{F}_t$, as well as the semi-discretizations $\mathcal{F}_t^n$ are globally bounded uniformly in $n \in \mathbb{N}$. This evidently extends to the general solutions φ_n for all $n \in \mathbb{N}_0$ and the proof is finished. ☐

3.3. Simulations. Consider a scalar Hammerstein IDE

$$u_{t+1}(x) = \int_a^b k_{\alpha_t}(x-y) g(u_t(y)) \, \mathrm{d}y \quad \text{for all } x \in [a,b] \tag{3.8}$$

with convolution kernels $k_\alpha : \mathbb{R} \to \mathbb{R}$ (see Tab. 1) depending on dispersal parameter $\alpha_t > 0$ and a (nonlinear) growth function $g : \mathbb{R} \to \mathbb{R}$.

The degenerate kernel semi-discretization (3.4) of (3.8) simplifies to

$$u_{t+1} = \sum_{j_1=0}^{n} \left(\sum_{j_2=0}^{n} k_{\alpha_t}(\eta_{j_1}^n - \eta_{j_2}^n) \int_a^b e_{j_2}(y) g(u_t(y)) \, \mathrm{d}y \right) e_{j_1}, \qquad \eta_j^n := a + j\tfrac{b-a}{n}.$$

If we discretize the remaining integrals by the trapezoidal rule (2.20), then the full discretization (2.21) has the right-hand side

$$\hat{\mathcal{F}}_t^n(\upsilon) := \frac{b-a}{2n} \left(k_{\alpha_t}(\eta_i^n - a) g(\upsilon(0)) + 2 \sum_{j=1}^{n-1} k_{\alpha_t}(\eta_i^n - \eta_j^n) g(\upsilon(j)) + k_{\alpha_t}(\eta_i^n - b) g(\upsilon(n)) \right)_{i=0}^{n}.$$

Here, the values $\upsilon_t(i)$ approximate $u_t(\eta_i)$ for $0 \leq i \leq n$.

We now consider a situation dual to Exam. 2.9 in the sense that (3.8) models populations which first grow and then disperse.

Example 3.2 (periodic Beverton-Holt equation). On $\Omega = [-2, 2]$ we study the Beverton-Holt function $g(z) := r\frac{(2 - \frac{3}{2}\cos\frac{x}{2})z}{1+|z|}$ to describe growth and use the 4-periodic sequence $(\alpha_t)_{t \in \mathbb{Z}}$ from Exam. 2.9 as dispersal parameters. Again the growth rate $r > 0$ is interpreted as bifurcation parameter. The trivial solution of (3.8) exhibits a transcritical bifurcation for some critical $r_1 > 0$. Due to [2, Thm. 5.1] the nontrivial 4-periodic solution ϕ^* is globally attractive for $r > r_1$. In particular for $r = 4$, Fig. 5 illustrates the orbit $\{\phi_0^*, \phi_1^*, \phi_2^*, \phi_3^*\}$. As theoretically predicted by Prop. 3.1, quadratic convergence is confirmed by the table in Fig. 6 (left). Again, the errors $c(n)$ for the smooth Gauß kernel are smaller than for the Laplace kernel (see Fig. 5 (right)).

Appendix A. Robustness of global stability. Assume $U \subseteq X$ is a nonempty, open, convex subset of a Banach space X and (Λ, d) denotes a metric space. The subsequent result is a quantitative version of [13, Thm. 2.1]:

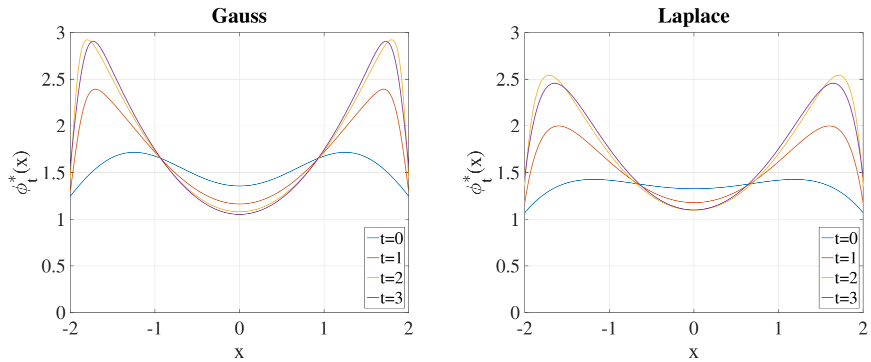

FIG. 5. *For Exam. 3.2 with $r = 4$: Globally attractive 4-periodic solutions of the Beverton-Holt IDE (3.8) with 4-periodic dispersal rates $(\alpha_t)_{t\in\mathbb{Z}}$: Gauß kernel (left) and Laplace kernel (right)*

n	Gauß	Laplace
16	4.094543296	1.96612629
32	1.993927677	2.006424007
64	2.018281087	2.013700027
128	2.012291764	2.009501587
256	2.006785445	2.005516125
512	2.003552025	2.002888726
1024	2.001813352	2.001476866
2048	2.000915199	2.000748714

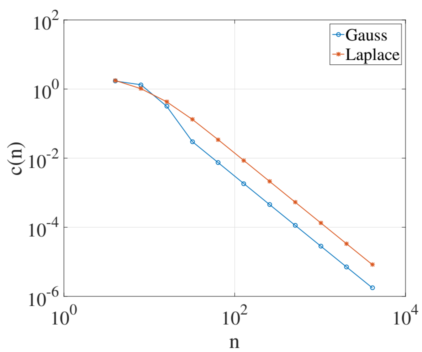

FIG. 6. *For Exam. 3.2 with $r = 4$: Approximations to the convergence rates $c(n)$ (left) and development of the error $\|\phi^{2n} - \phi^n\|$ (right) for $n \in \{2^2, \dots, 2^{11}\}$*

THEOREM A.1. *Let $q \in [0, 1)$, $\lambda_0 \in \Lambda$ and assume that $\Gamma_0 \in \mathfrak{N}$, $\gamma_0 : \mathbb{R}_+^2 \to \mathbb{R}_+$ are functions with $\lim_{\rho_1,\rho_2 \searrow 0} \gamma_0(\rho_1, \rho_2) = 0$. If the C^1-mappings $\Pi_\lambda : U \to U$, $\lambda \in \Lambda$, satisfy the following properties*

(i') there exists a $u_0 \in U$ with $\lim_{s\to\infty} \Pi_{\lambda_0}^s(u) = u_0$ for all $u \in U$,

(ii') $(u, \lambda) \mapsto D\Pi_\lambda(u)$ exists as continuous function with $\|D\Pi_{\lambda_0}(u_0)\| \le q$,

(iii') there exists a $\rho_0 > 0$ such that for all $u \in B_{\rho_0}(u_0) \cap U$, $\lambda \in \Lambda$ it holds

$$\|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \le \Gamma_0(d(\lambda, \lambda_0)), \tag{A.1}$$

$$\|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u_0)\| \le \gamma_0(\|u - u_0\|, d(\lambda, \lambda_0)), \tag{A.2}$$

(iv') for every $\lambda \in \Lambda$ there is a set $\tilde{B}_\lambda \subset U$ such that for each $u \in U$, there exists a $T \in \mathbb{N}$ such that $\Pi_\lambda^T(u) \in \tilde{B}_\lambda$,

(v') $\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda)$ is relatively compact in U

and $\rho \in (0, \rho_0)$, $\delta > 0$ are chosen so small that $\bar{B}_\rho(u_0) \subset U$,

$$\Gamma_0(\delta) \le \tfrac{1-q}{2}\rho, \qquad \gamma_0(\rho, \delta) \le \tfrac{1-q}{2}, \tag{A.3}$$

then there exists a continuous mapping $u^* : B_\delta(\lambda_0) \to \bar{B}_\rho(u_0)$ with

(a) $u^*(\lambda_0) = u_0$ and $\Pi_\lambda(u^*(\lambda)) \equiv u^*(\lambda)$ on $B_\delta(\lambda_0)$,

(b) $\|u^*(\lambda) - u_0\| \le \frac{2}{1-q}\Gamma_0(d(\lambda, \lambda_0))$,

(c) $\lim_{t\to\infty}\Pi_\lambda^t(u)=u^*(\lambda)$ *for all* $u\in U$, $\lambda\in B_\delta(\lambda_0)$.

Proof. (a) For all $u\in\bar B_\rho(u_0)$, $\lambda\in B_\delta(\lambda_0)$ one concludes the relation

$$\|D\Pi_\lambda(u)\|\leq\|D\Pi_{\lambda_0}(u_0)\|+\|D\Pi_\lambda(u)-D\Pi_{\lambda_0}(u_0)\|\stackrel{(A.2)}{\leq}q+\gamma_0(\rho,\delta)\stackrel{(A.3)}{\leq}\tfrac{q+1}{2}<1$$

from (ii'). The mean value theorem [8, p. 341, Thm. 4.2] and the convexity of U imply

$$\|\Pi_\lambda(\bar u)-\Pi_\lambda(u)\|\leq\int_0^1\|D\Pi_\lambda(u+\vartheta(\bar u-u))\|\,\mathrm{d}\vartheta\,\|u-\bar u\|\leq\tfrac{1+q}{2}\|u-\bar u\|$$

for all $u,\bar u\in\bar B_\rho(u_0)$, $\lambda\in B_\delta(\lambda_0)$. Referring to (i'), the continuity of Π_{λ_0} guarantees that $\Pi_{\lambda_0}(u_0)=u_0$ and thus

$$\begin{aligned}\|\Pi_\lambda(u)-u_0\| &\leq \|\Pi_\lambda(u)-\Pi_\lambda(u_0)\|+\|\Pi_\lambda(u_0)-\Pi_{\lambda_0}(u_0)\|\\ &\stackrel{(A.1)}{\leq}\tfrac{1+q}{2}\|u-u_0\|+\Gamma_0(d(\lambda,\lambda_0))\stackrel{(A.3)}{\leq}\tfrac{1+q}{2}\rho+\tfrac{1-q}{2}\rho=\rho.\end{aligned}$$

The latter two estimates imply that $\Pi_\lambda:\bar B_\rho(u_0)\to\bar B_\rho(u_0)$ is both well-defined and a contraction uniformly in $\lambda\in B_\delta(\lambda_0)$. The uniform contraction principle guarantees that there exists a unique fixed point function $u^*:B_\delta(\lambda_0)\to\bar B_\rho(u_0)$ satisfying (a).

(b) For all $\lambda\in B_\delta(\lambda_0)$ the estimate (b) readily results from

$$\begin{aligned}\|u^*(\lambda)-u_0\| &\leq \|\Pi_\lambda(u^*(\lambda))-\Pi_\lambda(u_0)\|+\|\Pi_\lambda(u_0)-\Pi_{\lambda_0}(u_0)\|\\ &\stackrel{(A.1)}{\leq}\tfrac{1+q}{2}\|u^*(\lambda)-u_0\|+\Gamma_0(d(\lambda,\lambda_0)).\end{aligned}$$

(c) The global attractivity of $u^*(\lambda)$ w.r.t. the mapping Π_λ for $\lambda\in B_\delta(\lambda_0)$ can be shown just as in [13, proof of Thm. 2.1]. ∎

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