p4 skip δ

$$\|x_{k+1}-x_*\|^2 - \|x_k - x_*\|^2 = \underbrace{\|x_{k+1}-x_k\|^2}_{=\|F'(x_k)^*(F(x_k)-y^\delta)\|^2 \le \|F(x_k)-y^\delta\|} + 2\langle x_{k+1}-x_k, x_k - x_*\rangle$$

$\underbrace{x_{k+1}-x_*}_{=x_{k+1}-x_k+x_k-x_*}$

$$= -\langle F(x_k)-y^\delta, \underbrace{F'(x_k)(x_k-x_*)}_{\approx F(x_k)-y^\delta}\rangle \approx -\|F(x_k)-y^\delta\|^2$$

$$= -\|F(x_k)-y^\delta\|^2$$
$$+2\langle F(x_k)-y^\delta, \underbrace{F(x_k)-y^\delta-F'(x_k)(x_k-x_*)}\rangle$$
$$\|.\| \le \eta\|F(x_k)-F(x_*)\| + \delta \le \eta\|F(x_k)-y^\delta\| + (1+\eta)\delta$$

$$\le -\|F(x_k)-y^\delta\|\left((1-2\eta)\|F(x_k)-y^\delta\| - 2(1+\eta)\delta\right)$$
$$= -\|F(x_k)-y^\delta\|(1-2\eta)\left(\|F(x_k)-y^\delta\| - 2\frac{1+\eta}{1-2\eta}\delta\right) = \smile$$

p5: $\forall k \le k_*-1$: $\|x_{k+1}-x_*\|^2 - \|x_k-x_*\|^2 \le -\underbrace{(1-2\eta)\left(1-2\frac{1+\eta}{(1-2\eta)\tau}\right)}_{=:c}\|F(x_k)-y^\delta\|^2$

$$\Rightarrow c\sum_{k=0}^{k_*-1}\underbrace{\|F(x_k)-y^\delta\|^2}_{\ge\tau^2\delta^2} \le \sum_{k=0}^{k_*-1}\left(\|x_k-x_*\|^2 - \|x_{k+1}-x_*\|^2\right) = \|x_0-x_*\|^2 - \|x_{k_*}-x_*\|^2$$

$$\Rightarrow k_* \cdot c\tau^2\delta^2 \le \|x_0-x_*\|$$

case $\delta = 0$: $\forall k \in \mathbb{N}: \|F(x_k)-y^\delta\|^2 \ge \tau\delta$: $\sum_{k=0}^{\infty}\|F(x_k)-y\|^2 \le \frac{\|x_0-x_*\|^2}{c}$

$$\Rightarrow F(x_k) \xrightarrow[k\to\infty]{} y$$

□

convergence with exact data

$e_k := x_k - x^+$; $\|e_k\|$ mon. decr., $\|e_k\| \ge 0 \Rightarrow \exists \varepsilon \ge 0: \|e_k\| \searrow \varepsilon$

We prove that e_k is a Cauchy sequence

$j > k$ arb. fixed; choose $l \in \{k,..j\}$ s.t. $\forall i \in \{k..j\}$ $\|F(x_l)-y\| \le \|F(x_i)-y\|$

$\|e_k - e_j\| \le \|e_k - e_l\| + \|e_l - e_j\|$, where $\|e_l - e_j\|^2 = 2\langle e_l - e_j, e_l\rangle + \underbrace{\|e_j\|^2 - \|e_l\|^2}_{\xrightarrow[k,l,j\to\infty]{}0}$

$$|\langle e_l - e_j, e_l\rangle| = |\langle x_l - x_j, x_l - x^+\rangle| = \left|\sum_{i=l}^{j-1}\langle \underbrace{x_{i+1}-x_i}_{=-F'(x_i)^*(F(x_i)-y)}, x_l - x^+\rangle\right|$$

$$= \left|\sum_{i=l}^{j-1}\langle F(x_i)-y, F'(x_i)(x_l-x_i) + F'(x_i)(x_i-x^+)\rangle\right|$$

$$\le \sum_{i=l}^{j-1}\|F(x_i)-y\|\left((1+\eta)\underbrace{\|F(x_l)-F(x_i)\|}_{\le\|F(x_l)-y\|+\|F(x_i)-y\| \le 2\|F(x_i)-y\|} + (1+\eta)\|F(x_i)-F(x^+)\|\right)$$

$$\le 3(1+\eta)\sum_{i=l}^{j-1}\|F(x_i)-y\|^2 \xrightarrow[k,l,j\to\infty]{} 0 \quad \text{since } \sum_{n=0}^{\infty}\|F(x_n)-y\|^2 < \infty$$

analogously: $\|e_k - e_l\| \to 0$

$\Rightarrow e_k$ converges to $e_* \Rightarrow x_k = e_k + x^+ \to e^* + x^+ = x^* \Rightarrow F(x^*) = y$ □

convergence with noisy data:

$x_*:=\lim_{k\to\infty} x_k$ (exact data) ; $\forall k\in\mathbb{N}: x_k^\delta \underset{\delta\to 0}{\to} x_k$ (finitely many applications of the continuous operators F, F'^*)

subsequence - subsequence argument

$(\delta_n)_{n\in\mathbb{N}}$ arbitrary fixed, $\delta_n \searrow 0$, $\|y_n - y\|\le\delta_n$; $k_n := k_*(\delta_n, y_n)$

case 1: $(k_n)_{n\in\mathbb{N}}$ has finite accumulation point k_0

$\Rightarrow \exists (k_{n_m})_{m\in\mathbb{N}}\ \forall m\in\mathbb{N}: k_{n_m}=k_0 \Rightarrow \forall m\in\mathbb{N}\ \|F(x_{k_0}^{\delta_{n_m}})-y_{n_m}\|\le\tau\delta_{n_m}$

$\underset{m\to\infty}{\Rightarrow} x_{k_0}^{\delta_{n_m}} \to x_{k_0}$ and $\|F(x_{k_0})-y\|\le 0$ ($x_k\to x_*$)

$\Rightarrow x_{k_0+1} = x_{k_0} - F'(x_{k_0})^*(F(x_{k_0})-y) = x_{k_0} \Rightarrow \dots \Rightarrow \forall k\ge k_0: x_k = x_{k_0} \Rightarrow x_{k_0}=x_*$

i.e. $x_{k_n}^{\delta_n}$ has a subsequence converging to x_*

case 2: $k_n \underset{n\to\infty}{\to} \infty$

$\Rightarrow \exists (k_{n_m})_{m\in\mathbb{N}}: k_{n_m} \underset{m\to\infty}{\nearrow} \infty$

$\Rightarrow \forall m\le l: \|x_{k_{n_l}}^{\delta_{n_l}} - x_*\| \le \|x_{k_{n_m}}^{\delta_{n_l}} - x_*\| \le \|x_{k_{n_m}}^{\delta_{n_l}} - x_{k_{n_m}}\| + \|x_{k_{n_m}} - x_*\|$

$\varepsilon>0$ arb. fixed

$x_k \underset{k\to\infty}{\to} x_* \Rightarrow \exists m: \|x_{k_{n_m}} - x^*\| < \frac{\varepsilon}{2}$

$x_k^\delta \underset{\delta\to 0}{\to} x_k \Rightarrow \exists l\ge m: \|x_{k_{n_m}}^\delta - x_{k_{n_m}}\| < \frac{\varepsilon}{2}$

i.e. $x_{k_n}^{\delta_n}$ has a subsequence converging to x_* □

$\langle (K^*K+\alpha I)^{-1} r, -2r + (KK^*+\alpha I)^{-1} KK^* r\rangle$

$= (K(K^*+\alpha I)^{-1}(KK^*r+\alpha r)$ [illegible]

$= -\langle (K^*K+\alpha I)^{-1} r, K(K^*K+\alpha I)^{-1}K^*r\rangle - 2\alpha \|(K^*K+\alpha I)^{-1} r\|^2$

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linear convergence of LM residuals:

$\|F(x_{k+1})-y\| = \|F(x_k)-y+F'(x_k)(x_{k+1}-x_k) + F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1}-x_k)\|$

$\le q\|F(x_k)-y\| + \eta \underbrace{\|F(x_{k+1})-F(x_k)\|}_{\le \|F(x_{k+1})-y\| + \|F(x_k)-y\|}$

$\Rightarrow (1-\eta)\|F(x_{k+1})-y\| \le (q+\eta)\|F(x_k)-y\|$

$\Rightarrow \|F(x_{k+1})-y\| \le \frac{q+\eta}{1-\eta}\|F(x_k)-y\|$