p 4 proof of well-definedness:

- $J_\alpha(x) \geq 0 \ \forall x \in X \Rightarrow \inf\limits_{x \in \mathcal{D}(F)} J_\alpha(x) =: I \geq 0$ exists

$\Rightarrow \exists (x_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(F): J_\alpha(x_n) \underset{n\to\infty}{\to} I$ (minimizing sequence)

- $(J_\alpha(x_n))_{n\in\mathbb{N}}$ converges $\Rightarrow \exists C > 0 \ \forall n \in \mathbb{N}: C \geq J_\alpha(x_n) = \|F(x_n) - y^\delta\| + \alpha \|x_n - x_0\|^2$

$$\Rightarrow \left.\begin{array}{l} \forall n \in \mathbb{N}: \|x_n - x_0\| \leq \sqrt{\frac{C}{\alpha}} \\ \|F(x_n) - y^\delta\| \leq \sqrt{C} \end{array}\right\} \Rightarrow \exists \text{subsequ. } n_k: \left.\begin{array}{l} x_{n_k} \rightharpoonup x \\ F(x_{n_k}) \rightharpoonup f \end{array}\right\} \underset{(3)}{\Rightarrow} \begin{array}{l} x \in \mathcal{D}(F) \\ F(x) = f \end{array}$$

- $\|.\|$ weakly lower semicont.

$$\Rightarrow J_\alpha(x) \leq \liminf_{k\to\infty} \|F(x_{n_k}) - y^\delta\|^2 + \liminf_{k\to\infty} \alpha \|x_{n_k} - x_0\|^2 \leq \limsup_{k\to\infty} \left(\|F(x_{n_k}) - y^\delta\|^2 + \alpha \|x_{n_k} - x_0\|^2\right) = I$$

□

p 5 proof of convergence: (skip subscript $_k$)

minimality $\Rightarrow \|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha \|x_\alpha^\delta - x_0\|^2 \leq \|F(x^+) - y^\delta\|^2 + \alpha \|x^+ - x_0\|^2$ (MIN)

$$\Rightarrow \left\{\begin{array}{l} \|F(x_{\alpha(\delta)}^\delta) - y^\delta\|^2 \leq \delta^2 + \alpha(\delta) \|x^+ - x_0\|^2 \to 0 \\ \|x_{\alpha(\delta)}^\delta - x_0\|^2 \leq \frac{\delta^2}{\alpha} + \|x^+ - x_0\|^2 \to \|x^+ - x_0\|^2 \end{array}\right\} \text{as } \delta \to 0$$

$\Rightarrow \exists$ subseq. $\delta_\ell \to 0: \left(x_\ell := x_{\alpha(\delta_\ell)}^{\delta_\ell} \underset{\ell\to\infty}{\rightharpoonup} x \ \wedge \ F(x_\ell) \underset{\ell\to\infty}{\rightharpoonup} f\right)$ (*)

For any subseq. $\delta_\ell \to 0$ such that (*) holds we have:

(3) $\Rightarrow x \in \mathcal{D}(F) \wedge F(x) = f \wedge \forall g \in Y: \langle f - y, g\rangle = \langle f - F(x_\ell), g\rangle + \langle F(x_\ell) - y^{\delta_\ell}, g\rangle + \langle y^{\delta_\ell} - y, g\rangle \underset{\ell\to\infty}{\to} 0$

that is: $x \in \mathcal{D}(F) \wedge F(x) = y \xRightarrow{x^+ \text{ unique}} x = x^+$

norm conv: $\limsup\limits_{\ell\to\infty} \|x_\ell - x^+\|^2 = \underbrace{\limsup\limits_{\ell\to\infty} \|x_\ell - x_0\|^2}_{\leq \|x^+ - x_0\|^2} + \|x_0 - x^+\|^2 + 2 \underbrace{\limsup\limits_{\ell\to\infty} \langle x_\ell - x_0, x_0 - x^+\rangle}_{\underset{\ell\to\infty}{\to} -\|x^+ - x_0\|} = 0$

subsequence-subsequence argument $\Rightarrow x_{\alpha(\delta)}^\delta \underset{\delta\to 0}{\to} x^+$ □

p 8 proof of convergence rates:

$$(\text{MIN}) \Rightarrow \underbrace{\|F(x_\alpha^\delta) - y^\delta\|}_{=:r}{}^2 + \alpha \underbrace{\|x_\alpha^\delta - x^+\|}_{e}{}^2 \leq \delta^2 + 2\alpha \underbrace{\langle x_\alpha^\delta - x^+, \overbrace{x^+ - x_0}^{= F'(x^+)^* w}\rangle}_{= \langle w, F'(x^+)(x_\alpha^\delta - x^+)\rangle}$$

$$\|F'(x^+)(x_\alpha^\delta - x^+)\| = \|F(x_\alpha^\delta) - y^\delta - (F(x_\alpha^\delta) - F(x^+) - F'(x^+)(x_\alpha^\delta - x^+)) + y^\delta - y\| \leq r + \tfrac{L}{2} e^2 + \delta$$

$\Rightarrow r^2 + \alpha e^2 \leq \delta^2 + 2\alpha \|w\| (r + \frac{L}{2} e^2 + \delta)$ (**)

$\Rightarrow (r - \alpha\|w\|)^2 + \alpha(1 - L\|w\|) e^2 \leq (\delta + \alpha \|w\|)^2$

$\Rightarrow r \leq \delta + 2\alpha \|w\|, \quad e \leq \frac{1}{\sqrt{1 - L\|w\|}} \left(\frac{\delta}{\sqrt{\alpha}} + \sqrt{\alpha} \|w\|\right)$

[a priori choice $\alpha = c\delta$
$\Rightarrow e \leq \tilde{C} \sqrt{\delta}$]

discrepancy principle: $\underline{\tau}\delta \leq r \leq \overline{\tau}\delta$

$\Rightarrow (\underline{\tau} - 1)\delta \leq 2\alpha\|w\| \Rightarrow \frac{1}{\alpha} \leq \frac{2\|w\|}{\underline{\tau} - 1} \cdot \frac{1}{\delta}$

(**) with $r^2 \geq \underline{\tau}^2 \delta^2 \geq \delta^2 \Rightarrow (1 - L\|w\|) e^2 \leq \frac{1}{\alpha} \left(\delta^2 + 2\alpha\|w\|(r + \delta)\right) \leq \delta \cdot \left(\frac{2\|w\|}{\underline{\tau} - 1} + 2\|w\|(\overline{\tau} + 1)\right)$

p 10 $\underbrace{F'(a)h}_{=:\bar{v}} = ?$ $\quad F(\tilde{a}) = \tilde{u}, \; F(a) = u$ $\quad v = \tilde{u} - u$ $\quad \begin{cases} -(\tilde{a}\tilde{u}')' = f = -(a u')' \Rightarrow -(\tilde{a} v')' = (hu')' & h : \tilde{a} - a \\ v = 0 \text{ on } \partial\Omega \end{cases}$

$h = \tilde{a} - a \to 0 \quad \rightsquigarrow v \to \bar{v}$ with $\begin{cases} -(a\bar{v}')' = (hu')' & \text{in } \Omega = (0,1) \\ \bar{v} = 0 & \text{on } \partial\Omega \end{cases}$

$\underbrace{F'(a)^* w}_{=:p} = ? \quad \langle F'(a)h, w\rangle_{L^2} \overset{!}{=} \langle h, F'(a)^* w\rangle_{H^1} \quad \forall h \in H^1(0,1)$, $w \in L^2(0,1)$

$$\underbrace{\int_0^1 (A(a))^{-1}[(hu')'](s)\, w(s)\, ds}_{} \overset{!}{=} \int_0^1 (h'(s)p'(s) + h(s)p(s))\, ds$$

$$= \int_0^1 (hu')'(A(a)^{-1}w)(s)\, ds$$

$$= -\int_0^1 h(s)\, u'(s)\, (A(a)^{-1}w)'(s)\, ds$$

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source condition: $a^+ - a_0 = F'(a^+)^* w \underset{\substack{\uparrow \\ u = F(a^+)}}{=} -B^{-1}[u'(A(a)^{-1}w)']$

$$\Leftrightarrow w = -A(a)\left[\frac{B(a^+ - a_0)}{u'}\right] \in L^2(\Omega) \Leftrightarrow (*)$$

assume $|u'(s)| \geq \gamma > 0 \; \forall s \in (0,1) \Rightarrow$ <u>in</u>homogeneous boundary cond. needed!

$(*) \Leftrightarrow a^+ - a_0 \in H^4(0,1) \wedge (a^+ - a_0)'(0) = 0 = (a^+ - a_0)'(1) \wedge \left(\frac{B(a^+ - a_0)}{u'}\right)(s) = 0, \; s \in \{0,1\}$
