p 12,13 $x \in \arg\min \|Tx - y^\delta\|^2 + \alpha \|x - x_0\|^2 =: J_\alpha(x)$

$$\Leftrightarrow \forall h \in X: 0 = J_\alpha'(x)h = 2\langle Tx - y^\delta, Th\rangle + 2\alpha\langle x - x_0, h\rangle$$
$$= 2\left(\langle T^*(Tx - y^\delta) + \alpha(x - x_0), h\rangle\right)$$
$$\Leftrightarrow (T^*T + \alpha I)x = T^*y^\delta + \alpha x_0$$

Tikh $(T^*T + \alpha I)^{-1} T^* y = q_\alpha(T^*T)T^*y \leftarrow q_\alpha(\lambda) = \frac{1}{\lambda + \alpha}$

$$r_\alpha(\lambda) = 1 - q_\alpha(\lambda)\lambda = \frac{\lambda + \alpha - \lambda}{\lambda + \alpha} = \frac{\alpha}{\lambda + \alpha}$$

it Tikh $x_{\alpha,n+1} = (T^*T + \alpha I)^{-1}(T^*y^\delta + \alpha x_{\alpha,n}^\delta) = \left(\frac{1}{\alpha}T^*T + I\right)^{-(n+1)} x_0 + (T^*T + \alpha I)^{-1} \sum_{j=0}^{n} \left(\frac{1}{\alpha}T^*T + I\right)^{-j} T^*y$

$$x_0 = 0 \quad q_{\alpha,n} = \frac{1}{\lambda + \alpha}\sum_{j=0}^{n-1}\left(\frac{\alpha}{\lambda+\alpha}\right)^j = \frac{1}{\lambda+\alpha}\,\frac{1 - \left(\frac{\alpha}{\lambda+\alpha}\right)^n}{1 - \frac{\alpha}{\lambda+\alpha}} = \frac{(\lambda+\alpha)^n - \alpha^n}{\lambda(\lambda+\alpha)^n}$$
$$r_\alpha(\lambda) = 1 - \frac{(\lambda+\alpha)^n - \alpha^n}{(\lambda+\alpha)^n} = \left(\frac{\alpha}{\lambda+\alpha}\right)^n$$

LW $x_{n+1} = x_n - T^*(Tx_n - y) = (I - T^*T)x_n + T^*y = (I - T^*T)^{n+1}x_0 + \sum_{j=0}^{n}(I - T^*T)^j T^*y$

$$x_0 = 0: \quad q_{\alpha,n}(\lambda) = \sum_{j=0}^{n-1}(1 - \lambda)^j = \frac{1 - (1-\lambda)^n}{\lambda}$$
$$r_\alpha(\lambda) = 1 - (1 - (1-\lambda)^n) = (1-\lambda)^n$$

TSVD: $x_\alpha = \sum_{\sigma_j^2 \ge \alpha} \frac{1}{\sigma_j}\langle y, v_j\rangle u_j = \sum_{j\in\mathbb{N}} 1\!\!1_{[\alpha,\infty)}(\sigma_j^2)\frac{1}{\sigma_j^2}\underbrace{\sigma_j\langle y, v_j\rangle u_j}_{\sum_j \cdot \,=\, T^*y}$

$$q_\alpha(\lambda) = 1\!\!1_{[\alpha,\infty)}(\lambda)\cdot\frac{1}{\lambda}$$
$$r_\alpha(\lambda) = 1 - 1\!\!1_{[\alpha,\infty)}(\lambda) = 1\!\!1_{[0,\alpha)}(\lambda)$$

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p 15: $\|R_\alpha y\|^2 = \sum_{j=1}^\infty |q_\alpha(\sigma_j^2)\sigma_j\langle y, v_j\rangle|^2 = \sum_{j=1}^\infty \underbrace{q_\alpha(\sigma_j^2)}_{\le \frac{C_q}{\alpha}} \underbrace{q_\alpha(\sigma_j^2)\sigma_j^2}_{\le 1 + C_r}\langle y, v_j\rangle^2$

$$\le \frac{C_q(1+C_r)}{\alpha}\underbrace{\sum_{j=1}^\infty \langle y, v_j\rangle^2}_{\le \|y\|^2}$$

$y \in D(T^\dagger) \Rightarrow \|T^\dagger y - R_\alpha y\|^2 = \|x^\dagger - x_\alpha\|^2 = \|r_\alpha(T^*T)x^\dagger\|^2 = \sum_{j=1}^\infty \underbrace{r_\alpha(\sigma_j^2)^2\langle x^\dagger, u_j\rangle^2}_{\begin{cases}\le C_r^2\langle x^\dagger, u_j\rangle^2 \\ \xrightarrow{\alpha\to 0} 0 \ \forall j\end{cases}}$

Lebesgue's Dom Conv. Thm $\Rightarrow \|T^\dagger y - R_\alpha y\| \xrightarrow{\alpha \to 0} 0$

$$\|R_{\bar\alpha(\delta,y^\delta)}y^\delta - T^\dagger y\| \le \underbrace{\|R_{\bar\alpha(\delta,y^\delta)}(y^\delta - y)\|}_{\le \sqrt{C_q(1+C_r)}\frac{\delta}{\sqrt{\bar\alpha}} \to 0} + \underbrace{\|R_{\bar\alpha(\delta,y^\delta)}y - T^\dagger y\|}_{\xrightarrow{\delta\to 0} 0 \text{ wg } \bar\alpha(\delta,y^\delta)\to 0}$$

p16: Tikh: $\sup_{\lambda\in[0,\|T^*T\|]} \underbrace{|\lambda^\mu \frac{\alpha}{\lambda+\alpha}|}_{=\phi(\lambda)} = ?$

$\phi(0)=0;\ \phi(\|T^*T\|) = \frac{\|T^*T\|^\mu \alpha}{\|T^*T\|+\alpha}$

$0=\phi'(\lambda) = \mu\lambda^{\mu-1}\frac{1}{\lambda+\alpha} + \lambda^\mu\left(-\frac{1}{\lambda+\alpha}\right)^2 = \frac{\lambda^{\mu-1}}{(\lambda+\alpha)^2}\underbrace{(\mu(\lambda+\alpha)-\lambda)}_{=\mu\alpha-(1-\mu)\lambda}$

$\Leftrightarrow \bar\lambda = \frac{\mu}{1-\mu}\alpha\ ;\ \phi(\bar\lambda) = \left(\frac{\mu}{1-\mu}\right)^\mu \alpha^\mu (1-\mu)\frac{\alpha}{\alpha}$

$= \mu^\mu (1-\mu)^{1-\mu}\alpha^\mu$

case $\mu=1$: $\phi'(\lambda) = \frac{1}{(\lambda+\alpha)^2}\mu\alpha \geq 0 \Rightarrow \bar\lambda = \frac{\|T^*T\|}{\|T^*T\|+\alpha}\cdot\alpha$

p17: $\|x^+-x_\alpha\| = \|r_\alpha(T^*T)x^+\| = \|r_\alpha(T^*T)(T^*T)^\mu w\|$

$= \sqrt{\sum_{j=1}^\infty \left(\underbrace{r_\alpha(\sigma_j^2)(\sigma_j^2)^\mu}_{\leq C_\mu\alpha^\mu}\langle w,u_j\rangle\right)^2} \leq C_\mu\alpha^\mu\|w\|$

$\Rightarrow \|x^+-x_\alpha^\delta\| \leq \|x^+-x_\alpha\| + \|R_\alpha(y^\delta-y)\| \leq \underbrace{C_\mu\alpha^\mu\|w\| + \tilde C\frac{\delta}{\sqrt\alpha}}_{=:\phi(\alpha)}$

optimal choice of $\alpha$: minimize rhs.

$\min_{\alpha>0}\phi$: $\lim_{\alpha\to 0}\phi(\alpha) = +\infty$ $\quad 0=\phi'(\alpha) = C_\mu\|w\|\mu\alpha^{\mu-1} - \frac12\tilde C\delta\alpha^{-\frac32}$

$\lim_{\alpha\to+\infty}\phi(\alpha) = +\infty$ $\quad = \alpha^{-\frac32}(C_\mu\|w\|\mu\alpha^{\mu+\frac12} - \frac12\tilde C\delta)$

$\rightsquigarrow \alpha := \left(\frac{\tilde C}{2C_\mu\|w\|\mu}\delta\right)^{\frac{1}{\mu+\frac12}}$

$\bar\alpha = c\,\delta^{\frac{2}{2\mu+1}} \Rightarrow \|x^+-x_\alpha^\delta\| \leq \phi(\bar\alpha) = C_\mu c^\mu\delta^{\frac{2\mu}{2\mu+1}}\|w\| + \tilde C\delta^{\underbrace{1-\frac{1}{2\mu+1}}_{=\frac{2\mu}{2\mu+1}}}$

source cond in terms of SVD

$x^+ = (T^*T)^\mu w \Leftrightarrow \langle y, v_j\rangle = \sigma_j\langle x^+, u_j\rangle = \sigma_j^{1+2\mu}\langle w, u_j\rangle$

$w\in X \Leftrightarrow \infty > \sum_{j=1}^\infty \langle w,u_j\rangle^2 = \sum_{j=1}^\infty \frac{1}{\sigma_j^{2+4\mu}}\langle y, v_j\rangle^2$

$\rightsquigarrow$ more restrictive than Picard