$J'(x)[h] =$

$$2\langle Tx - y^\delta, Th\rangle + 2\alpha\langle x - x_0, h\rangle = 0$$

$$\sum_{j=1}^{\infty} (\sigma_j^2 + \alpha)\langle x_\alpha^\delta, u_j\rangle u_j = \sum_{j=1}^{\infty}\left(\sigma_j \langle y^\delta, v_j\rangle + \alpha\langle x_0, u_j\rangle\right) u_j$$

$$\Leftrightarrow \langle x_\alpha^\delta, u_j\rangle =$$

$$(-\Delta)^{-1} x = \sigma x$$

$$\Leftrightarrow \sigma x = -\Delta x$$

$$\Omega = (0,1) : \ -x''(s) = \frac{1}{\sigma} x(s)$$

$$\rightsquigarrow x(s) = a \sin\left(\tfrac{1}{\sqrt{\sigma}} s\right) + b \cos\left(\tfrac{1}{\sqrt{\sigma}} s\right)$$

$$x(0) = x(1) = 0 \Rightarrow b = 0, \ \sqrt{\sigma} = \pi j \quad j \in \mathbb{N}$$

$$a = \left(\int_0^1 \sin\left(\tfrac{1}{\pi j} s\right)^2 ds\right)^{-1} \ \ldots \text{normalisation}$$

Lemma: $(\forall \lambda \in [0, \|A\|] \ |f(\lambda)| \le C) \Rightarrow \|f(A)\| \le C$

Proof
$$\underbrace{\Big\|\sum_{j=1}^{\infty} f(\lambda_j)\langle x, u_j\rangle u_j\Big\|_X^2}_{= \|f(A)x\|_X} \underset{\text{Parseval}}{=} \sum_{j=1}^{\infty} \underbrace{f(\lambda_j)^2}_{\le C} |\langle x, u_j\rangle|^2 \le C \sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 \underset{\text{Bessel}}{\le} C\|x\|_X$$

Lemma $f(T^*T)T^* = T^* f(TT^*)$

Proof
$$f(T^*T)T^*y = \sum_{j=1}^{\infty} f(\sigma_j^2)\underbrace{\langle T^*y, u_j\rangle}_{= \langle y, Tu_j\rangle = \sigma_j\langle y, v_j\rangle} u_j = \sum_{j=1}^{\infty} f(\sigma_j^2)\sigma_j \langle y, v_j\rangle u_j$$

$$T^* f(TT^*) y = \sum_{j=1}^{\infty} \sigma_j \underbrace{\langle f(TT^*)y, v_j\rangle}_{= \langle \sum_{\ell=1}^{\infty} f(\sigma_\ell^2)\langle y, v_\ell\rangle v_\ell, v_j\rangle = f(\sigma_j^2)\langle y, v_j\rangle} = \sum_{j=1}^{\infty} \sigma_j f(\sigma_j^2)\langle y, v_j\rangle v_j$$

Lemma $x = (T^*T)^{\frac{1}{2}} w = T^* v \qquad \|w\| = \|v\|$

$$\Rightarrow x \overset{(T^*T)^{\frac{1}{2}} w}{=} \sum_{j=1}^{\infty} (\sigma_j^2)^{\frac{1}{2}} \langle w, u_j\rangle u_j = \sum_{j=1}^{\infty} \sigma_j \langle w, u_j\rangle u_j$$

$$v := \sum_{j=1}^{\infty} \langle w, u_j\rangle v_j \in Y \qquad \|v\|_Y^2 = \sum_{j=1}^{\infty} \langle w, u_j\rangle^2 = \|w\|_X^2$$

$$T^* v = \sum_{j=1}^{\infty} \sigma_j \langle v, v_j\rangle u_j = \sum_{j=1}^{\infty} \sigma_j \Big\langle \sum_{\ell=1}^{\infty} \langle w, u_\ell\rangle v_\ell, v_j\Big\rangle u_j = x$$