Exercise Sheet 3

Implementation of some iterative regularization method for the reconstruction of a diffusion coefficient

Consider in the following the model of groundwater filtration in a given domain Ω with known (vanishing) potential u at the boundary. The *forward problem* is to determine the potential (or piezometric head) u after a sufficiently long time, i.e. the solution of the boundary value problem

$$
\begin{aligned}
-div(a(x,y,z)\nabla u) &= f(x,y,z), \quad (x,y,z) \in \omega && (1)\\
u &= 0 \quad \in \partial\Omega && (2)
\end{aligned}
$$

where f denotes internal sinks and sources and a is the (spatially varying) transmissivity. The *inverse problem* is now to determine a from internal measurements of the potential u.

The task is to implement a scheme which determines $a = a(x)$ in the one-dimensional case, i.e.

$$-(a(x)u_x)_x = f(x), \quad x \in (0,1) \tag{3}$$

from 100 equidistant point measurements in the interior in $(0,1)$. Note that if the first spatial derivative of u does not vanish, a can be computed explicitly as $a(x) = \frac{1}{u_x(x)}\left[a(0)u_x(0) - \int_0^x f(s)ds\right]$ and is therefore uniquely determined.

For the solution of this inverse problem one of the following schemes should be implemented and tested:

- Nonlinear Landweber iteration

$$a_{k+1}^\delta = a_k^\delta + F'(a_k^\delta)^*(u^\delta - F(a_k^\delta)) \tag{4}$$

- Iteratively regularized Gauss-Newton method

$$a_{k+1}^\delta = a_k^\delta + (F'(a_k^\delta)^* F'(a_k^\delta) + \alpha_k I)^{-1} F'(a_k^\delta)^*(u^\delta - F(a_k^\delta) + \alpha_k(a_0 - a_k^\delta)) \tag{5}$$

  where a_0 denotes the initial guess for $a(x)$.

In both methods F denotes the parameter to solution map

$$F : D(F) := \{a \in H^1[0,1] | a(x) \geq \underline{a} > 0\} \to L^2[0,1]$$

$$a \mapsto F(a) := u(a)$$

where $u(a)$ is the solution of the forward problem (1). It can be shown that the derivative and its adjoint are given by

$$F'(a)h = A(a)^{-1}[(hu_x(a))_x]$$

and

$$F'(a)^* w = -B^{-1}[u_x(a)(A(a)^{-1}w)_x]$$

where

$$\begin{aligned} A(a) : H^2[0,1] \cap H_0^1[0,1] &\to L^2[0,1] \\ u &\mapsto A(a)u := -(au_x)_x \end{aligned}$$

and

$$\begin{aligned} B : \mathcal{D}(B) := \{\psi \in H^2[0,1] \mid \psi'(0) = \psi'(1) = 0\} &\to L^2[0,1] \\ \psi &\to B\psi := -\psi'' + \psi \end{aligned}$$

Both methods should be stopped by the discrepancy principle.

Solution of the forward problem (also of its linearization as well as of the PDE defining B) can be done either numerically (by finite differences or 1-d finite elements) or — in this 1-d case — analytically.

In order to obtain measurements u^δ solve the forward problem analytically with a given $a(x)$ and add random noise of 1, 2, 4 and 8 per cent, which you can do, by e.g, the folloing commands in (pseudo)matlab, N being the size of the discretized version of u):
`noise = randn(N,1);`
$u^\delta = u +$ `percent * (norm(u,2)/norm(noise,2)) * noise;`
with `percent` $\in \{0.01, 0.02, 0.04, 0.08\}$.

Try out the implemented methods for

- $a(x) \equiv 1$
- $a(x) = 1 + \frac{1}{5}|x - \frac{1}{2}|$
- $a(x) = 1 + \frac{1}{5}(x - \frac{1}{2})^2$

For the (prescribed) source term f you may try to use just a constant function; alternatively to this, you can prescribe u and compute the resulting f from (3).