Exercise Sheet 1

Regularization by projection

Let X and Y be Hilbert spaces and $T \in L(X,Y)$ compact with $\mathcal{N}(T) = \{0\}$ and $\overline{\mathcal{R}(T)} = Y$.

Consider a sequence $Y_0 \subset Y_1 \subset \ldots$ of finite dimensional subspaces of Y with orthogonal projections $Q_n : Y \to Y_n$, such that $\overline{\bigcup_{n\in\mathbb{N}} Y_n} = Y$, hence $\lim_{n\to\infty} Q_n y = y$ for all $y \in Y$.

The operator equation $Tx = y$ with $y \in R(T)$ is approximated by

$$Q_n T x_n = Q_n y. \tag{1}$$

We abbreviate $T_n := Q_n T$ and $X_n := T^* Y_n$ and define an approximation x_n^δ of $x^\dagger$ by the best approximate solution

$$x_n^\delta := T_n^\dagger Q_n y^\delta \in \mathcal{N}(T_n)^\perp = X_n$$

of (1).

1. Prove that: In case $\delta = 0$, $y \in \mathcal{D}(T^\dagger)$, the approximation $x_n := T_n^\dagger Q_n y$ is the orthogonal projection of $x^\dagger$ onto X_n. Moreover, $x_n \to x^\dagger = T^\dagger y$ as $n \to \infty$.

2. Prove that: The family $\{T_n^\dagger Q_n\}$ with an a prior parameter choice strategy $\overline{n}(\delta)$ is a regularization method iff

$$\overline{n}(\delta) \to \infty \text{ and } \frac{\delta}{\rho_{\overline{n}(\delta)}} \to 0 \text{ as } \delta \to 0\,,$$

   where ρ_n is the smallest nonzero singular value of T_n, i.e., the smallest singular value of $T_n|_{X_n}$.

3. Let $\dim Y_n = n$. Prove that

$$\rho_n \le \sigma_n, \qquad n \in \mathbb{N}.$$

   and equality holds iff $Y_n = \operatorname{span}\{v_1, \ldots, v_n\}$, where $\{(\sigma_j; u_j, v_j)\}_{j\in\mathbb{N}}$ is a singular system of T.

   (Hint: The smallest eigenvalue of a positive definite selfadjoint operator $A \in L(Z,Z)$ can be characterized as $\lambda_{\min}(A) = \min_{z\in Z\setminus\{0\}} \frac{\langle Az,z\rangle}{\|z\|^2}$. How are the singular values of T_n related to the eigenvalues of $T_n^* T_n$?)

   In this case the method described above coincides with truncated singular value decomposition

$$R_n y = \sum_{j=1}^{n} \frac{1}{\sigma_j} \langle y, v_j\rangle u_j$$

   or alternatively, in terms of a threshold value α

$$R_\alpha y = \sum_{\sigma_j^2 \ge \alpha} \frac{1}{\sigma_j} \langle y, v_j\rangle u_j \tag{2}$$

4. Derive the functions q_α and r_α for TSVD in the formulation (2) and verify conditions (11), (12), (13), and (18) (with $\mu_0 = \infty$)

Remark

Alternatively to projecting onto finite dimensional subspaces in image space Y, one could consider a sequence of finite dimensional subspaces X_n of X and define x_n as the bestapproximate solution in X_n of $Tx = y$, i.e., with noisy data

$$x_n^\delta \in \operatorname{argmin}\{\|\tilde{x}_n\| \, : \, \tilde{x}_n \in \operatorname{argmin}\{\|T\hat{x} - y^\delta\| \, : \, \hat{x} \in X_n\}\}$$

(i.e., $T\tilde{x}_n \in Y_n := TX_n$ is the metric projection of y^δ onto Y_n). However, this method converges only under certain conditions; see e.g., [Engl, Hanke, Neubauer 1996], [Kirsch, 1996] for further details on regularization by projection.