Methods for Inverse Problems:
III. Landweber iteration

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overview

1. Landweber iteration for nonlinear problems
2. Monotonicity of the error
3. Convergence
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Landweber iteration for nonlinear problems

Gradient methods for the minimization of

$$\min \tfrac{1}{2}\|F(x) - y\|^2 \qquad \text{over} \quad \mathcal{D}(F)\,.$$

$$x_{k+1}^\delta = x_k^\delta + \omega_k^\delta F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\,, \tag{1}$$

Landweber iteration: $\omega_k^\delta \equiv 1$

$$x_{k+1}^\delta = x_k^\delta + F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\,, \qquad k \in \mathbb{N}_0\,. \tag{2}$$

Assumptions:

scaling:

$$\|F'(x)\| \le 1\,, \qquad x \in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)\,. \tag{3}$$

tangential cone condition:

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \le \eta \|F(\tilde{x}) - F(x)\| \tag{4}$$

[Hanke&Neubauer&Scherzer 1995]

Monotonicity of the error

Theorem

Assume that the conditions (3) *and* (4) *hold and that the equation* $F(x) = y$ *has a solution* $x_* \in \mathcal{B}_\rho(x_0)$. *If* $x_k^\delta \in \mathcal{B}_\rho(x_*)$ *and*

$$\|y^\delta - F(x_k^\delta)\| > 2\,\frac{1+\eta}{1-2\eta}\,\delta\,.$$

then $x_k^\delta, x_{k+1}^\delta \in \mathcal{B}_\rho(x_*) \subset \mathcal{B}_{2\rho}(x_0)$ *and*

$$\|x_{k+1}^\delta - x_*\| \leq \|x_k^\delta - x_*\|$$

$\rightsquigarrow$ choose

$$\tau \geq 2\,\frac{1+\eta}{1-2\eta} \tag{5}$$

in the stopping rule according to the discrepancy principle:

$$\|y^\delta - F(x_{k_*}^\delta)\| \leq \tau\delta < \|y^\delta - F(x_k^\delta)\|\,, \qquad 0 \leq k < k_*\,, \tag{6}$$

Square summability of the residuals

Corollary

Let the assumptions of Proposition 1 hold and let k_* be chosen according to the stopping rule (6), (5). Then

$$k_*(\tau\delta)^2 < \sum_{k=0}^{k_*-1} \|y^\delta - F(x_k^\delta)\|^2 \le \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)} \|x_0 - x_*\|^2.$$

In particular, if $y^\delta = y$ (i.e., if $\delta = 0$), then

$$\sum_{k=0}^{\infty} \|y - F(x_k)\|^2 < \infty\,. \tag{7}$$

Convergence

Theorem

Assume that the conditions (3) *and* (4) *hold and that the equation* $F(x) = y$ *is solvable in* $\mathcal{B}_\rho(x_0)$*. Then the nonlinear Landweber itertaion applied to exact data* y *converges to a solution of* $F(x) = y$*. If* $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ *for all* $x \in \mathcal{B}_\rho(x^\dagger)$*, then* x_k *converges to the* x_0*-minimum-norm-solution* $x^\dagger$ *as* $k \to \infty$*.*

Theorem

Let the assumptions of Theorem 3 *hold and let* $k_* = k_*(\delta, y^\delta)$ *be chosen according to the stopping rule* (6)*,* (5)*. Then the Landweber iterates* $x_{k_*}^\delta$ *converge to a solution of* $F(x) = y$*. If* $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ *for all* $x \in \mathcal{B}_\rho(x^\dagger)$*, then* $x_{k_*}^\delta$ *converges to* $x^\dagger$ *as* $\delta \to 0$*.*

Convergence rates

Theorem

Let additionally to the assumptions of Theorem 4

$$F'(x) = R_x F'(x^\dagger) \quad \textit{and} \quad \|R_x - I\| \le c\,\|x - x^\dagger\|\,, \qquad x \in \mathcal{B}_{2\rho}(x_0)\,.$$

(stronger than (4)*) hold. If* $\tau > 2$ *and if* $x^\dagger - x_0$ *satisfies*

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^\mu v\,, \quad v \in \mathcal{N}(F'(x^\dagger))^\perp \tag{8}$$

with some $0 < \mu \le 1/2$ *and* $\|v\|$ *sufficiently small, then* $k_* = O\Big(\|v\|^{\frac{2}{2\mu+1}} \delta^{-\frac{2}{2\mu+1}}\Big)$ *and*

$$\|x_{k_*}^\delta - x^\dagger\| = \begin{cases} o\Big(\|v\|^{\frac{1}{2\mu+1}} \delta^{\frac{2\mu}{2\mu+1}}\Big), & \mu < \frac{1}{2}\,, \\ O\Big(\sqrt{\|v\|\delta}\Big), & \mu = \frac{1}{2}\,. \end{cases}$$

Steepest descent and minimal error method

$$\omega_k^\delta := \frac{\|s_k^\delta\|^2}{\|F'(x_k^\delta)s_k^\delta\|^2} \qquad \text{and} \qquad \omega_k^\delta := \frac{\|y^\delta - F(x_k^\delta)\|^2}{\|s_k^\delta\|^2}$$

- monotonicity of the errors and well-definedness
- convergence can be shown for perturbed data
- convergence rates for exact data [Neubauer&Scherzer 1995]

Further Literature

- Landweber in Hilbert scales, preconditioning [Egger&Neubauer 2005]
- iteratively regularized Landweber

$$x_{k+1}^\delta = x_k^\delta + F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)) + \beta_k(x_0 - x_k^\delta)$$
$$\text{with} \quad 0 < \beta_k \le \overline{\beta} < \tfrac{1}{2} .$$

  convergence rates results under weaker restrictions on the nonlinearity of F [Scherzer 1998]
- generalization to Banach space setting: [Schöpfer&Louis&Schuster 2006, Schöpfer&Schuster&Louis 2008, BK &Schöpfer&Schuster 2009]