Mathematics of nonlinear acoustics: modeling, analysis and inverse problems

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joint work with

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Modeling - Analysis - Optimization
DOC.FUNDS DOCTORAL SCHOOL

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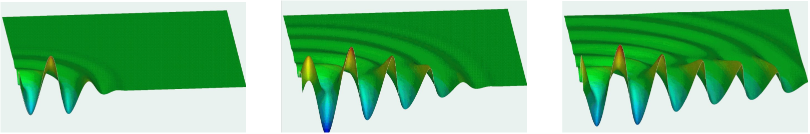
Outline

- modeling:
  - models of nonlinear acoustics
  - fractional damping models in ultrasonics
- parameter asymptotics
- some inverse problems

Nonlinear Acoustic Wave Propagation



nonlinear wave propagation:

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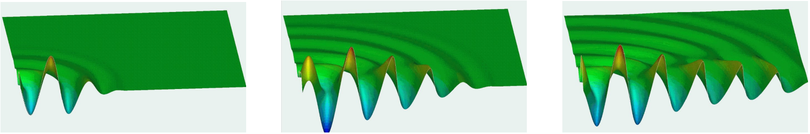

nonlinear wave propagation:

sound speed depends on (signed) amplitude $\Rightarrow$ sawtooth profile

models of nonlinear acoustics

Physical Principles

main physical quantities:

- acoustic particle velocity $\mathbf{v}$;
- acoustic pressure p;
- mass density ϱ;
- absolute temperature ϑ;
- heat flux $\boldsymbol{q}$;
- entropy η;

decomposition into mean and fluctuating part:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\sim = \mathbf{v}\,, \qquad p = p_0 + p_\sim\,, \qquad \varrho = \varrho_0 + \varrho_\sim$$

Physical Principles

- acoustic particle velocity $\mathbf{v}$;
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governing equations:

- momentum conservation = Navier Stokes equation (with $\nabla \times \mathbf{v} = 0$):

$$\varrho\Big(\mathbf{v}_t + \nabla(\mathbf{v}\cdot\mathbf{v})\Big) + \nabla p = \Big(\frac{4\mu_V}{3} + \zeta_V\Big)\Delta\mathbf{v}$$

- mass conservation = equation of continuity: $\nabla\cdot(\varrho\mathbf{v}) = -\varrho_t$
- entropy equation: $\varrho\vartheta(\eta_t + \mathbf{v}\cdot\nabla\eta) = -\nabla\cdot\boldsymbol{q}$
- equation of state: $\dfrac{p}{p_0} = \varrho^\gamma \exp\left(\dfrac{\eta-\eta_0}{c_v}\right)$
- Gibbs equation: $\vartheta d\eta = c_v d\vartheta - p\dfrac{1}{\varrho^2}d\varrho$

$\gamma = \frac{c_p}{c_v}$ . . . adiabatic index;

c_p / c_v . . . specific heat at constant pressure / volume;

ζ_V / μ_V . . . bulk / shear viscosity

Physical Principles

So far, 5 equations for 6 unknowns $\mathbf{v}$, p, ϱ, ϑ, $\boldsymbol{q}$, η.
Still need a constitutive relation between temperature and heat flux.

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leads to infinite speed of propagation paradox.

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leads to infinite speed of propagation paradox.

Maxwell-Cattaneo law $\tau\boldsymbol{q}_t + \boldsymbol{q} = -K\nabla\vartheta$

τ... relaxation time
allows for "thermal waves" (second sound phenomenon)

Classical Models of Nonlinear Acoustics

- **Kuznetsov's equation** [Lesser & Seebass 1968, Kuznetsov 1971]

$$p_{\sim tt} - c^2 \Delta p_\sim - \delta \Delta p_{\sim t} = \left(\frac{B}{2A\varrho_0 c^2} p_\sim^2 + \varrho_0 |\mathbf{v}|^2 \right)_{tt}$$

where $\varrho_0 \mathbf{v}_t = -\nabla p \quad \rightsquigarrow \quad \varrho_0 \psi_t = p$
for the **particle velocity v** and the **pressure** p, i.e.,

$$\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left(\frac{B}{2A\, c^2} (\psi_t)^2 + |\nabla \psi|^2 \right)_t$$

since $\nabla \times \mathbf{v} = 0$ hence $\mathbf{v} = -\nabla \psi$ for a **velocity potential** ψ

$\delta = \kappa(\mathrm{Pr}(\frac{4}{3} + \frac{\zeta_V}{\mu_V}) + \gamma - 1)$ ... diffusivity of sound; κ... thermal diffusivity
$\frac{B}{A} \hat{=} \gamma - 1$ ... nonlinearity parameter (in liquids / gases)

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for the **particle velocity v** and the **pressure** p

- **Westervelt equation** [Westervelt 1963] via $\varrho_0 |\mathbf{v}|^2 \approx \frac{1}{\varrho_0 c^2}(p_\sim)^2$

$$p_{\sim tt} - c^2 \Delta p_\sim - \delta \Delta p_{\sim t} = \frac{1}{\varrho_0 c^2} \left(1 + \tfrac{B}{2A} \right) p^2_{\sim tt}$$

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Advanced Models of Nonlinear Acoustics (Examples)

- **Jordan-Moore-Gibson-Thompson** equation [Jordan 2009, 2014], [Christov 2009], [Straughan 2010]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta + \tau c^2)\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

$\tau\ldots$ relaxation time

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$z := \psi_t + \frac{c^2}{\delta+\tau c^2}\psi$ solves weakly damped wave equation

$$z_{tt} - \tilde{c}\Delta z + \gamma z_t = r(z, \psi)$$

with $\tilde{c} = c^2 + \frac{\delta}{\tau}$, $\gamma = \frac{1}{\tau} - \frac{c^2}{\delta+\tau c^2} > 0$
$\rightsquigarrow$ second sound phenomenon

Advanced Models of Nonlinear Acoustics (Examples)

- **Blackstock-Crighton** equation [Brunnhuber & Jordan 2016], [Blackstock 1963], [Crighton 1979]

$$(\partial_t - a\Delta)\left(\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t\right) - ra\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_{tt}$$

$a = \frac{\nu}{\mathrm{Pr}} \ldots$ thermal conductivity

Advanced versus Classical Models of Nonlinear Acoustics

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  τ . . . relaxation time

- cf. Kuznetsov:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

- further models:[Angel & Aristegui 2014], [Christov & Christov & Jordan 2007], [Kudryashov & Sinelshchikov 2010], [Ockendon & Tayler 1983], [Makarov & Ochmann 1996], [Rendón & Ezeta &Pérez-López 2013], [Rasmussen & Sørensen & Christiansen 2008], [Soderholm 2006], ...
- resonances, shock waves:[Ockendon & Ockendon & Peake & Chester 1993], [Ockendon & Ockendon 2001, 2004, 2016],...
- traveling waves solutions:[Jordan 2004], [Chen & Torres & Walsh 2009], [Keiffer & McNorton & Jordan & Christov, 2014], [Gaididei & Rasmussen & Christiansen & Sørensen, 2016],...
- well-posendness and asymptotic behaviour:
  for KZK: [Rozanova-Pierrat 2007, 2008, 2009, 2010]
  for Westervelt, Kuznetsov, Blackstock-Crighton, JMGT on bounded domain Ω:
  based on semigroup theory and energy estimates:[BK & Lasiecka 2009, 2012], [BK & Lasiecka & Veljović 2011], [BK & Lasiecka & Marchand 2012], [BK & Lasiecka & Pospiezalska 2012], [Lasiecka & Wang 2015], [Liu & Triggiani 2013], [Marchand & McDevitt & Triggiani 2012], [Nikolić 2015], [Nikolić & BK 2016], [Pellicer & Solá-Morales 2019], , [Dell'Oro&Lasiecka&Pata 2020]
  based on maximal L_p regularity:[Meyer & Wilke 2011, 2013], [Meyer & Simonett 2016], [Brunnhuber & Meyer 2016], [BK 2016]
  Cauchy problem (on $\Omega = \mathbb{R}^\varkappa$)
  for Kuznetsov: [Dekkers & Rozanova-Pierrat 2019]
  for JMGT: [Pellicer & Said-Houari 2017], [Nikolić & Said-Houari 2021]
- control of JMGT [Bucci&Lasiecka 2020], [Bucci&Pandolfi 2020]

The Westervelt equation: potential degeneracy

with $\kappa := \frac{1+\frac{B}{2A}}{\varrho_0 c^2}$, $\quad u = p_\sim$

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa (u^2)_{tt}$$

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This also illustrates state dependence of the effective wave speed:

$$u_{tt} - \tilde{c}(u)^2 \Delta u - \tilde{b}(u) \Delta u_t = f(u)$$

with $\tilde{c}(u) = \dfrac{c}{\sqrt{1-2\kappa u}}$, $\tilde{b}(u) = \dfrac{b}{1-2\kappa u}$, $f(u) = \dfrac{2\kappa (u_t)^2}{1-2\kappa u}$

as long as $2\kappa u < 1$ (otherwise the model loses its validity) parameter asymptotics

Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation $\quad (b = \delta + \tau c^2)$

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

Kuznetsov's equation:

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Does ψ^0 solve Kuznetsov's equation?

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[Bongarti&Charoenphon&Lasiecka; BK& Nikolić, 2019-21]

Remarks

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- For $\tau = 0$ (classical Westervelt and Kuznetsov equation) the reformulation of the linearization as a first order system leads to an analytic semigroup and maximal parabolic regularity.
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$$
\begin{aligned}
\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = & \left(\frac{k}{2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t \\
= & k\psi^\tau_t\psi^\tau_{tt} + |\nabla\psi^\tau|^2_t
\end{aligned}
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$$\iff \quad \tau\psi^\tau_{ttt} + (1 - k\psi^\tau_t)\psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = |\nabla\psi^\tau|^2_t$$

Plan of the analysis

- Establish well-posedness of the linearized equation along with energy estimates.
- Use these results to prove well-posedness of the Westervelt type JMGT equation for $\tau > 0$ by a fixed point argument.
- Establish additional higher order energy estimates.
- Use these results to prove well-posedness of the Kuznetsov type JMGT equation for $\tau > 0$ (sufficiently small) by a fixed point argument.
- Take limits as $\tau \to 0$

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BK & Vanja Nikolić. On the Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. *Math. Meth. Mod. Appl. Sci. (M3AS)*, 29:2523–2556, 2019.

BK & Vanja Nikolić. Vanishing relaxation time limit of the Jordan–Moore–Gibson–Thompson wave equation with Neumann and absorbing boundary conditions. *Pure and Applied Functional Analysis*, 5:1–26, 2020.

shortcut to limit result

The linearized problem

$$\begin{cases} \tau\psi_{ttt} + \alpha(x,t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = f & \text{in } \Omega \times (0,T), \\ \psi = 0 \quad \text{on } \partial\Omega \times (0,T), \\ (\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) \quad \text{in } \Omega \times \{0\}, \end{cases}$$

under the assumptions

$$\alpha(x,t) \geq \underline{\alpha} > 0 \ \text{ on } \Omega \quad \text{a.e. in } \Omega \times (0,T). \tag{1}$$

$$\begin{aligned} &\alpha \in L^\infty(0,T; L^\infty(\Omega)) \cap L^\infty(0,T; W^{1,3}(\Omega)), \\ &f \in H^1(0,T; L^2(\Omega)). \end{aligned} \tag{2}$$

$$(\psi_0, \psi_1, \psi_2) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega). \tag{3}$$

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Theorem (lin)

Let c^2, b, $\tau > 0$, and let $T > 0$. Let the assumptions (1), (2), (3) hold. Then there exists a unique solution

$$\psi \in X^W := W^{1,\infty}(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap W^{2,\infty}(0,T;H_0^1(\Omega))\cap H^3(0,T;L^2(\Omega)).$$

The solution fullfils the estimate

$$\begin{aligned} \|\psi\|^2_{W,\tau} :=& \tau^2\|\psi_{ttt}\|^2_{L^2L^2} + \tau\|\psi_{tt}\|^2_{L^\infty H^1} + \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2} \\ &\leq C(\alpha,T,\tau)\left(|\psi_0|^2_{H^2} + |\psi_1|^2_{H^2} + \tau|\psi_2|^2_{H^1} + \|f\|^2_{L^\infty L^2} + \|f_t\|^2_{L^2L^2}\right). \end{aligned}$$

If additionally $\|\nabla\alpha\|_{L^\infty L^3} < \frac{\underline{\alpha}}{C^\Omega_{H^1,L^6}}$ holds, then $C(\alpha,T,\tau)$ is independent of τ.

Well-posedness of the Westervelt type JMGT equation

$$\begin{cases} \tau\psi_{ttt} + (1 - k\psi_t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0 & \text{in } \Omega \times (0, T), \\ \psi = 0 \quad \text{on } \partial\Omega \times (0, T), \\ (\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\}, \end{cases}$$

Theorem

Let c^2, $b > 0$, $k \in \mathbb{R}$ and let $T > 0$. There exist $\rho, \rho_0 > 0$ such that for all $(\psi_0, \psi_1, \psi_2) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ satisfying

$$\|\psi_0\|^2_{H^2(\Omega)} + \|\psi_1\|^2_{H^2(\Omega)} + \tau\|\psi_2\|^2_{H^1(\Omega)} \leq \rho_0^2\,,$$

there exists a unique solution $\psi \in X^W$ and $\|\psi\|^2_{W,\tau} \leq \rho^2$.

Banach's Contraction Principle for $\mathcal{T} : \phi \mapsto \psi$ solution ψ of (4) with $\alpha = 1 - k\phi_t$, $f = 0$:
self-mapping on $B_\rho^{X^W}$: energy estimate from Theorem (lin).
contractivity: $\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{W,\tau} \leq q\|\phi_1 - \phi_2\|_{W,\tau}$ by estimate from Theorem (lin):
$\hat{\psi} = \psi_1 - \psi_2 = \mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)$ solves (4) with $\alpha = 1 - k\phi_{1\,t}$ and
$f = k\hat{\phi}_t\psi_{2\,tt}$ where $\hat{\phi} = \phi_1 - \phi_2$.

Limits for vanishing relaxation time

Consider the τ-independent part of the norms

$$\|\psi\|^2_{W,\tau} :=$$
$$\tau^2\|\psi_{ttt}\|^2_{L^2L^2} + \tau\|\psi_{tt}\|^2_{L^\infty H^1} + \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2}$$

namely

$$\|\psi\|^2_{\bar{X}^W} := \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2}\,,$$

since these norms will be uniformly bounded, independently of τ.

Limits for vanishing relaxation time

Consider the τ-independent part of the norms

$$\|\psi\|^2_{\bar{X}^W} := \|\psi_{tt}\|^2_{L^2 H^1} + \|\psi\|^2_{W^{1,\infty} H^2} \,,$$

and recall the spaces for the initial data

$$X_0^W := H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega).$$

Theorem

Let c^2, b, $T > 0$, and $k \in \mathbb{R}$. Then there exist $\bar{\tau}$, $\rho_0 > 0$ such that for all $(\psi_0, \psi_1, \psi_2) \in X_0^W$, the family $(\psi^\tau)_{\tau \in (0,\bar{\tau})}$ of solutions to the Westervelt type JMGT equation converges weakly in $\bar{X}^W$ to a solution $\bar{\psi} \in \bar{X}^W$ of the Westervelt equation with initial conditions $\bar{\psi}(0) = \psi_0$, $\bar{\psi}_t(0) = \psi_1$.*

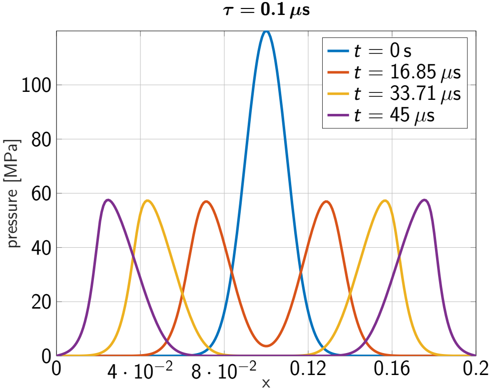
Numerical Experiments

- comparison of Westervelt-JMGT and Westervelt solutions
- numerical experiments for water in a 1-d channel geometry

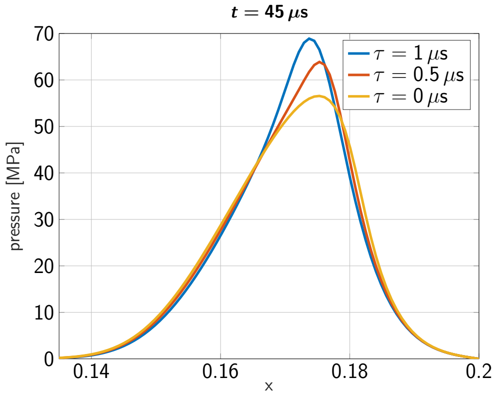
$$c = 1500\,\mathrm{m/s},\ \delta = 6\cdot 10^{-9}\,\mathrm{m}^2/\mathrm{s},\ \rho = 1000\,\mathrm{kg/m}^3,\ B/A = 5;$$

- space discretization with B-splines (Isogeometric Analysis): quadratic basis functions, globally C^2; 251 dofs on $\Omega = [0, 0.2m]$
- time discretization by Newmark scheme, adapted to 3rd order equation; 800 time steps on $[0, T] = [0, 45\mu s]$
- initial conditions $(\psi_0, \psi_1, \psi_2) = \left(0,\ \mathcal{A}\exp\left(-\frac{(x-0.1)^2}{2\sigma^2}\right),\ 0\right)$ with $\mathcal{A} = 8\cdot 10^4\,\mathrm{m}^2/\mathrm{s}^2$ and $\sigma = 0.01$,

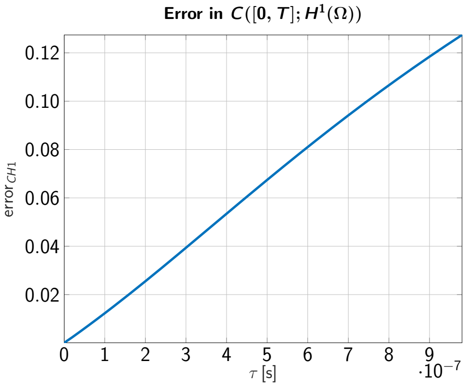
Snapshots of pressure $p = \varrho\psi_t$ for fixed relaxation time $\tau = 0.1\,\mu s$



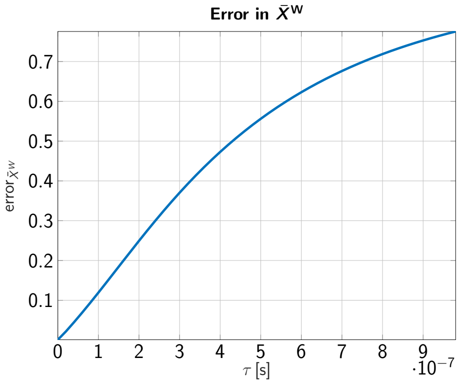
Pressure wave for different relaxation parameters τ at final time $t = 45\mu s$.



Relative errors as $\tau \to 0$



in $C([0,T];H^1(\Omega))$



in $\bar{X}^W = H^2(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^2(\Omega))$.

Recap: Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - (\delta + \tau c^2)\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

versus Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\tfrac{B}{2Ac^2}((\psi_t)^2) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^τ as $\tau \searrow 0$? Yes
Does ψ^0 solve Kuznetsov's equation? Yes

[Bongarti&Charoenphon&Lasiecka; BK& Nikolić, 2019-21] limit in JMGT/Kuznetsov/Westervelt

for vanishing diffusivity of sound δ

Vanishing diffusivity of sound

Jordan-Moore-Gibson-Thompson equation

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and Kuznetsov's equation:

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[BK& Nikolić, SIAP 2021]
recover results (in particular on required regularity of initial data) from [Dörfler Gerner Schnaubelt 2016] for $\delta = 0$

limit in Blackstock-Crighton

for vanishing thermal conductivity a

Vanishing thermal conductivity

Blackstock-Crighton equation

$$(\partial_t - a\Delta)\left(\psi^a_{tt} - c^2\Delta\psi^a - \delta\Delta\psi^a_t\right) - ra\Delta\psi^a_t = \left(\frac{B}{2Ac^2}({\psi^a_t}^2) + |\nabla\psi^a|^2\right)_{tt}$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

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Integrate once wrt time: Consistency of initial data needed:
$\psi_2 - c^2\Delta\psi_0 - \delta\Delta\psi_1 = \frac{B}{Ac^2}\psi_1\psi_2 + 2\nabla\psi_0 \cdot \nabla\psi_1$

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[BK& Thalhammer, M3AS 2018]

limit in time fractional JMGT

for differentiation order $\alpha \nearrow 1$

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^{\alpha}+\psi^{\alpha}_{tt}-c^2\Delta\psi^{\alpha}-(\delta+\tau^{\alpha}c^2)\Delta D_t^{\alpha}\psi^{\alpha}=\left(\frac{B}{2Ac^2}(\psi^{\delta}_t)^2+|\nabla\psi^{\delta}|^2\right)_t$$

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[BK& Nikolić, M3AS 2022]

fractional damping models in ultrasonics

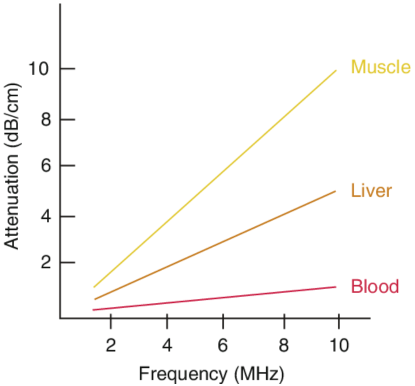

Figure 2.6 in [Chan&Perlas, Basics of Ultrasound Imaging, 2011]

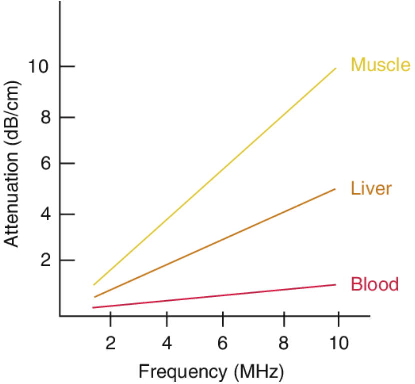

Figure 2.6 in [Chan&Perlas, Basics of Ultrasound Imaging, 2011]

$\rightsquigarrow$ constitutive modeling of

- pressure – density relation
- temperature – heat flux relation ▸▸ shortcut

Fractional Models of (Linear) Viscoelasticity

- equation of motion (resulting from balance of forces)

$$\varrho \mathbf{u}_{tt} = \mathrm{div}\sigma + \mathbf{f}$$

- strain as symmetric gradient of displacements:

$$\epsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

- constitutive model: stress-strain relation

$\mathbf{u}$... displacements
σ... stress tensor
ϵ... strain tensor
ϱ... mass density

Fractional Models of (Linear) Viscoelasticity 1-d setting

- equation of motion (resulting from balance of forces)

$$\varrho u_{tt} = \sigma_x + f$$

- strain as symmetric gradient of displacements:

$$\epsilon = u_x.$$

- constitutive model: stress-strain relation:

$$\begin{aligned}
\text{Hooke's law (pure elasticity):}\quad & \sigma = b_0\epsilon \\
\text{Newton model:}\quad & \sigma = b_1\epsilon_t \\
\text{Kelvin-Voigt model:}\quad & \sigma = b_0\epsilon + b_1\epsilon_t \\
\text{Maxwell model:}\quad & \sigma + a_1\sigma_t = b_0\epsilon \\
\text{Zener model:}\quad & \sigma + a_1\sigma_t = b_0\epsilon + b_1\epsilon_t
\end{aligned}$$

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fractional Newton model: $\sigma = b_1 \partial_t^\beta \epsilon$

fractional Kelvin-Voigt model: $\sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

fractional Maxwell model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon$

fractional Zener model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

general model class: $\sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \sigma = \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \epsilon$

[Caputo 1967, Atanackovic, Pilipović, Stanković, Zorica 2014]

Fractional Models of (Linear) Acoustics via $p - \varrho$

balance of momentum
$$\varrho_0 \mathbf{v}_t = -\nabla p + \mathbf{f}$$

balance of mass
$$\varrho \nabla \cdot \mathbf{v} = -\varrho_t$$

equation of state
$$\frac{\varrho_\sim}{\varrho_0} = \frac{p_\sim}{p_0}$$

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insert constitutive equations into combination of balance laws
$\rightsquigarrow$ fractional acoustic wave equations [Holm 2019, Szabo 2004]:

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insert constitutive equations into combination of balance laws
$\rightsquigarrow$ fractional acoustic wave equations [Holm 2019, Szabo 2004]:

- Caputo-Wismer-Kelvin wave equation (fractional Kelvin-Voigt):
$$p_{tt} - b_0 \triangle p - b_1 \partial_t^{\beta} \triangle p = \tilde{f}\,,$$
- modified Szabo wave equation (fractional Maxwell):
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \triangle p = \tilde{f}\,,$$
- fractional Zener wave equation:
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \triangle p + b_1 \partial_t^{\beta} \triangle p = \tilde{f}\,,$$
- general fractional model:
$$\textstyle\sum_{n=0}^{N} a_n \partial_t^{2+\alpha_n} p - \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \triangle p = \tilde{f}\,.$$

Fractional Models of (Linear) Acoustics via $\vartheta - \boldsymbol{q}$

recall:

Classically: Fourier's law $\quad \boldsymbol{q} = -K\nabla\vartheta$

leads to infinite speed of propagation paradox.

Maxwell-Cattaneo law $\quad \tau\boldsymbol{q}_t + \boldsymbol{q} = -K\nabla\vartheta$

allows for "thermal waves" (second sound phenomenon)
can lead to violation of the 2nd law of thermodynamics

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"interpolate" by using fractional derivatives
[Compte & Metzler 1997, Povstenko 2011]:

(GFE I) $\qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\tau_\vartheta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\vartheta;$

(GFE II) $\qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\tau_\vartheta^{\alpha-1}\mathrm{D}_t^{\alpha-1}\nabla\vartheta;$

(GFE III) $\qquad (1+\tau\partial_t)\boldsymbol{q}(t) = -K\tau_\vartheta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\vartheta;$

(GFE) $\qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\nabla\vartheta.$

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$${}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$${}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

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- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

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some recent books on fractional PDEs: [Kubica & Ryszewska & Yamamoto 2020], [Jin 2021], [BK & Rundell 2022]

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some inverse problems

Photoacoustic tomography PAT with fractional attenuation

- attenuation of ultrasound in human tissue follows a power law frequency dependence ω^α
  $\rightsquigarrow$ fractional derivative ∂_t^α term in time domain
- PAT acoustic (sub)problem: Reconstruct initial pressure from observations of pressure at some transducer array over time
  see, e.g., [Kuchment & Kunyanski 2011]
- only mildly ill-posed without attenuation
- severely ill-posed in with integer (1st) order damping

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The inverse problem of PAT and TAT

Identify $u_0(x)$ in

$$u_{tt} + c^2 \mathcal{A} u + D u = 0 \text{ in } \Omega \times (0, T)$$
$$u(0) = u_0, \quad u_t(0) = 0 \text{ in } \Omega$$

where $\mathcal{A} u = -\triangle$ with homogeneous Dirichlet boundary conditions from observations

$$g = u \quad \text{on } \Sigma \times (0, T)$$

$\Sigma \subset \overline{\Omega}$... transducer array (surface or collection of discrete points)

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Caputo-Wismer-Kelvin:

$$D = b\mathcal{A}\partial_t^\beta \quad \text{with } \beta \in [0,1], \quad b \geq 0$$

fractional Zener:

$$D = a\partial_t^{2+\alpha} + b\mathcal{A}\partial_t^\beta \quad \text{with } a > 0,\ b \geq ac^2,\ 1 \geq \beta \geq \alpha > 0,$$

space fractional Chen-Holm:

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FZ fractional Zener:

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Uniqueness

Linear independence assumption:
For each eigenvalue λ of $\mathcal{A}$ with eigenfunctions $(\varphi_k)_{k\in K^\lambda}$, the restrictions of the eigenfunctions to the observation manifold are linear independent: For any coefficient set $(b_k)_{k\in K^\lambda}$

$$\left(\sum_{k\in K^\lambda} b_k\varphi_k(x) = 0 \quad \text{for all } x\in\Sigma\right) \Longrightarrow \left(b_k = 0 \text{ for all } k\in K^\lambda\right).$$

Theorem

Suppose the domain Ω and the operator $\mathcal{A}$ are known. Then under the linear independence assumption we can uniquely recover the initial value $u_0(x)$ from time trace measurements g on Σ.

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- Uniqueness of $c_0(x)$ from the same observations can be shown by Sturm-Liouville theory.
- tools of proof:
  separation of variables (solution representation),
  analysis in Laplace domain (location of poles),
  uniqueness of eigenvalues from poles.

[BK&Rundell. Inverse Problems, 37(4):045002]

Nonlinearity parameter imaging

- B/A parameter is sensitive to differences in tissue properties, thus appropriate for characterization of biological tissues
- viewing $\kappa = \frac{1}{\varrho c^2}(\frac{B}{2A} + 1)$ as a spatially varying coefficient in the Westervelt equation, it can be used for medical imaging
- $\rightsquigarrow$ acoustic nonlinearity parameter tomography [Bjørnø 1986; Burov, Gurinovich, Rudenko, Tagunov 1994; Cain 1986; Ichida, Sato, Linzer 1983; Varray, Basset, Tortoli, Cachard 2011; Zhang, Gong et al 1996, 2001]...

The inverse problem of nonlinearity parameter imaging

Identify $\kappa(x)$ in

$$\left(u-\kappa(x)u^2\right)_{tt}-c_0^2\triangle u+Du=r \quad \text{in } \Omega\times(0,T)$$

$$u=0 \text{ on } \partial\Omega\times(0,T), \quad u(0)=0, \quad u_t(0)=0 \quad \text{in } \Omega$$

(with excitation r) from observations

$$g=u \quad \text{on } \Sigma\times(0,T)$$

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fractional damping

Caputo-Wismer-Kelvin:

$$D = -b\Delta\partial_t^\beta \quad \text{with } \beta \in [0, 1], \quad b \geq 0$$

fractional Zener:

$$D = a\partial_t^{2+\alpha} - b\Delta\partial_t^\beta \quad \text{with } a > 0, \ b \geq ac^2, \ 1 \geq \beta \geq \alpha > 0,$$

space fractional Chen-Holm:

$$D = b(-\Delta)^{\tilde{\beta}}\partial_t \quad \text{with } \tilde{\beta} \in [0, 1], \ b \geq 0,$$

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.

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  nonlinear case: higher harmonics
  see also asymptotics argument in [Kurylev & Lassas & Uhlmann 2019]

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[Yamamoto &BK 2021] BCBJ equation

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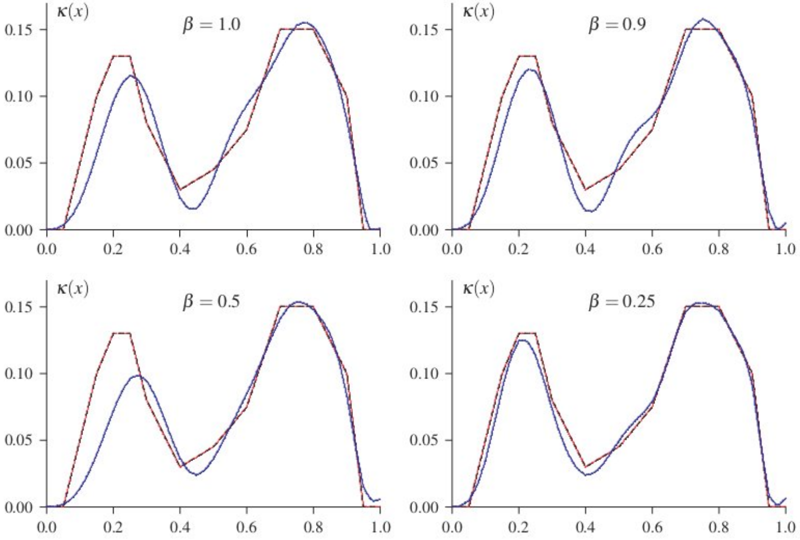
[Acosta & Uhlmann & Zhai 2022] Westervelt equation:

- uniqueness from Neumann-Dirichlet map

Reconstructions of $\kappa(x)$

Caputo-Wismer-Kelvin: $D = -b\Delta\partial_t^\beta$, 0.1% noise



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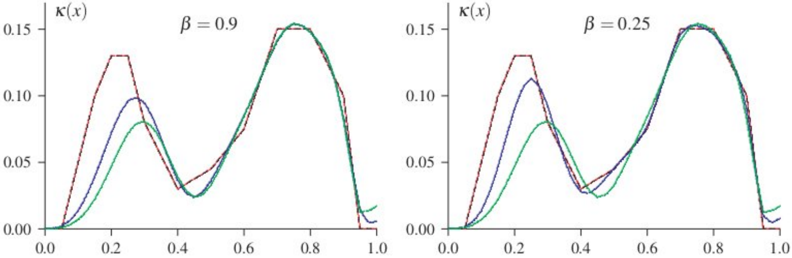
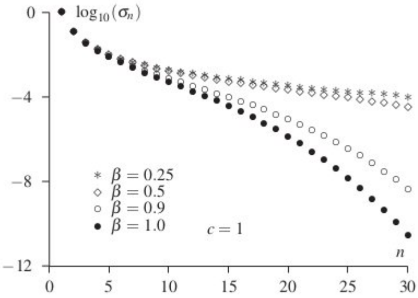
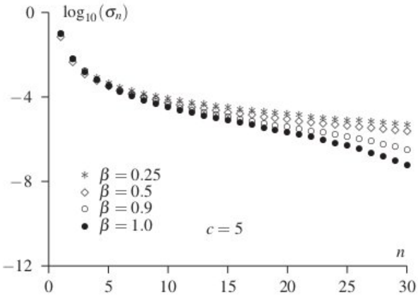


0.5% (blue) and 1% (green) noise

Singular values of linearized forward operator





Outlook: Some further inverse problems

- Determine fractional differentiation orders α_n, β_m in wave type eq.

$$\sum_{n=0}^{N} a_n \partial_t^{2+\alpha_n} p - \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \triangle p = \tilde{f} \,.$$

[BK& Rundell 2022];
for subdiffusion, see [Hatano& Nakagawa& Wang& Yamamoto 2013]
. . . [Jin& Kian 2022]

Outlook: Some further inverse problems

- Determine nonlinearity f in generalized Westervelt equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = -\kappa(f(u))_{tt}$$

[BK& Rundell 2021]

Outlook: Some further inverse problems

- Determine kernels k_ε, $k_{\mathrm{tr}\,\varepsilon}$ in viscoelastic model

$$\rho \mathbf{u}_{tt} - \mathrm{div}[\mathbb{C}\varepsilon(\mathbf{u}) + k_\varepsilon * \mathbb{A}\varepsilon(\mathbf{u}_t) + k_{\mathrm{tr}\,\varepsilon} * \mathrm{tr}\varepsilon(\mathbf{u}_t)\mathbb{I}] = \mathbf{f}$$

[BK & Khristenko & Nikolić & Rajendran & Wohlmuth 2022]

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