Forward and Inverse Problems in Nonlinear Acoustics

Barbara Kaltenbacher

Alpen-Adria-Universität Klagenfurt

ECM, Sevilla, July 2024

joint work with

Vanja Nikolić, Radboud University
William Rundell, Texas A&M University
Teresa Rauscher, University of Klagenfurt
Benjamin Rainer, University of Klagenfurt

FWF Der Wissenschaftsfonds.

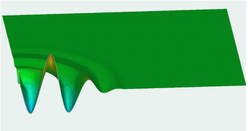
Modeling - Analysis - Optimization
DOC.FUNDS DOCTORAL SCHOOL

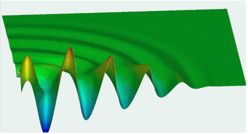
UNIVERSITÄT KLAGENFURT

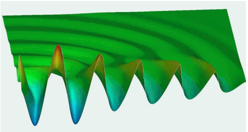
Outline

- modeling:
  - models of nonlinear acoustics
  - fractional damping models in ultrasonics
- analysis
  - parameter asymptotics
  - multiharmonic expansion
- inverse problems
  - nonlinearity parameter imaging

Nonlinear Acoustic Wave Propagation

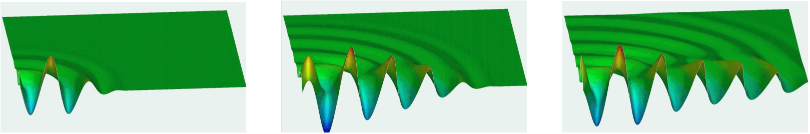






nonlinear wave propagation:

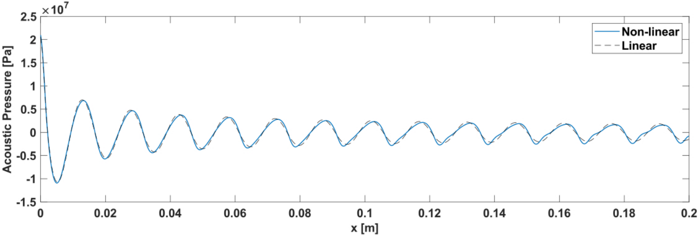
Nonlinear Acoustic Wave Propagation



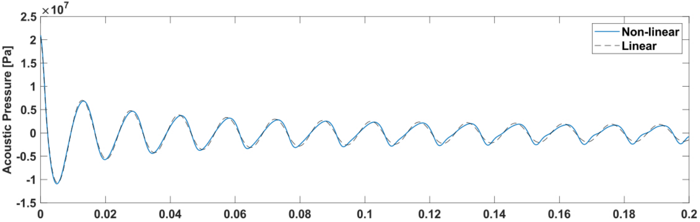
nonlinear wave propagation:

sound speed depends on (signed) amplitude $\Rightarrow$ sawtooth profile

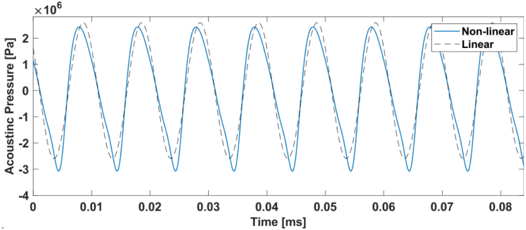
space:
[m]



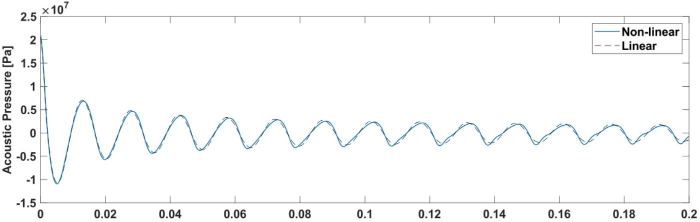
space:
[m]



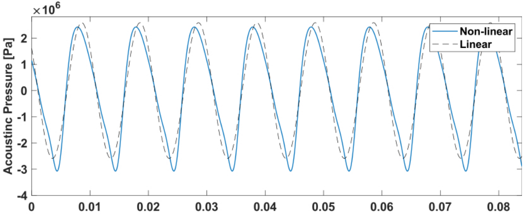
time:
[ms]



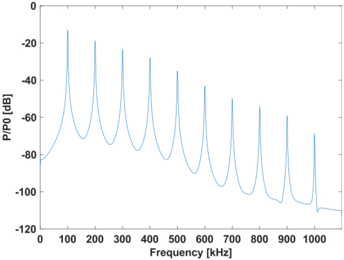
space:
[m]



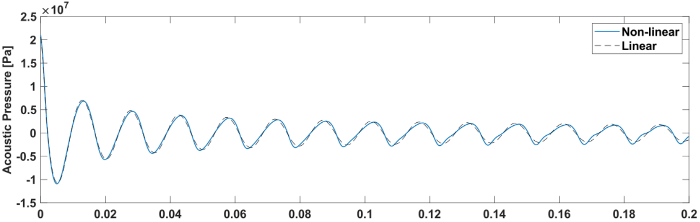
time:
[ms]



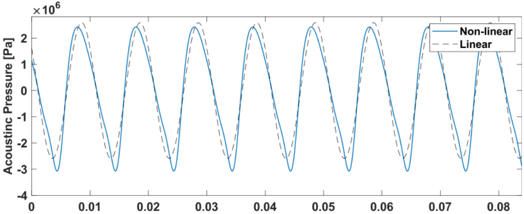
frequency:
[kHz]



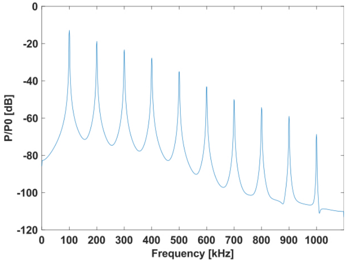
space:
[m]



time:
[ms]



frequency:
[kHz]



(taken from Master thesis by Benjamin Rainer, University of Klagenfurt, 2024) models of nonlinear acoustics

Physical Principles

main physical quantities:

- acoustic particle velocity $\mathbf{v}$;
- acoustic pressure p;
- mass density ϱ;
- absolute temperature ϑ;
- heat flux $\boldsymbol{q}$;
- entropy η;

decomposition into mean and fluctuating part:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\sim = \mathbf{v}\,, \qquad p = p_0 + p_\sim\,, \qquad \varrho = \varrho_0 + \varrho_\sim$$

Physical Principles

- acoustic particle velocity $\mathbf{v}$;
- acoustic pressure p;
- mass density ϱ;
- absolute temperature ϑ;
- heat flux $\boldsymbol{q}$;
- entropy η;

governing equations:

- momentum conservation = Navier Stokes equation (with $\nabla \times \mathbf{v} = 0$):

$$\varrho\Big(\mathbf{v}_t + \nabla(\mathbf{v}\cdot\mathbf{v})\Big) + \nabla p = \Big(\frac{4\mu_V}{3} + \zeta_V\Big)\Delta\mathbf{v}$$

- mass conservation = equation of continuity:

$$\varrho_t + \nabla\cdot(\varrho\mathbf{v}) = 0$$

- entropy equation: $$\varrho\vartheta(\eta_t + \mathbf{v}\cdot\nabla\eta) = -\nabla\cdot\boldsymbol{q}$$
- equation of state: $$\frac{p}{p_0} = \varrho^\gamma \exp\left(\frac{\eta-\eta_0}{c_v}\right)$$
- Gibbs equation: $$\vartheta d\eta = c_v d\vartheta - p\frac{1}{\varrho^2}d\varrho$$

$\gamma = \frac{c_p}{c_v}$ . . . adiabatic index;

c_p / c_v . . . specific heat at constant pressure / volume;

Physical Principles

So far, 5 equations for 6 unknowns $\mathbf{v}$, p, ϱ, ϑ, $\boldsymbol{q}$, η.
Still need a constitutive relation between temperature and heat flux.

Physical Principles

So far, 5 equations for 6 unknowns $\mathbf{v}$, p, ϱ, ϑ, $\boldsymbol{q}$, η.
Still need a constitutive relation between temperature and heat flux.

Classically: Fourier's law $\quad \boldsymbol{q} = -K\nabla\vartheta$

K ... thermal conductivity
leads to infinite speed of propagation paradox.

Physical Principles

So far, 5 equations for 6 unknowns $\mathbf{v}$, p, ϱ, ϑ, $\boldsymbol{q}$, η.
Still need a constitutive relation between temperature and heat flux.

Classically: Fourier's law $\quad \boldsymbol{q} = -K\nabla\vartheta$

K... thermal conductivity
leads to infinite speed of propagation paradox.

Maxwell-Cattaneo law $\quad \tau\boldsymbol{q}_t + \boldsymbol{q} = -K\nabla\vartheta$

τ... relaxation time
allows for "thermal waves" (second sound phenomenon)

Classical Models of Nonlinear Acoustics

- **Kuznetsov's equation** [Lesser & Seebass 1968, Kuznetsov 1971]

$$p_{\sim tt} - c^2 \Delta p_\sim - \delta \Delta p_{\sim t} = \left(\frac{B}{2A\varrho_0 c^2} p_\sim^2 + \varrho_0 |\mathbf{v}|^2 \right)_{tt}$$

where $\varrho_0 \mathbf{v}_t = -\nabla p \quad \rightsquigarrow \quad \varrho_0 \psi_t = p$
for the **particle velocity v** and the **pressure** p, i.e.,

$$\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left(\frac{B}{2A\, c^2} (\psi_t)^2 + |\nabla \psi|^2 \right)_t$$

since $\nabla \times \mathbf{v} = 0$ hence $\mathbf{v} = -\nabla \psi$ for a **velocity potential** ψ

$\delta = \kappa(\mathrm{Pr}(\frac{4}{3} + \frac{\zeta_V}{\mu_V}) + \gamma - 1)$ ... diffusivity of sound;
$\frac{B}{A} \hat{=} \gamma - 1$ ... nonlinearity parameter (in liquids / gases)

Classical Models of Nonlinear Acoustics

- **Kuznetsov's equation** [Lesser & Seebass 1968, Kuznetsov 1971]

$$p_{\sim tt} - c^2\Delta p_\sim - \delta\Delta p_{\sim t} = \left(\tfrac{B}{2A\varrho_0 c^2} p_\sim^2 + \varrho_0|\mathbf{v}|^2\right)_{tt}$$

  where $\varrho_0 \mathbf{v}_t = -\nabla p$ $\quad\rightsquigarrow\quad$ $\varrho_0\psi_t = p$
  for the **particle velocity v** and the **pressure** p

- **Westervelt equation** [Westervelt 1963] via $\varrho_0|\mathbf{v}|^2 \approx \frac{1}{\varrho_0 c^2}(p_\sim)^2$

$$p_{\sim tt} - c^2\Delta p_\sim - \delta\Delta p_{\sim t} = \frac{1}{\varrho_0 c^2}\left(1 + \tfrac{B}{2A}\right) p^2_{\sim tt}$$

$\delta = \kappa(\mathrm{Pr}(\frac{4}{3} + \frac{\zeta_V}{\mu_V}) + \gamma - 1)$ ... diffusivity of sound;
$\frac{B}{A} \,\hat{=}\, \gamma - 1$ ... nonlinearity parameter (in liquids / gases)

Advanced Models of Nonlinear Acoustics (Examples)

- **Jordan-Moore-Gibson-Thompson** equation [Jordan 2009, 2014], [Christov 2009], [Straughan 2010]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta + \tau c^2)\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

τ . . . relaxation time

Advanced Models of Nonlinear Acoustics (Examples)

- **Jordan-Moore-Gibson-Thompson** equation [Jordan 2009, 2014], [Christov 2009], [Straughan 2010]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta + \tau c^2)\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

$\tau \ldots$ relaxation time

$z := \psi_t + \frac{c^2}{\delta+\tau c^2}\psi$ solves weakly damped wave equation

$$z_{tt} - \tilde{c}\Delta z + \gamma z_t = r(z,\psi)$$

with $\tilde{c} = c^2 + \frac{\delta}{\tau}$, $\gamma = \frac{1}{\tau} - \frac{c^2}{\delta+\tau c^2} > 0$
$\rightsquigarrow$ second sound phenomenon

Advanced Models of Nonlinear Acoustics (Examples)

- **Blackstock-Crighton** equation [Brunnhuber & Jordan 2016], [Blackstock 1963], [Crighton 1979]

$$(\partial_t - a\Delta)\left(\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t\right) - ra\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_{tt}$$

$a = \frac{\nu}{\mathrm{Pr}} \ldots$ thermal conductivity

Advanced versus Classical Models of Nonlinear Acoustics

- **Blackstock-Crighton** equation [Brunnhuber & Jordan 2016], [Blackstock 1963], [Crighton 1979]

$$(\partial_t - a\Delta)\left(\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t\right) - ra\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_{tt}$$

$a = \frac{\nu}{\mathrm{Pr}} \ldots$ thermal conductivity

- **Jordan-Moore-Gibson-Thompson** equation [Jordan 2009, 2014], [Christov 2009], [Straughan 2010]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta + \tau c^2)\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

$\tau \ldots$ relaxation time

- cf. Kuznetsov:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

- further models:[Angel & Aristegui 2014], [Christov & Christov & Jordan 2007], [Kudryashov & Sinelshchikov 2010], [Ockendon & Tayler 1983], [Makarov & Ochmann 1996], [Rendón & Ezeta &Pérez-López 2013], [Rasmussen & Sørensen & Christiansen 2008], [Soderholm 2006], ...
- resonances, shock waves:[Ockendon & Ockendon & Peake & Chester 1993], [Ockendon & Ockendon 2001, 2004, 2016],...
- traveling waves solutions:[Jordan 2004], [Chen & Torres & Walsh 2009], [Keiffer & McNorton & Jordan & Christov, 2014], [Gaididei & Rasmussen & Christiansen & Sørensen, 2016],...
- well-posendness and asymptotic behaviour:
  for KZK: [Rozanova-Pierrat 2007, 2008, 2009, 2010]
  for Westervelt, Kuznetsov, Blackstock-Crighton, JMGT on bounded domain Ω:
  based on semigroup theory and energy estimates:[BK & Lasiecka 2009, 2012], [BK & Lasiecka & Veljović 2011], [BK & Lasiecka & Marchand 2012], [BK & Lasiecka & Pospiezalska 2012], [Lasiecka & Wang 2015], [Liu & Triggiani 2013], [Marchand & McDevitt & Triggiani 2012], [Nikolić 2015], [Nikolić & BK 2016], [Pellicer & Solá-Morales 2019], , [Dell'Oro&Lasiecka&Pata 2020]
  based on maximal L_p regularity:[Meyer & Wilke 2011, 2013], [Meyer & Simonett 2016], [Brunnhuber & Meyer 2016], [BK 2016]
  Cauchy problem (on $\Omega = \mathbb{R}^\varkappa$)
  for Kuznetsov: [Dekkers & Rozanova-Pierrat 2019]
  for JMGT: [Pellicer & Said-Houari 2017], [Nikolić & Said-Houari 2021]
- control of JMGT [Bucci&Lasiecka 2020], [Bucci&Pandolfi 2020]

Analysis of initial-boundary value problems

consider:
Westervelt / Kuznetsov / Jordan-Moore-Gibson-Thompson / Blackstock-Crighton equation on some domain $\Omega \subseteq \mathbb{R}^d$
+boundary conditions on $\partial\Omega$
+initial conditions at $t = 0$

e.g.,

$$\begin{aligned} u_{tt} - c^2 \Delta u - b \Delta u_t &= \tfrac{\kappa}{2} (u^2)_{tt} && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= g && \text{on } \partial\Omega \\ u(t=0) = u_0 \,, \; u_t(t=0) &= u_1 && \text{in } \Omega \end{aligned}$$

where u ... pressure

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\kappa}{2}(u^2)_{tt}$$

Degeneracy

e.g., for Westervelt (u … pressure)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa(u_t)^2$$

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \tfrac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa(u_t)^2$$

$$(1 - \kappa u)u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(u_t)^2$$

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\kappa}{2}(u^2)_{tt} = \kappa u \, u_{tt} + \kappa (u_t)^2$$

$$(1 - \kappa u) u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa (u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa (u_t)^2$$

$$(1 - \kappa u) u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa (u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson, Blackstock-Crighton.

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa (u_t)^2$$

$$(1 - \kappa u) u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa (u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson, Blackstock-Crighton.

- $\rightsquigarrow$ employ energy estimates to obtain bound on u in $C(0, T; H^2(\Omega))$
- $\rightsquigarrow$ use smallness of u in $C(0, T; H^2(\Omega))$ and $H^2(\Omega) \to L_\infty(\Omega)$ embedding to guarantee $1 - \kappa u \geq \underline{\alpha} > 0$

Degeneracy

e.g., for Westervelt (u … pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa (u_t)^2$$

$$(1 - \kappa u) u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa (u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson, Blackstock-Crighton.

- $\rightsquigarrow$ employ energy estimates to obtain bound on u in $C(0,T; H^2(\Omega))$
- $\rightsquigarrow$ use smallness of u in $C(0,T; H^2(\Omega))$ and $H^2(\Omega) \to L_\infty(\Omega)$ embedding to guarantee $1 - \kappa u \geq \underline{\alpha} > 0$
- $\rightsquigarrow$ fixed point argument

Degeneracy – State dependent wave speed

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \tfrac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa(u_t)^2$$

$$(1-\kappa u)u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson, Blackstock-Crighton.

Degeneracy – State dependent wave speed

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \tfrac{\kappa}{2}(u^2)_{tt} = \kappa u\, u_{tt} + \kappa(u_t)^2$$

$$(1-\kappa u)u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(u_t)^2$$

$\Rightarrow$ degeneracy for $u \geq \frac{1}{\kappa}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson, Blackstock-Crighton.

This also illustrates state dependence of the effective wave speed:

$$u_{tt} - \tilde{c}^2\Delta u - \tilde{b}(u)\Delta u_t = f(u)$$

with $\tilde{c}(u) = \frac{c}{\sqrt{1-\kappa u}}$, $\tilde{b}(u) = \frac{b}{1-\kappa u}$, $f(u) = \frac{\kappa(u_t)^2}{1-\kappa u}$ as long as $1-\kappa u > 0$ (otherwise the model loses its validity) parameter asymptotics

Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation $\quad (b = \delta + \tau c^2)$

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation $\quad (b = \delta + \tau c^2)$

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^τ as $\tau \searrow 0$?
Does ψ^0 solve Kuznetsov's equation?

Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation $\quad (b = \delta + \tau c^2)$

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^τ as $\tau \searrow 0$?
Does ψ^0 solve Kuznetsov's equation?

- As $\tau \to 0$ the PDE changes from hyperbolic (group) to parabolic (analytic semigroup, maximal parabolic regularity)

Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation $\quad (b = \delta + \tau c^2)$

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t^2) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^τ as $\tau \searrow 0$?
Does ψ^0 solve Kuznetsov's equation?

- As $\tau \to 0$ the PDE changes from hyperbolic (group) to parabolic (analytic semigroup, maximal parabolic regularity)

[Bongarti&Charoenphon&Lasiecka; BK& Nikolić, 2019-21]

shortcut to limit result

Remarks

- We will consider the "Westervelt type" and the "Kuznetsov type" equation; without and with the gradient nonlinearity $|\nabla\psi|_t^2$

Remarks

- We will consider the "Westervelt type" and the "Kuznetsov type" equation; without and with the gradient nonlinearity $|\nabla\psi|_t^2$
- For $\tau = 0$ (classical Westervelt and Kuznetsov equation) the reformulation of the linearization as a first order system leads to an analytic semigroup and maximal parabolic regularity.
  These properties get lost with $\tau > 0$; the equation loses its "parabolic nature".
  This is consistent with physics: infinite $\to$ finite propagation speed.

Remarks

- We will consider the "Westervelt type" and the "Kuznetsov type" equation; without and with the gradient nonlinearity $|\nabla\psi|_t^2$
- For $\tau = 0$ (classical Westervelt and Kuznetsov equation) the reformulation of the linearization as a first order system leads to an analytic semigroup and maximal parabolic regularity.
  These properties get lost with $\tau > 0$; the equation loses its "parabolic nature".
  This is consistent with physics: infinite $\rightarrow$ finite propagation speed.
- As in the classical models, potential degeneracy can be an issue

$$
\begin{aligned}
\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t &= \left(\frac{\kappa}{2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t \\
&= \kappa\psi^\tau_t\psi^\tau_{tt} + |\nabla\psi^\tau|^2_t
\end{aligned}
$$

Remarks

- We will consider the "Westervelt type" and the "Kuznetsov type" equation; without and with the gradient nonlinearity $|\nabla\psi|_t^2$
- For $\tau = 0$ (classical Westervelt and Kuznetsov equation) the reformulation of the linearization as a first order system leads to an analytic semigroup and maximal parabolic regularity.
  These properties get lost with $\tau > 0$; the equation loses its "parabolic nature".
  This is consistent with physics: infinite $\to$ finite propagation speed.
- As in the classical models, potential degeneracy can be an issue

$$
\begin{aligned}
\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t &= \left(\frac{\kappa}{2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t \\
&= \kappa\psi^\tau_t\psi^\tau_{tt} + |\nabla\psi^\tau|^2_t
\end{aligned}
$$

$$
\Longleftrightarrow \quad \tau\psi^\tau_{ttt} + (1 - \kappa\psi^\tau_t)\psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = |\nabla\psi^\tau|^2_t
$$

Plan of the analysis

- Establish well-posedness of the linearized equation along with energy estimates.
- Use these results to prove well-posedness of the Westervelt type JMGT equation for $\tau > 0$ by a fixed point argument.
- Establish additional higher order energy estimates.
- Use these results to prove well-posedness of the Kuznetsov type JMGT equation for $\tau > 0$ (sufficiently small) by a fixed point argument.
- Take limits as $\tau \to 0$

Plan of the analysis

- Establish well-posedness of the linearized equation along with energy estimates.
- Use these results to prove well-posedness of the Westervelt type JMGT equation for $\tau > 0$ by a fixed point argument.
- Establish additional higher order energy estimates.
- Use these results to prove well-posedness of the Kuznetsov type JMGT equation for $\tau > 0$ (sufficiently small) by a fixed point argument.
- Take limits as $\tau \to 0$

BK & Vanja Nikolić. On the Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. *Math. Meth. Mod. Appl. Sci. (M3AS)*, 29:2523–2556, 2019.

BK & Vanja Nikolić. Vanishing relaxation time limit of the Jordan–Moore–Gibson–Thompson wave equation with Neumann and absorbing boundary conditions. *Pure and Applied Functional Analysis*, 5:1–26, 2020.

The linearized problem

$$\begin{cases} \tau\psi_{ttt} + \alpha(x,t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = f & \text{in } \Omega \times (0,T), \\ \psi = 0 \quad \text{on } \partial\Omega \times (0,T), \\ (\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\}, \end{cases}$$

under the assumptions

$$\alpha(x,t) \geq \underline{\alpha} > 0 \;\text{ on } \Omega \quad \text{a.e. in } \Omega \times (0,T). \tag{1}$$

$$\begin{aligned} &\alpha \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{1,3}(\Omega)), \\ &f \in H^1(0,T;L^2(\Omega)). \end{aligned} \tag{2}$$

$$(\psi_0, \psi_1, \psi_2) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega). \tag{3}$$

The linearized problem

$$
\begin{cases}
\tau\psi_{ttt} + \alpha(x,t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = f \quad \text{in } \Omega \times (0,T),\\
\psi = 0 \quad \text{on } \partial\Omega \times (0,T),\\
(\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) \quad \text{in } \Omega \times \{0\},
\end{cases}
\tag{4}
$$

Theorem (lin)

Let c^2, b, $\tau > 0$, and let $T > 0$. Let the assumptions (1), (2), (3) hold. Then there exists a unique solution

$$\psi \in X^W := W^{1,\infty}(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{2,\infty}(0,T; H_0^1(\Omega)) \cap H^3(0,T; L^2(\Omega)).$$

The solution fullfils the estimate

$$
\begin{aligned}
\|\psi\|_{W,\tau}^2 :=& \tau^2 \|\psi_{ttt}\|_{L^2L^2}^2 + \tau\|\psi_{tt}\|_{L^\infty H^1}^2 + \|\psi_{tt}\|_{L^2H^1}^2 + \|\psi\|_{W^{1,\infty}H^2}^2 \\
&\leq C(\alpha, T, \tau)\left(|\psi_0|_{H^2}^2 + |\psi_1|_{H^2}^2 + \tau|\psi_2|_{H^1}^2 + \|f\|_{L^\infty L^2}^2 + \|f_t\|_{L^2L^2}^2\right).
\end{aligned}
$$

If additionally $\|\nabla\alpha\|_{L^\infty L^3} < \frac{\alpha}{C_{H^1,L^6}^{\Omega}}$ holds, then $C(\alpha, T, \tau)$ is independent of τ.

Well-posedness of the Westervelt type JMGT equation

$$\begin{cases} \tau\psi_{ttt} + (1 - k\psi_t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0 \quad \text{in } \Omega \times (0, T), \\ \psi = 0 \quad \text{on } \partial\Omega \times (0, T), \\ (\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) \quad \text{in } \Omega \times \{0\}, \end{cases}$$

Theorem

Let c^2, $b > 0$, $k \in \mathbb{R}$ and let $T > 0$. There exist ρ,$\rho_0 > 0$ such that for all $(\psi_0, \psi_1, \psi_2) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ satisfying

$$\|\psi_0\|^2_{H^2(\Omega)} + \|\psi_1\|^2_{H^2(\Omega)} + \tau\|\psi_2\|^2_{H^1(\Omega)} \le \rho_0^2 \,,$$

there exists a unique solution $\psi \in X^W$ and $\|\psi\|^2_{W,\tau} \le \rho^2$.

Banach's Contraction Principle for $\mathcal{T} : \phi \mapsto \psi$ solution ψ of (4) with $\alpha = 1 - k\phi_t$, $f = 0$:
self-mapping on $B_\rho^{X^W}$: energy estimate from Theorem (lin).
contractivity: $\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{W,\tau} \le q\|\phi_1 - \phi_2\|_{W,\tau}$ by estimate from Theorem (lin):
$\hat{\psi} = \psi_1 - \psi_2 = \mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)$ solves (4) with $\alpha = 1 - k\phi_{1\,t}$ and
$f = k\hat{\phi}_t\psi_{2\,tt}$ where $\hat{\phi} = \phi_1 - \phi_2$.

Limits for vanishing relaxation time

Consider the τ-independent part of the norms

$$\|\psi\|^2_{W,\tau} := \\ \tau^2\|\psi_{ttt}\|^2_{L^2L^2} + \tau\|\psi_{tt}\|^2_{L^\infty H^1} + \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2}$$

namely

$$\|\psi\|^2_{\bar{X}^W} := \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2}\,,$$

since these norms will be uniformly bounded, independently of τ.

Limits for vanishing relaxation time

Consider the τ-independent part of the norms

$$\|\psi\|^2_{\bar{X}^W} := \|\psi_{tt}\|^2_{L^2H^1} + \|\psi\|^2_{W^{1,\infty}H^2}\,,$$

and impose smallness of initial data in the space

$$X_0^W := H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega).$$

Theorem (BK&Nikolić M3AS 2019)

Let c^2, b, $T>0$, and $k\in\mathbb{R}$. Then there exist $\bar{\tau}$, $\rho_0>0$ such that for all $(\psi_0,\psi_1,\psi_2)\in B_{\rho_0}^{X_0^W}$, the family $(\psi^\tau)_{\tau\in(0,\bar{\tau})}$ of solutions to the Westervelt type JMGT equation converges weakly in $\bar{X}^W$ to a solution $\bar{\psi}\in\bar{X}^W$ of the Westervelt equation with initial conditions $\bar{\psi}(0)=\psi_0$, $\bar{\psi}_t(0)=\psi_1$.*

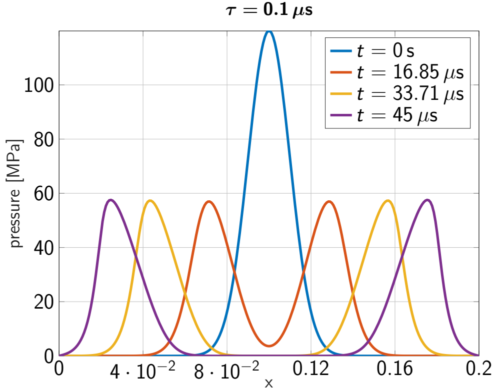
Numerical Experiments

- comparison of Westervelt-JMGT and Westervelt solutions
- numerical experiments for water in a 1-d channel geometry

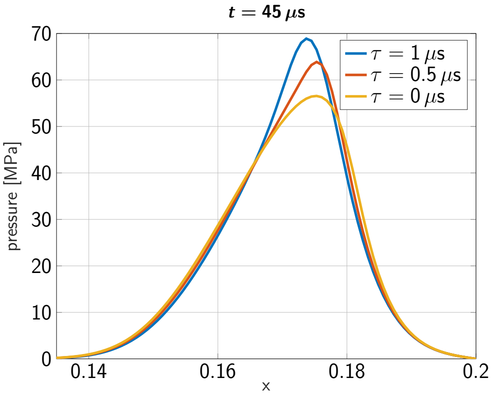
$$c = 1500\,\mathrm{m/s},\ \delta = 6 \cdot 10^{-9}\,\mathrm{m^2/s},\ \rho = 1000\,\mathrm{kg/m^3},\ B/A = 5;$$

- space discretization with B-splines (Isogeometric Analysis): quadratic basis functions, globally C^2; 251 dofs on $\Omega = [0, 0.2m]$
- time discretization by Newmark scheme, adapted to 3rd order equation; 800 time steps on $[0, T] = [0, 45\mu s]$
- initial conditions $(\psi_0, \psi_1, \psi_2) = \left(0,\ \mathcal{A} \exp\left(-\frac{(x-0.1)^2}{2\sigma^2}\right),\ 0\right)$ with $\mathcal{A} = 8 \cdot 10^4\,\mathrm{m^2/s^2}$ and $\sigma = 0.01$,

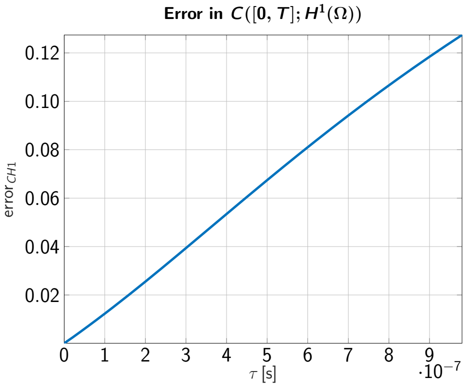
Snapshots of pressure $p = \varrho\psi_t$ for fixed relaxation time $\tau = 0.1\,\mu s$

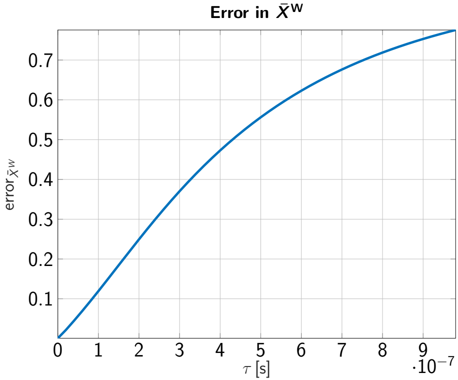


Pressure wave for different relaxation parameters τ at final time $t = 45\mu s$.



Relative errors as $\tau \to 0$





in $C([0,T];H^1(\Omega))$

in $\bar{X}^W = H^2(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^2(\Omega))$.

Recap: Vanishing relaxation time

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - (\delta + \tau c^2)\Delta\psi^\tau_t = \left(\tfrac{B}{2Ac^2}(\psi^\tau_t)^2 + |\nabla\psi^\tau|^2\right)_t$$

versus Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\tfrac{B}{2Ac^2}((\psi_t)^2) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^τ as $\tau \searrow 0$? Yes
Does ψ^0 solve Kuznetsov's equation? Yes

[Bongarti&Charoenphon&Lasiecka; BK& Nikolić, 2019-21] limit in JMGT/Kuznetsov/Westervelt

for vanishing diffusivity of sound δ

Vanishing diffusivity of sound

Kuznetsov's equation (likewise for Jordan-Moore-Gibson-Thompson):

$$\psi^\delta_{tt} - c^2\Delta\psi^\delta - \delta\Delta\psi^\delta_t = \left(\tfrac{B}{2Ac^2}(\psi^\delta_t)^2 + \left|\nabla\psi^\delta\right|^2\right)_t$$

undamped quasilinear wave equation:

$$\psi_{tt} - c^2\Delta\psi = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^δ as $\delta \searrow 0$?
Does ψ^0 solve the respective inviscid ($\delta = 0$) equation?

Challenge: $\delta > 0$ is crucial for global in time well-posedness and exponential decay in $d \in \{2, 3\}$ space dimensions.

Vanishing diffusivity of sound

Kuznetsov's equation (likewise for Jordan-Moore-Gibson-Thompson):

$$\psi^\delta_{tt} - c^2\Delta\psi^\delta - \delta\Delta\psi^\delta_t = \left(\tfrac{B}{2Ac^2}(\psi^\delta_t)^2 + \left|\nabla\psi^\delta\right|^2\right)_t$$

undamped quasilinear wave equation:

$$\psi_{tt} - c^2\Delta\psi = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^δ as $\delta \searrow 0$?
Does ψ^0 solve the respective inviscid ($\delta = 0$) equation?

Challenge: $\delta > 0$ is crucial for global in time well-posedness and exponential decay in $d \in \{2,3\}$ space dimensions.

[BK& Nikolić, SIAP 2021]
recover results (in particular on required regularity of initial data) from [Dörfler Gerner Schnaubelt 2016] for $\delta = 0$

limit in Blackstock-Crighton

for vanishing thermal conductivity a

Vanishing thermal conductivity

Blackstock-Crighton equation

$$(\partial_t - a\Delta)\left(\psi^a_{tt} - c^2\Delta\psi^a - \delta\Delta\psi^a_t\right) - ra\Delta\psi^a_t = \left(\frac{B}{2Ac^2}(\psi^{a2}_t) + |\nabla\psi^a|^2\right)_{tt}$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi^2_t) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^a as $a \searrow 0$?
Does ψ^0 solve Kuznetsov's equation?

Integrate once wrt time: Consistency of initial data needed:
$\psi_2 - c^2\Delta\psi_0 - \delta\Delta\psi_1 = \frac{B}{Ac^2}\psi_1\psi_2 + 2\nabla\psi_0 \cdot \nabla\psi_1$

Vanishing thermal conductivity

Blackstock-Crighton equation

$$(\partial_t - a\Delta)\left(\psi^a_{tt} - c^2\Delta\psi^a - \delta\Delta\psi^a_t\right) - ra\Delta\psi^a_t = \left(\frac{B}{2Ac^2}(\psi^{a2}_t) + |\nabla\psi^a|^2\right)_{tt}$$

Kuznetsov's equation:

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2Ac^2}(\psi^2_t) + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^0 of ψ^a as $a \searrow 0$?
Does ψ^0 solve Kuznetsov's equation?

Integrate once wrt time: Consistency of initial data needed:
$\psi_2 - c^2\Delta\psi_0 - \delta\Delta\psi_1 = \frac{B}{Ac^2}\psi_1\psi_2 + 2\nabla\psi_0 \cdot \nabla\psi_1$

[BK& Thalhammer, M3AS 2018]

limit in time fractional JMGT

for differentiation order $\alpha \nearrow 1$

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^{\alpha}+\psi_{tt}^{\alpha}-c^2\Delta\psi^{\alpha}-(\delta+\tau^{\alpha}c^2)\Delta D_t^{\alpha}\psi^{\alpha}=\left(\tfrac{B}{2Ac^2}(\psi_t^{\alpha})^2+|\nabla\psi^{\alpha}|^2\right)_t$$

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi_{ttt}+\psi_{tt}-c^2\Delta\psi-(\delta+\tau c^2)\Delta\psi_t=\left(\tfrac{B}{2Ac^2}(\psi_t)^2+|\nabla\psi|^2\right)_t$$

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^\alpha + \psi^\alpha_{tt} - c^2\Delta\psi^\alpha - (\delta+\tau^{\alpha} c^2)\Delta D_t^{\alpha}\psi^\alpha = \left(\tfrac{B}{2Ac^2}(\psi^\alpha_t)^2 + |\nabla\psi^\alpha|^2\right)_t$$

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta+\tau c^2)\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^1 of ψ^α as $\alpha \nearrow 1$?
Does ψ solve the respective integer order equation?

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^\alpha+\psi^\alpha_{tt}-c^2\Delta\psi^\alpha-(\delta+\tau^\alpha c^2)\Delta D_t^\alpha\psi^\alpha=\left(\tfrac{B}{2Ac^2}(\psi^\alpha_t)^2+|\nabla\psi^\alpha|^2\right)_t$$

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi_{ttt}+\psi_{tt}-c^2\Delta\psi-(\delta+\tau c^2)\Delta\psi_t=\left(\tfrac{B}{2Ac^2}(\psi_t)^2+|\nabla\psi|^2\right)_t$$

Existence of a limit ψ^1 of ψ^α as $\alpha\nearrow 1$?
Does ψ solve the respective integer order equation?

- Derivation of proper models from physical balance and constitutive laws

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^{\alpha}+\psi^{\alpha}_{tt}-c^2\Delta\psi^{\alpha}-(\delta+\tau^{\alpha}c^2)\Delta D_t^{\alpha}\psi^{\alpha}=\Big(\tfrac{B}{2Ac^2}(\psi^{\alpha}_t)^2+|\nabla\psi^{\alpha}|^2\Big)_t$$

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi_{ttt}+\psi_{tt}-c^2\Delta\psi-(\delta+\tau c^2)\Delta\psi_t=\Big(\tfrac{B}{2Ac^2}(\psi_t)^2+|\nabla\psi|^2\Big)_t$$

Existence of a limit ψ^1 of ψ^{α} as $\alpha \nearrow 1$?
Does ψ solve the respective integer order equation?

- Derivation of proper models from physical balance and constitutive laws
- Leading derivative order in PDE changes with α.

Fractional to integer damping

fractional Jordan-Moore-Gibson-Thompson equation

$$\tau^{\alpha} D_t^{2+\alpha}\psi^\alpha + \psi^\alpha_{tt} - c^2\Delta\psi^\alpha - (\delta + \tau^\alpha c^2)\Delta D_t^\alpha \psi^\alpha = \left(\tfrac{B}{2Ac^2}(\psi^\alpha_t)^2 + |\nabla\psi^\alpha|^2\right)_t$$

Jordan-Moore-Gibson-Thompson equation

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (\delta + \tau c^2)\Delta\psi_t = \left(\tfrac{B}{2Ac^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

Existence of a limit ψ^1 of ψ^α as $\alpha \nearrow 1$?
Does ψ solve the respective integer order equation?

- Derivation of proper models from physical balance and constitutive laws
- Leading derivative order in PDE changes with α.

[BK& Nikolić, M3AS 2022]

fractional damping models in ultrasonics

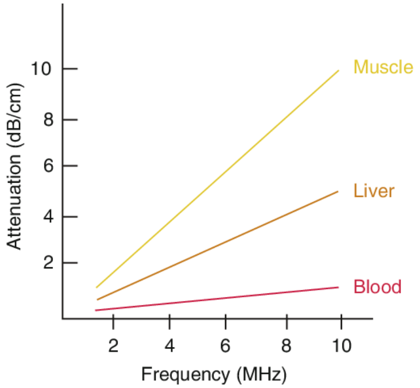


Figure 2.6 in [Chan&Perlas, Basics of Ultrasound Imaging, 2011]

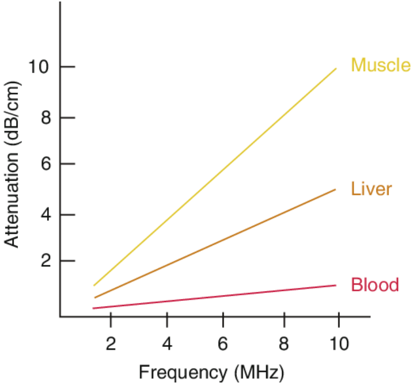


Figure 2.6 in [Chan&Perlas, Basics of Ultrasound Imaging, 2011]

$\rightsquigarrow$ constitutive modeling of

- pressure – density relation
- temperature – heat flux relation shortcut

Fractional Models of (Linear) Viscoelasticity

- equation of motion (resulting from balance of forces)

$$\varrho \mathbf{u}_{tt} = \mathrm{div}\sigma + \mathbf{f}$$

- strain as symmetric gradient of displacements:

$$\epsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

- constitutive model: stress-strain relation

$\mathbf{u}$. . . displacements
σ. . . stress tensor
ϵ. . . strain tensor
ϱ. . . mass density

Fractional Models of (Linear) Viscoelasticity 1-d setting

- equation of motion (resulting from balance of forces)

$$\varrho u_{tt} = \sigma_x + f$$

- strain as symmetric gradient of displacements:

$$\epsilon = u_x .$$

- constitutive model: stress-strain relation:

$$\begin{aligned} \text{Hooke's law (pure elasticity):} \quad & \sigma = b_0 \epsilon \\ \text{Newton model:} \quad & \sigma = b_1 \epsilon_t \\ \text{Kelvin-Voigt model:} \quad & \sigma = b_0 \epsilon + b_1 \epsilon_t \\ \text{Maxwell model:} \quad & \sigma + a_1 \sigma_t = b_0 \epsilon \\ \text{Zener model:} \quad & \sigma + a_1 \sigma_t = b_0 \epsilon + b_1 \epsilon_t \end{aligned}$$

Fractional Models of (Linear) Viscoelasticity 1-d setting

- equation of motion (resulting from balance of forces)

$$\varrho u_{tt} = \sigma_x + f$$

- strain as symmetric gradient of displacements:

$$\epsilon = u_x .$$

- constitutive model: stress-strain relation:

fractional Newton model: $\sigma = b_1 \partial_t^\beta \epsilon$

fractional Kelvin-Voigt model: $\sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

fractional Maxwell model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon$

fractional Zener model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

general model class: $\displaystyle\sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \sigma = \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \epsilon$

[Caputo 1967, Atanackovic, Pilipović, Stanković, Zorica 2014]

Fractional Models of (Linear) Acoustics via $p - \varrho$

balance of momentum

$$\varrho_0 \mathbf{v}_t = -\nabla p + \mathbf{f}$$

balance of mass

$$\varrho \nabla \cdot \mathbf{v} = -\varrho_t$$

equation of state

$$\frac{\varrho_\sim}{\varrho_0} = \frac{p_\sim}{p_0}$$

Fractional Models of (Linear) Acoustics via $p - \varrho$

balance of momentum
$$\varrho_0 \mathbf{v}_t = -\nabla p + \mathbf{f}$$

balance of mass
$$\varrho \nabla \cdot \mathbf{v} = -\varrho_t$$

equation of state
$$\sum_{m=0}^{M} b_m \partial_t^{\beta_m} \frac{\varrho_\sim}{\varrho_0} = \sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \frac{p_\sim}{p_0}$$

Fractional Models of (Linear) Acoustics via $p - \varrho$

balance of momentum
$$\varrho_0 \mathbf{v}_t = -\nabla p + \mathbf{f}$$

balance of mass
$$\varrho \nabla \cdot \mathbf{v} = -\varrho_t$$

equation of state
$$\sum_{m=0}^{M} b_m \partial_t^{\beta_m} \frac{\varrho_\sim}{\varrho_0} = \sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \frac{p_\sim}{p_0}$$

insert constitutive equations into combination of balance laws
$\rightsquigarrow$ fractional acoustic wave equations [Holm 2019, Szabo 2004]:

Fractional Models of (Linear) Acoustics via $p - \varrho$

balance of momentum
$$\varrho_0 \mathbf{v}_t = -\nabla p + \mathbf{f}$$

balance of mass
$$\varrho \nabla \cdot \mathbf{v} = -\varrho_t$$

equation of state
$$\sum_{m=0}^{M} b_m \partial_t^{\beta_m} \frac{\varrho_\sim}{\varrho_0} = \sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \frac{p_\sim}{p_0}$$

insert constitutive equations into combination of balance laws
$\rightsquigarrow$ fractional acoustic wave equations [Holm 2019, Szabo 2004]:

- Caputo-Wismer-Kelvin wave equation (fractional Kelvin-Voigt):
$$p_{tt} - b_0 \Delta p - b_1 \partial_t^{\beta} \Delta p = \tilde{f} \,,$$
- modified Szabo wave equation (fractional Maxwell):
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \Delta p = \tilde{f} \,,$$
- fractional Zener wave equation:
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \Delta p + b_1 \partial_t^{\beta} \Delta p = \tilde{f} \,,$$
- general fractional model:
$$\textstyle\sum_{n=0}^{N} a_n \partial_t^{2+\alpha_n} p - \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \Delta p = \tilde{f} \,.$$

Fractional Models of (Linear) Acoustics via $\vartheta - \boldsymbol{q}$

recall:

Classically: Fourier's law $\quad \boldsymbol{q} = -K\nabla\vartheta$

leads to infinite speed of propagation paradox.

Maxwell-Cattaneo law $\quad \tau\boldsymbol{q}_t + \boldsymbol{q} = -K\nabla\vartheta$

allows for "thermal waves" (second sound phenomenon)
can lead to violation of the 2nd law of thermodynamics

Fractional Models of (Linear) Acoustics via $\vartheta - \boldsymbol{q}$

recall:

Classically: Fourier's law $\quad \boldsymbol{q} = -K\nabla\vartheta$

leads to infinite speed of propagation paradox.

Maxwell-Cattaneo law $\quad \tau\boldsymbol{q}_t + \boldsymbol{q} = -K\nabla\vartheta$

allows for "thermal waves" (second sound phenomenon)
can lead to violation of the 2nd law of thermodynamics

"interpolate" by using fractional derivatives
[Compte & Metzler 1997, Povstenko 2011]:

$$\text{(GFE I)} \qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\tau_\vartheta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\vartheta;$$

$$\text{(GFE II)} \qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\tau_\vartheta^{\alpha-1}\mathrm{D}_t^{\alpha-1}\nabla\vartheta;$$

$$\text{(GFE III)} \qquad (1+\tau\partial_t)\boldsymbol{q}(t) = -K\tau_\vartheta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\vartheta;$$

$$\text{(GFE)} \qquad (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -K\nabla\vartheta.$$

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$${}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$${}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$${}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$${}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

- R-L is defined on a larger function space, but derivative of constant is nonzero; singularity at initial time a
- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$${}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$${}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

- R-L is defined on a larger function space, but derivative of constant is nonzero; singularity at initial time a
- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

some recent books on fractional PDEs: [Kubica & Ryszewska & Yamamoto 2020], [Jin 2021], [BK & Rundell 2022]

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$$ {}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$$ {}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

- R-L is defined on a larger function space, but derivative of constant is nonzero; singularity at initial time a
- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

Nonlocal and causal character of these derivatives provides them with a "memory"

Fractional derivatives

Abel fractional integral operator

$$I_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$$ {}_a^R D_t^\alpha f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$$ {}_a^C D_t^\alpha f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

- R-L is defined on a larger function space, but derivative of constant is nonzero; singularity at initial time a
- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

Nonlocal and causal character of these derivatives provides them with a "memory"
$\rightsquigarrow$ initial values are tied to later values and can therefore be better reconstructed backwards in time.

inverse problems

Nonlinearity parameter imaging

- B/A parameter is sensitive to differences in tissue properties, thus appropriate for characterization of biological tissues
- viewing $\kappa = \frac{1}{\varrho c^2}(\frac{B}{2A} + 1)$ as a spatially varying coefficient in the Westervelt equation, it can be used for medical imaging
- $\rightsquigarrow$ acoustic nonlinearity parameter tomography [Bjørnø 1986; Burov, Gurinovich, Rudenko, Tagunov 1994; Cain 1986; Ichida, Sato, Linzer 1983; Varray, Basset, Tortoli, Cachard 2011; Zhang, Gong et al 1996, 2001]...

The inverse problem of nonlinearity parameter imaging

Identify $\kappa(x)$ in

$$\left(u - \kappa(x)u^2\right)_{tt} - c_0^2 \Delta u + Du = r \quad \text{in } \Omega \times (0, T)$$

$$\partial_\nu u + \gamma u = 0 \text{ on } \partial\Omega \times (0, T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega$$

(with excitation r) from observations

$$g = u \quad \text{on } \Sigma \times (0, T)$$

$\Sigma \subset \overline{\Omega}$... transducer array (surface or collection of discrete points)

The inverse problem of nonlinearity parameter imaging

Identify $\kappa(x)$ in

$$\left(u - \kappa(x)u^2\right)_{tt} - c_0^2\Delta u + Du = r \quad \text{in } \Omega \times (0,T)$$

$$\partial_\nu u + \gamma u = 0 \text{ on } \partial\Omega \times (0,T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega$$

(with excitation r) from observations

$$g = u \quad \text{on } \Sigma \times (0,T)$$

$\Sigma \subset \overline{\Omega}$... transducer array (surface or collection of discrete points)

fractional damping

Caputo-Wismer-Kelvin:

$$D = -b\Delta\partial_t^\beta \quad \text{with } \beta \in [0,1], \quad b \geq 0$$

fractional Zener:

$$D = a\partial_t^{2+\alpha} - b\Delta\partial_t^\beta \quad \text{with } a > 0,\ b \geq ac^2,\ 1 \geq \beta \geq \alpha > 0,$$

space fractional Chen-Holm:

$$D = b(-\Delta)^{\tilde{\beta}}\partial_t \quad \text{with } \tilde{\beta} \in [0,1],\ b \geq 0,$$

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":
  linear case: double excitation $\Rightarrow$ double observation
  linear case: exitation at freq. ω $\Rightarrow$ observation at freq. ω

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":
  linear case: double excitation $\Rightarrow$ double observation
  linear case: exitation at freq. $\omega \ \Rightarrow$ observation at freq. ω
  nonlinear case: higher harmonics

Chances and Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":
  linear case: double excitation $\Rightarrow$ double observation
  linear case: exitation at freq. ω $\Rightarrow$ observation at freq. ω
  nonlinear case: higher harmonics $\rightsquigarrow$ information multiplies

[Acosta & Uhlmann & Zhai 2022] Westervelt equation:

- uniqueness from Neumann-Dirichlet map

[Eptaminitakis & Stefanov 2022] Westervelt equation:

- geometric optics solutions

[BK&Rundell IPI 2021, Math.Comp. 2022, Inv.Prob. 2023, IPI 2023] Westervelt eq.; [BK EECT 2023] JMGT eq.

- Well-definedness and Fréchet differentiability of forward operator $F : \kappa \mapsto u|_\Sigma$
- uniqueness for linearised problem from a single boundary observation
- reconstructions by Newton's method
- convergence of frozen Newton via range invariance of linearised forward operator
- uniqueness of $\kappa(x)$ from a single boundary observation (for unknown $c_0(x)$)
- linearised uniqueness of $\kappa(x)$ *and* $c_0(x)$ from two boundary observations

[Yamamoto &BK 2021] BCBJ eq. (a higher order model)

- linearised uniqueness and conditional stability via Carleman estimates

[Acosta & Uhlmann & Zhai 2022] Westervelt equation:

- uniqueness from Neumann-Dirichlet map

[Eptaminitakis & Stefanov 2022] Westervelt equation:

- geometric optics solutions

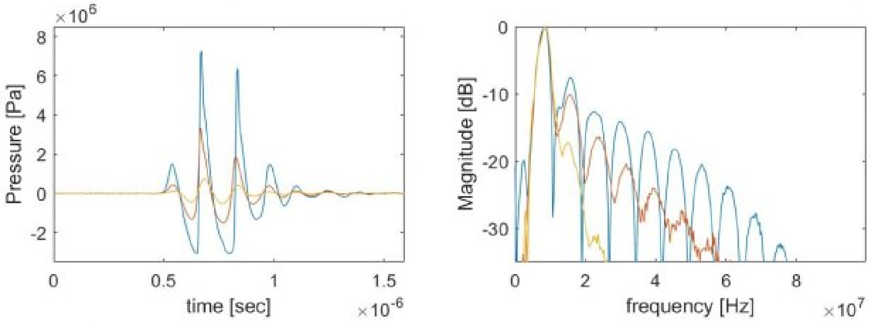
[BK&Rundell IPI 2021, Math.Comp. 2022, Inv.Prob. 2023, IPI 2023] Westervelt eq.; [BK EECT 2023] JMGT eq.

- Well-definedness and Fréchet differentiability of forward operator $F : \kappa \mapsto u|_\Sigma$
- uniqueness for linearised problem from a single boundary observation
- reconstructions by Newton's method
- convergence of frozen Newton via range invariance of linearised forward operator
- uniqueness of $\kappa(x)$ from a single boundary observation (for unknown $c_0(x)$)
- linearised uniqueness of $\kappa(x)$ *and* $c_0(x)$ from two boundary observations

[Yamamoto &BK 2021] BCBJ eq. (a higher order model)

- linearised uniqueness and conditional stability via Carleman estimates multiharmonic expansion

Multiharmonics in nonlinear acoustics



time signal from excitation at several amplitudes;
(taken from Master thesis by Teresa Rauscher, University of Klagenfurt, 2021)

Modeling wave propagation in frequency domain

linear wave equation

$$u_{tt} - c^2 \Delta u = r$$

Modeling wave propagation in frequency domain

linear wave equation

$$u_{tt} - c^2 \Delta u = r$$

Using a harmonic excitation $r(x,t) = \Re(\hat{r}(x) e^{\imath\omega t})$
and a harmonic ansatz for u

$$u(x,t) = \Re\left(\hat{u}(x) e^{\imath\omega t}\right)$$

Modeling wave propagation in frequency domain

linear wave equation

$$u_{tt} - c^2 \Delta u = r$$

Using a harmonic excitation $r(x,t) = \Re(\hat{r}(x)e^{\imath\omega t})$
and a harmonic ansatz for u

$$u(x,t) = \Re\left(\hat{u}(x)e^{\imath\omega t}\right)$$

leads to the Helmholtz equation

$$-\omega^2 \hat{u} - c^2 \Delta \hat{u} = \hat{r}.$$

Modeling nonlinear wave propagation in frequency domain

Westervelt equation:

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa(x)(u^2)_{tt} + r$$

Modeling nonlinear wave propagation in frequency domain

Westervelt equation:

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa(x)(u^2)_{tt} + r$$

Using a harmonic excitation $r(x,t) = \Re(\hat{r}(x) e^{\imath \omega t})$
and a multiharmonic expansion of u

$$u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x) e^{\imath k \omega t} \right)$$

Modeling nonlinear wave propagation in frequency domain

Westervelt equation:

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa(x)(u^2)_{tt} + r$$

Using a harmonic excitation $r(x,t) = \Re(\hat{r}(x) e^{\imath \omega t})$
and a multiharmonic expansion of u

$$u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x) e^{\imath k \omega t} \right)$$

leads to the coupled system

$$m = 1: \qquad -\omega^2 \hat{u}_1 - (c^2 + \imath\omega b)\Delta \hat{u}_1 = \hat{r} - \frac{\kappa}{2}\omega^2 \sum_{k=3:2}^{\infty} \overline{\hat{u}_{\frac{k-1}{2}}}\, \hat{u}_{\frac{k+1}{2}}$$

$$m = 2, 3 \ldots: \quad -\omega^2 m^2 \hat{u}_m - (c^2 + \imath\omega m b)\Delta \hat{u}_m = -\frac{\kappa}{4}\omega^2 m^2 \sum_{\ell=1}^{m-1} \hat{u}_\ell \hat{u}_{m-\ell}$$

$$-\frac{\kappa}{2}\omega^2 m^2 \sum_{k=m+2:2}^{\infty} \overline{\hat{u}_{\frac{k-m}{2}}}\, \hat{u}_{\frac{k+m}{2}}$$

Modeling nonlinear wave propagation in frequency domain

Theorem (time periodic solutions; [BK, EECT 2021])

For $b, c^2, \beta, \gamma, T > 0$, $\kappa \in L^\infty(\Omega)$, there exists $\rho > 0$ such that for all $r \in L^2(0,T;L^2(\Omega))$ with $\|r\|_{L^2(0,T;L^2(\Omega))} \leq \rho$ there exists a unique solution

$$u \in X := H^2(0,T;L^2(\Omega)) \cap H^1(0;T;H^{3/2}(\Omega)) \cap L^2(0;T;H^2(\Omega))$$

of

$$\begin{cases} u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(x)(u^2)_{tt} + g \quad \text{in } \Omega \times (0,T), \\ \beta u_t + \gamma u + \partial_\nu u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u(T)\,,\ u_t(0) = u_t(T) \quad \text{in } \Omega, \end{cases}$$

and the solution fulfills the estimate

$$\|u\|_X \leq \tilde{C}\|g\|_{L^2(0,T;L^2(\Omega))}$$

Back to the inverse problem

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":
  linear case: double excitation $\Rightarrow$ double observation
  linear case: exitation at freq. $\omega \;\Rightarrow\;$ observation at freq. ω
  nonlinear case: higher harmonics $\rightsquigarrow$ information multiplies

Back to the inverse problem

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.
- nonlinearity helps by "adding information":
  linear case: double excitation $\Rightarrow$ double observation
  linear case: exitation at freq. ω $\Rightarrow$ observation at freq. ω
  nonlinear case: higher harmonics $\rightsquigarrow$ information multiplies

$\Rightarrow$ enhanced uniqueness results for the inverse problem:
linear case: piecewise constant coefficients
nonlinear case: general spatially variable coefficients
from a single observation

linearised uniqueness

The inverse problem in frequency domain

model: $u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa(x)\,(u^2)_{tt} + r$

observation: $u(x_0) = g(x) \quad x_0 \in \Sigma$

The inverse problem in frequency domain

model: $u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(x)\,(u^2)_{tt} + r$

observation: $u(x_0) = g(x) \quad x_0 \in \Sigma$

$$r(x,t) = \Re(\hat{r}(x)e^{\imath\omega t}) \;\rightsquigarrow\; u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x)e^{\imath k\omega t}\right)$$

The inverse problem in frequency domain

model: $\quad u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(x)\,(u^2)_{tt} + r$

observation: $\quad u(x_0) = g(x) \quad x_0 \in \Sigma$

$$r(x,t) = \Re(\hat{r}(x)e^{\imath\omega t}) \;\rightsquigarrow\; u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x)e^{\imath k\omega t}\right)$$

$$\vec{u} = (\hat{u}_j)_{j\in\mathbb{N}} \qquad B_m(\vec{u}) = \tfrac{1}{4}\sum_{\ell=1}^{m-1} \hat{u}_\ell \hat{u}_{m-\ell} + \tfrac{1}{2}\sum_{k=m+2:2}^{\infty} \overline{\hat{u}_{\frac{k-m}{2}}}\,\hat{u}_{\frac{k+m}{2}}$$

The inverse problem in frequency domain

$$\begin{array}{ll} \text{model:} & u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(x)\,(u^2)_{tt} + r \\ \text{observation:} & u(x_0) = g(x) \quad x_0 \in \Sigma \end{array}$$

$$r(x,t) = \Re(\hat{r}(x)e^{\imath\omega t}) \;\rightsquigarrow\; u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x)e^{\imath k\omega t}\right)$$

$$\vec{u} = (\hat{u}_j)_{j\in\mathbb{N}} \qquad B_m(\vec{u}) = \tfrac{1}{4}\sum_{\ell=1}^{m-1} \hat{u}_\ell \hat{u}_{m-\ell} + \tfrac{1}{2}\sum_{k=m+2:2}^{\infty} \overline{\hat{u}_{\frac{k-m}{2}}}\,\hat{u}_{\frac{k+m}{2}}$$

$$F_m(\kappa,\vec{u}) := \begin{pmatrix} -\big(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\big)\hat{u}_m + \omega^2 m^2\,\kappa\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma\,\hat{u}_m \end{pmatrix}$$

$$y_m := \begin{pmatrix} \hat{r} \text{ if } m=1 \;/\; 0 \text{ if } m \geq 2 \\ \hat{g}_m \end{pmatrix}$$

The inverse problem in frequency domain

$$\begin{array}{ll} \text{model:} & u_{tt} - c^2\Delta u - b\Delta u_t = \kappa(x)\,(u^2)_{tt} + r \\ \text{observation:} & u(x_0) = g(x) \quad x_0 \in \Sigma \end{array}$$

$$r(x,t) = \Re(\hat{r}(x)e^{\imath\omega t}) \;\rightsquigarrow\; u(x,t) = \Re\left(\sum_{k=1}^{\infty} \hat{u}_k(x)e^{\imath k\omega t}\right)$$

$$\vec{u} = (\hat{u}_j)_{j\in\mathbb{N}} \qquad B_m(\vec{u}) = \tfrac{1}{4}\sum_{\ell=1}^{m-1} \hat{u}_\ell \hat{u}_{m-\ell} + \tfrac{1}{2}\sum_{k=m+2:2}^{\infty} \overline{\hat{u}_{\frac{k-m}{2}}}\,\hat{u}_{\frac{k+m}{2}}$$

$$F_m(\kappa,\vec{u}) := \begin{pmatrix} -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\hat{u}_m + \omega^2 m^2\,\kappa\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma\,\hat{u}_m \end{pmatrix}$$

$$y_m := \begin{pmatrix} \hat{r} \text{ if } m=1 \;/\; 0 \text{ if } m \geq 2 \\ \hat{g}_m \end{pmatrix}$$

$$\vec{F}(\kappa,\vec{u}) = y$$

The linearised inverse problem

$$\vec{F}(\kappa, \vec{u}) = y$$

where

$$F_m(\kappa, \vec{u}) := \begin{pmatrix} -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\hat{u}_m + \omega^2 m^2 \,\kappa\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma \hat{u}_m \end{pmatrix}$$

The linearised inverse problem

$$\vec{F}(\kappa, \vec{u}) = y$$

where

$$F_m(\kappa, \vec{u}) := \begin{pmatrix} -\big(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\big)\hat{u}_m + \omega^2 m^2 \kappa B_m(\vec{u}) \\ \mathrm{tr}_\Sigma \hat{u}_m \end{pmatrix}$$

linearisation:

$$\vec{F}'(\kappa, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) \approx F(\kappa, \vec{u}) - y$$

where

$$F'_m(\kappa, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) =$$
$$\begin{pmatrix} -\big(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\big)\underline{d\hat{u}_m} + \omega^2 m^2 \kappa B'_m(\vec{u})\underline{d\vec{u}} + \omega^2 m^2 \underline{d\kappa} B_m(\vec{u}) \\ \mathrm{tr}_\Sigma \underline{d\hat{u}_m} \end{pmatrix}$$

The linearised inverse problem

$$\vec{F}(\kappa, \vec{u}) = y$$

where

$$F_m(\kappa, \vec{u}) := \begin{pmatrix} -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\hat{u}_m + \omega^2 m^2\, \kappa\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma\, \hat{u}_m \end{pmatrix}$$

linearisation:

$$\vec{F}'(\kappa, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) \approx F(\kappa, \vec{u}) - y$$

where

$$F'_m(0, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) = \begin{pmatrix} -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\underline{d\hat{u}}_m + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma \underline{d\hat{u}}_m \end{pmatrix}$$

Linearised uniqueness

Show that

$$\vec{F}'(\kappa, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) = 0 \quad \Rightarrow \quad \underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0$$

where

$$F_m'(0, \vec{u})(\underline{d\kappa}, \underline{d\vec{u}}) = \begin{pmatrix} -\big(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\big)\underline{d\hat{u}}_m + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) \\ \mathrm{tr}_\Sigma \underline{d\hat{u}}_m \end{pmatrix}$$

$$\boxed{\begin{array}{lll} \text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\underline{d\hat{u}_m} + \omega^2 m^2\,\underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N} \end{array}}$$

$$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$$

$$\boxed{\begin{array}{lll} \text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right) \underline{d\hat{u}_m} + \omega^2 m^2 \, \underline{d\kappa} \, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N} \end{array}}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m$, $\psi_m \in \mathbb{R}$, $\phi \in H^2(\Omega, \mathbb{R})$, $\phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$

$$\boxed{\begin{array}{lll} \text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right) \underline{d\hat{u}_m} + \omega^2 m^2 \, \underline{d\kappa} \, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \operatorname{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N} \end{array}}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m, \;\; \psi_m \in \mathbb{R}, \;\; \phi \in H^2(\Omega, \mathbb{R}), \;\; \phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j \in \mathbb{N}, k \in K^j}$ of $-\Delta$ (with impedance b.c.)

$$\begin{aligned} 0 &= \langle \varphi_j^k, (\text{mod}) \rangle \\ &= -\left(\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j\right) \underbrace{\langle \varphi_j^k, \underline{d\hat{u}_m} \rangle}_{=: b_{m,j}^k} + \omega^2 m^2 \, B_m(\vec{\psi}) \underbrace{\langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle}_{=: a_j^k} \end{aligned}$$

$$\Rightarrow b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 \, B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}$$

$$\text{(obs)} \;\Rightarrow\; 0 = \operatorname{tr}_\Sigma \underline{d\hat{u}_m} = \sum_{j \in \mathbb{N}} \sum_{k \in K^j} b_{m,j}^k \operatorname{tr}_\Sigma \varphi_j^k = \sum_{j \in \mathbb{N}} M_{m,j} \sum_{k \in K^j} a_j^k \operatorname{tr}_\Sigma \varphi_j^k$$

$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$

$$\boxed{\begin{array}{lll} \text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\underline{d\hat{u}_m} + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N} \end{array}}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m$, $\psi_m \in \mathbb{R}$, $\phi \in H^2(\Omega, \mathbb{R})$, $\phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}_m} \rangle =: b_{m,j}^k$, $\quad \langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle =: a_j^k$

$$b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}$$

$$0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k \,\mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}$$

$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$

$$\begin{array}{lll} \text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\underline{d\hat{u}}_m + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}}_m = 0 & m \in \mathbb{N} \end{array}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m,\ \ \psi_m \in \mathbb{R},\ \ \phi \in H^2(\Omega,\mathbb{R}),\ \ \phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N},k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}}_m \rangle =: b_{m,j}^k, \quad \langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle =: a_j^k$

$$b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2\, B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}$$

$$0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k\, \mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}$$

Lemma: The infinite matrix M is nonsingular.

$$\Rightarrow \quad 0 = \sum_{k\in K^j} a_j^k\, \mathrm{tr}_\Sigma \varphi_j^k \quad j \in \mathbb{N}$$

$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$

$$\begin{array}{|lll|}\hline \text{(mod)} & -(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta)\underline{d\hat{u}_m} + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N} \\ \hline \end{array}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m$, $\psi_m \in \mathbb{R}$, $\phi \in H^2(\Omega, \mathbb{R})$, $\phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}_m} \rangle =: b_{m,j}^k$, $\quad \langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle =: a_j^k$

$$b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}$$

$$0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k \,\mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}$$

Lemma: The infinite matrix M is nonsingular.

$$\Rightarrow \quad 0 = \sum_{k\in K^j} a_j^k \,\mathrm{tr}_\Sigma \varphi_j^k \quad j \in \mathbb{N} \quad \Rightarrow \quad 0 = \mathrm{tr}_\Sigma \underbrace{\mathrm{Proj}_{E_j}[\phi^2 \underline{d\kappa}]}_{=: v_j}$$

$$\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$$

$$
\boxed{\begin{array}{lll}
\text{(mod)} & -\left(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta\right)\underline{d\hat{u}}_m + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\
\text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}}_m = 0 & m \in \mathbb{N}
\end{array}}
$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m$, $\psi_m \in \mathbb{R}$, $\phi \in H^2(\Omega, \mathbb{R})$, $\phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}}_m \rangle =: b_{m,j}^k$, $\quad \langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle =: a_j^k$

$$
b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}
$$

$$
0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k \, \mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}
$$

Lemma: The infinite matrix M is nonsingular.

$$
\Rightarrow \quad 0 = \sum_{k\in K^j} a_j^k \, \mathrm{tr}_\Sigma \varphi_j^k \quad j \in \mathbb{N} \quad \Rightarrow \quad 0 = \mathrm{tr}_\Sigma \underbrace{\mathrm{Proj}_{E_j}[\phi^2 \underline{d\kappa}]}_{=: v_j}
$$

Unique Continuation:

$$
-\Delta v_j = \lambda_j v_j \text{ and } \partial_\nu v_j + \gamma v_j = 0 \text{ and } \mathrm{tr}_\Sigma v_j = 0 \quad \Rightarrow \quad v_j = 0
$$

$$
\boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}
$$

$$
\boxed{\begin{array}{lll}
\text{(mod)} & -(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta)\underline{d\hat{u}_m} + \omega^2 m^2 \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\
\text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}_m} = 0 & m \in \mathbb{N}
\end{array}}
$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m$, $\psi_m \in \mathbb{R}$, $\phi \in H^2(\Omega, \mathbb{R})$, $\phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}_m} \rangle =: b_{m,j}^k$, $\quad \langle \varphi_j^k, \phi^2 \underline{d\kappa} \rangle =: a_j^k$

$$
b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}
$$

$$
0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k \, \mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}
$$

Lemma: The infinite matrix M is nonsingular.

$$
\Rightarrow \quad 0 = \sum_{k\in K^j} a_j^k \, \mathrm{tr}_\Sigma \varphi_j^k \quad j \in \mathbb{N} \quad \Rightarrow \quad 0 = \mathrm{tr}_\Sigma \underbrace{\mathrm{Proj}_{E_j}[\phi^2 \underline{d\kappa}]}_{=: v_j}
$$

Unique Continuation:

$$
-\Delta v_j = \lambda_j v_j \text{ and } \partial_\nu v_j + \gamma v_j = 0 \text{ and } \mathrm{tr}_\Sigma v_j = 0 \quad \Rightarrow \quad v_j = 0
$$

$$
\Rightarrow \quad 0 = a_j^k \quad k \in K^j,\ j \in \mathbb{N} \quad \Rightarrow \quad b_{m,j}^k = M_{m,j} a_j^k = 0 \quad k \in K^j,\ j \in \mathbb{N}
$$

$$\boxed{\begin{array}{lll} \text{(mod)} & -(\omega^2 m^2 + (c^2 + \imath\omega m b)\Delta)\underline{d\hat{u}}_m + \omega^2 m^2\, \underline{d\kappa}\, B_m(\vec{u}) = 0 & m \in \mathbb{N} \\ \text{(obs)} & \mathrm{tr}_\Sigma \underline{d\hat{u}}_m = 0 & m \in \mathbb{N} \end{array}}$$

Choose $\hat{u}_m(x) := \phi(x)\psi_m,\ \ \psi_m \in \mathbb{R},\ \ \phi \in H^2(\Omega, \mathbb{R}),\ \ \phi \neq 0$ a.e. in Ω

Expand wrt. eigensystem $(\varphi_j^k, \lambda_j)_{j\in\mathbb{N}, k\in K^j}$ of $-\Delta$ (with impedance b.c.)

$\langle \varphi_j^k, \underline{d\hat{u}_m}\rangle =: b_{m,j}^k, \quad \langle \varphi_j^k, \phi^2 \underline{d\kappa}\rangle =: a_j^k$

$$b_{m,j}^k = M_{m,j} a_j^k \qquad \text{with } M_{m,j} := \frac{\omega^2 m^2 B_m(\vec{\psi})}{\omega^2 m^2 - (c^2 + \imath\omega m b)\lambda_j}$$

$$0 = \sum_{j\in\mathbb{N}} M_{m,j} \sum_{k\in K^j} a_j^k\, \mathrm{tr}_\Sigma \varphi_j^k \quad m \in \mathbb{N}$$

Lemma: The infinite matrix M is nonsingular.

$$\Rightarrow \quad 0 = \sum_{k\in K^j} a_j^k\, \mathrm{tr}_\Sigma \varphi_j^k \quad j \in \mathbb{N} \quad \Rightarrow \quad 0 = \mathrm{tr}_\Sigma \underbrace{\mathrm{Proj}_{E_j}[\phi^2 \underline{d\kappa}]}_{=: v_j}$$

Unique Continuation:

$$-\Delta v_j = \lambda_j v_j \text{ and } \partial_\nu v_j + \gamma v_j = 0 \text{ and } \mathrm{tr}_\Sigma v_j = 0 \quad \Rightarrow \quad v_j = 0$$

$$\Rightarrow \quad 0 = a_j^k \quad k \in K^j,\ j \in \mathbb{N} \quad \Rightarrow \quad b_{m,j}^k = M_{m,j} a_j^k = 0 \quad k \in K^j,\ j \in \mathbb{N}$$

$$\Rightarrow \quad \boxed{\underline{d\kappa} = 0 \text{ and } \underline{d\vec{u}} = 0}$$

Linearised uniqueness

Theorem (BK&Rundell, IPI 2023)

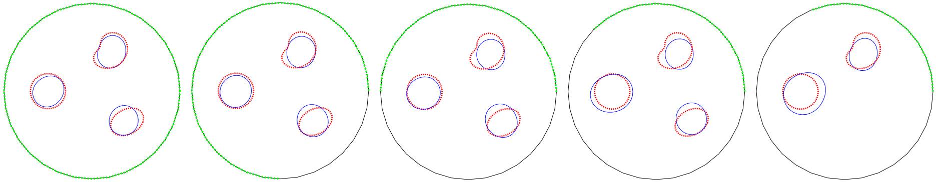
The homogeneous linearised
(at $\kappa = 0$, $\vec{u} = \phi(x)\vec{\psi}$ *with* $B_m(\vec{\psi}) \neq 0$, $m \in \mathbb{N}$*)*
inverse problem of nonlinearity coefficient imaging in frequency domain only has the trivial solution.

⏩ shortcut to the end

reconstructions from two harmonics

Reconstruction of three inclusions from partial data

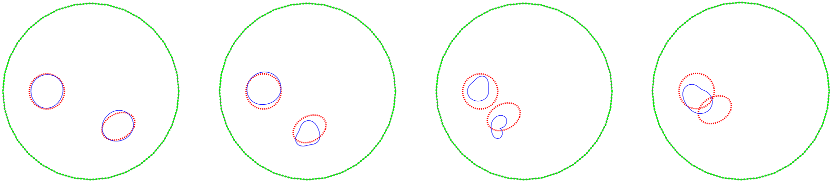
synthetic measurements with 1% noise
measurements taken on the green part of the boundary;
$\alpha \ldots$ observation angle
reconstruction by regularized Newton iterations
[BK& Rundell, IPI 2023]



(a) $\frac{\alpha}{2\pi} = 1$ (b) $\frac{\alpha}{2\pi} = 0.75$ (c) $\frac{\alpha}{2\pi} = 0.5$ (d) $\frac{\alpha}{2\pi} = 0.4$ (e) $\frac{\alpha}{2\pi} = 0.3$

Reconstruction of two inclusions at different distances

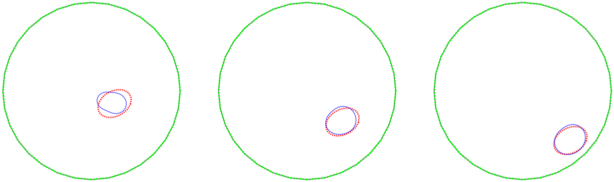
synthetic measurements with 1%noise



(a) $\frac{\theta}{2\pi} = 0.3$ (b) $\frac{\theta}{2\pi} = 0.2$ (c) $\frac{\theta}{2\pi} = 0.1$ (d) $\frac{\theta}{2\pi} = 0.09$

Reconstruction of one inclusion at different distances from the boundary

synthetic measurements with 1%noise



Outlook: Some further inverse problems

Reconstruct differentiation order(s)

- Determine fractional differentiation orders α_n, β_m in wave type eq.

$$\sum_{n=0}^{N} a_n \partial_t^{2+\alpha_n} p - \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \Delta p = \tilde{f} \,.$$

[BK& Rundell 2022];
for subdiffusion, see [Hatano& Nakagawa& Wang& Yamamoto 2013]
... [Jin& Kian 2022]

Reconstruct nonlinearity

- Determine nonlinearity f in generalized Westervelt equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = -\kappa(f(u))_{tt}$$

[BK& Rundell 2021]

Reconstruct general memory kernels

- Determine kernels k_ε, $k_{\mathrm{tr}\,\varepsilon}$ in viscoelastic model

$$\rho \mathbf{u}_{tt} - \mathrm{div}[\mathbb{C}\varepsilon(\mathbf{u}) + k_\varepsilon * \mathbb{A}\varepsilon(\mathbf{u}_t) + k_{\mathrm{tr}\,\varepsilon} * \mathrm{tr}\varepsilon(\mathbf{u}_t)\mathbb{I}] = \mathbf{f}$$

[BK & Khristenko & Nikolić & Rajendran & Wohlmuth 2022]

Thank you for your attention!