Iterative regularization of nonlinear ill-posed problems in Banach space

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joint work with

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Parameter identification in PDEs (I)

- e.g., electrical impedance tomography (EIT)

$$-\nabla(\sigma\nabla\phi) = 0 \text{ in } \Omega\,.$$

  Identify conductivity σ from measurements of the Dirichlet-to-Neumann map Λ_σ, i.e., all possible pairs $(\phi\ ,\ \sigma\partial_n\phi)$ on $\partial\Omega$.

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$$\nabla\times\left(\mu^{-1}\nabla\times\mathbf{E}\right)+\sigma\frac{\partial}{\partial t}\mathbf{E}+\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E}=\mathbf{J} \text{ in } \Omega\,.$$

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  Identify a from measurements of u in Ω.
- e.g. "c-example" (potential in stat. Schrödinger equation)

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- $$|\mathcal{F}f| = r$$

  Reconstruct the real-valued function $f : \mathbb{R} \to \mathbb{R}$ from measurements of the intensity $r : \mathbb{R} \to \mathbb{R}^+$ of its Fourier transform.

Forward operator F

- EIT: $F : \sigma \mapsto \Lambda_\sigma$
  where $\Lambda_\sigma : \phi \mapsto \sigma\partial_n\phi$ and $-\nabla(\sigma\nabla\phi) = 0$ in Ω.
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Nonlinear ill-posed problems

nonlinear operator equation

$$F(x) = y$$

$F : \mathcal{D}(F)(\subseteq X) \to Y$ ... nonlinear operator;
F not continuously invertible;

X, Y ... Banach spaces;

$y^\delta \approx y$ ... noisy data, $\|y^\delta - y\| \leq \delta$... noise level.

$\rightsquigarrow$ regularization necessary

Motivation for working in Banach space

- $X = L^P$ with $P \approx 1 \rightsquigarrow$ sparse solutions
  $\rightsquigarrow$ MS08 Optimization in Banach Spaces with
  Sparsity Constraints
- $X = L^P$ with $P \approx \infty \rightsquigarrow$ ellipticity and boundedness in the context of parameter id. in PDEs (e.g. $\nabla(a\nabla u) = 0$); avoid artificial increase of ill-posedness, that would result from a Hilbert space choice $X = H^{d/2+\epsilon}$

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c-example in Banach spaces

$$-\Delta u + c\, u = 0 \text{ in } \Omega\,.$$

Identify c from measurements of u in $\Omega \subseteq \mathbb{R}^d$.

(theoretical) reconstruction formula: $c = \dfrac{\Delta u}{u}$

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$\rightsquigarrow$ abstract stability result: Assume $|u| \geq \underline{u} > 0$, use $Y = L^\infty$:

$$\|c_1 - c_2\|_{L^P} \leq \frac{1}{\underline{u}} \|u(c_1) - u(c_2)\|_{W^{2,P}} + \frac{\|u(c_2)\|_{W^{2,P}}}{\underline{u}^2} \|u(c_1) - u(c_2)\|_{L^\infty}\,,$$

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Phase retrieval in Banach spaces

$$|\mathcal{F}f| = r$$

Reconstruct the real-valued function $f : \mathbb{R} \to \mathbb{R}$ from measurements of the intensity $r : \mathbb{R} \to \mathbb{R}^+$ of its Fourier transform.

natural preimage- and image spaces (Hausdorff-Young Theorem):

$$X = L^P_{\mathbb{R}}(\mathbb{R}), \quad Y = L^{\frac{P}{P-1}}_{\mathbb{R}}(\mathbb{R}) \quad \text{with } P \in [1,2]$$

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Regularization in Banach space

- case $Y = X$: iterative and variational regularization methods for linear and nonlinear problems
  [Plato'92,'94,'95, Bakushinskii&Kokurin'04]
- Tikhonov regularization for linear and nonlinear problems

  $$\mathcal{S}(F(x), y^\delta) + \alpha\mathcal{R}(x) = \min!$$

  [Burger&Osher'04, Resmerita&Scherzer'06,
  Hofmann&BK&Pöschl&Scherzer'07,
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$\rightsquigarrow$ motivates iterative regularization for nonlinear problems [Schöpfer&Louis&Schuster'06, Hein&Kazimierski'10, BK&Schöpfer&Schuster'09, BK&Hofmann'10,...]

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Outline

- short review on iterative regularization for nonlinear problems in Hilbert space
- some Banach space tools
- Landweber for nonlinear problems in Banach space
- Newton for nonlinear problems in Banach space
- numerical tests

Iterative regularization for nonlinear problems in Hilbert space

- gradient method for $\min_X \|F(x) - y^\delta\|^2$
  $\rightsquigarrow$ Landweber iteration
  $$x_{k+1}^\delta = x_k^\delta - \mu_k F'(x_k^\delta)^* \left(F(x_k^\delta) - y^\delta \right)$$

  [Hanke&Neubauer&Scherzer'96]

Iterative regularization for nonlinear problems in Hilbert space

- Newton's method for $F(x) = y^\delta$ plus regularization
  $\rightsquigarrow$ Levenberg Marquardt method

$$x_{k+1}^\delta = x_k^\delta - \underbrace{(F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1} F'(x_k^\delta)^*}_{\approx F'(x_k^\delta)^{-1}} \Big(F(x_k^\delta) - y^\delta \Big)$$

[Hanke'97,'10], Newton-CG: [Hanke'97], inexact Newton [Rieder'01]

$\rightsquigarrow$ iteratively regularized Gauss-Newton method (IRGN)

$$x_{k+1}^\delta = x_k^\delta - \underbrace{(F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1} F'(x_k^\delta)^*}_{\approx F'(x_k^\delta)^{-1}} \Big(F(x_k^\delta) - y^\delta + \alpha_k (x_k^\delta - x_0) \Big)$$

[Bakushinskii'92, BK&Neubauer&Scherzer'96,'08, BK'97, Hohage'97]

Some Banach space tools (I)

Smoothness

- X ... smooth $\iff$ norm Gâteaux differentiable on $X \setminus \{0\}$;
- X ... uniformly smooth $\iff$ norm Fréchet differentiable on unit sphere;

Convexity

- X ... strictly convex $\iff$ boundary of unit ball contains no line segment;
- X ... uniformly convex $\iff$ modulus of convexity $\delta_X(\epsilon) > 0 \; \forall \epsilon \in (0,2]$;

$$\delta_X(\epsilon) = \inf\{1 - \left\|\tfrac{1}{2}(x+y)\right\| : \|x\| = \|y\| = 1\,,\ \|x-y\| \geq \epsilon\}$$

$L^P(\Omega)$, $P \in (1,\infty)$ is uniformly convex (Hanner's ineq.) and uniformly smooth

Some Banach space tools (II)

- **Dual space:**

  $X^* = L(X, \mathbb{R}) \ldots$ bounded linear functionals on X
  $x^* : x \mapsto \langle x^*, x \rangle$
- X uniformly smooth $\Leftrightarrow$ X^* uniformly convex
- X reflexive: X smooth $\Leftrightarrow$ X^* strictly convex

Some Banach space tools (III)

- **Duality mapping:**

  $J_p : X \to 2^{X^*}$,
  $J_p(x) = \{x^* \in X^* \ : \ \langle x^*, x\rangle = \|x^*\| \, \|x\| = \|x\|^p\}$
  J_p set valued;
  j_p ... single valued selection of J_p;
- $J_p = \partial \frac{1}{p} \|\cdot\|^p$ (Asplund)
- X smooth $\Leftrightarrow$ J_p single valued
- X reflexive, smooth, strictly convex $\Rightarrow$ $J_p^{-1} = J^*_{\frac{p}{p-1}}$

$L^P(\Omega)$, $P \in (1,\infty)$: $\quad J_P(x) = \|x\|_{L_P}^{p-P} |x|^{P-1} \mathrm{sign}(x)$

$\rightsquigarrow$ J_P possibly nonlinear and nonsmooth

Some Banach space tools (IV)

Bregman distance:

$D_p(x,\tilde{x}) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x}-x\rangle \ : \ \xi \in J_p(\tilde{x})\}$,
$x,\tilde{x} \in X$;

smooth X:
$D_p(x,\tilde{x}) = \frac{p-1}{p}(\|\tilde{x}\|^p - \|x\|^p) + \langle J_p(x) - J_p(\tilde{x}), x\rangle$;

smooth and uniformly convex X:
convergence in D_p $\Leftrightarrow$ convergence in $\|\cdot\|$

Hilbert space case: $D_2(x,\tilde{x}) = \frac{1}{2}\|x-\tilde{x}\|^2$

Assumptions on pre-image and image space X, Y

- X smooth, uniformly convex
  $\Rightarrow$ X reflexive (Milman-Pettis) and strictly convex
  J_p single valued, norm-to-weak-continuous, bijective
- Y arbitrary Banach space

Further assumptions for convergence proofs

- closeness to a solution $x^\dagger$: $\|x_0 - x^\dagger\|$ sufficiently small
- F' Lipschitz continuous and $x^\dagger$ sufficiently smooth
  or
- tangential cone condition: For all $x \in \mathcal{D}(F)$ there exists $F'(x) \in L(X, Y)$ ($F'(x)$ not necess. Fréchet derivative) s.t.

  $$\|F(x) - F(\bar{x}) - F'(x)(x - \bar{x})\| \leq c_{tc}\|F(x) - F(\bar{x})\| \qquad \forall x, \bar{x} \in \mathcal{B}$$

- for Landweber:
  F, F' continuous and interior of $\mathcal{D}(F)$ nonempty
- for IRGN:
  F (weakly) sequentially closed

Stopping rule

discrepancy principle

$$k_*(\delta) = \min\{k \in \mathbb{N} \ : \ \left\| F(x_k^\delta) - y^\delta \right\| \leq C_{dp}\delta\}$$

$C_{dp} > 1$, $\|y^\delta - y\| \leq \delta \ldots$ noise level

Trade off between stability and approximation:

Stop as early as possible (stability) such that
the resudual is lower than the noise level (approximation)

Landweber for inverse problems in Banach space

$$
\begin{aligned}
J_p(x_{k+1}^\delta) &= J_p(x_k^\delta) - \mu_k F'(x_k^\delta)^* j_r(F'(x_k^\delta) - y^\delta)\,, \\
x_{k+1}^\delta &= J^*_{\frac{p}{p-1}}(J_p(x_{k+1}^\delta))
\end{aligned}
$$

$p, r \in (1, \infty)$.
for comparison: Landweber in Hilbert space:

$$
x_{k+1}^\delta = x_k^\delta - \mu_k F'(x_k^\delta)^* \Big(F(x_k^\delta) - y^\delta \Big)
$$

Convergence results for Landweber (I)

Theorem (monotonicity of the error)
μ_k appropriately chosen (suff. small), c_{tc} suff. small, C_{dp} suff. large.
Then for all $k \leq k_*(\delta) - 1$, $x_{k+1}^\delta \in \mathcal{D}(F)$ and

$$D_p(x^\dagger, x_{k+1}^\delta) - D_p(x^\dagger, x_k^\delta) \leq -C \frac{\left\| F(x_k) - y^\delta \right\|^p}{\left\| F'(x_k^\delta) \right\|^p} < 0$$

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{p-1}{p}(\|\tilde{x}\|^p - \|x\|^p) + (J_p(x) - J_p(\tilde{x}), x)$

$$D_p(x^\dagger, x_{k+1}^\delta) - D_p(x^\dagger, x_k^\delta)$$

$$= \frac{p-1}{p}\left(\left\|x_{k+1}^\delta\right\|^p - \left\|x_k^\delta\right\|^p\right) - \langle \underbrace{J_p(x_{k+1}^\delta) - J_p(x_k^\delta)}_{=\mu_k F'(x_k^\delta)^* j_r(F(x_k^\delta)-y^\delta)}, x^\dagger\rangle$$

$$= D_p(x_k^\delta, x_{k+1}^\delta) - \mu_k \langle j_r(F(x_k^\delta) - y^\delta), \underbrace{F'(x_k^\delta)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta)-y^\delta}\rangle$$

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$$\leq \underbrace{D_p(x_k^\delta, x_{k+1}^\delta)}_{=O(\mu_k^{1+\epsilon})} - \mu_k\Big(1 - c(c_{tc}, C_{dp})\Big) \left\|F(x_k^\delta) - y^\delta\right\|^r$$

Idea of proof

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$$
\begin{aligned}
&D_p(x^\dagger, x_{k+1}^\delta) - D_p(x^\dagger, x_k^\delta) \\
&= \frac{p-1}{p}\left(\left\|x_{k+1}^\delta\right\|^p - \left\|x_k^\delta\right\|^p\right) - \langle \underbrace{J_p(x_{k+1}^\delta) - J_p(x_k^\delta)}_{=\mu_k F'(x_k^\delta)^* j_r(F(x_k^\delta)-y^\delta)}, x^\dagger\rangle \\
&= D_p(x_k^\delta, x_{k+1}^\delta) - \mu_k \langle j_r(F(x_k^\delta) - y^\delta), \underbrace{F'(x_k^\delta)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta)-y^\delta}\rangle \\
&\leq \underbrace{D_p(x_k^\delta, x_{k+1}^\delta)}_{=O(\mu_k^{1+\epsilon})} - \mu_k\left(1 - c(c_{tc}, C_{dp})\right)\left\|F(x_k^\delta) - y^\delta\right\|^r \\
&\leq 0
\end{aligned}
$$

by the choice of μ_k.

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{p-1}{p}(\|\tilde{x}\|^p - \|x\|^p) + (J_p(x) - J_p(\tilde{x}), x)$

$$
\begin{aligned}
& D_p(x^\dagger, x_{k+1}^\delta) - D_p(x^\dagger, x_k^\delta) \\
&= \frac{p-1}{p}\left(\left\|x_{k+1}^\delta\right\|^p - \left\|x_k^\delta\right\|^p\right) - \langle \underbrace{J_p(x_{k+1}^\delta) - J_p(x_k^\delta)}_{=\mu_k F'(x_k^\delta)^* j_r(F(x_k^\delta)-y^\delta)}, x^\dagger\rangle \\
&= D_p(x_k^\delta, x_{k+1}^\delta) - \mu_k \langle j_r(F(x_k^\delta) - y^\delta), \underbrace{F'(x_k^\delta)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta)-y^\delta}\rangle \\
&\le \underbrace{D_p(x_k^\delta, x_{k+1}^\delta)}_{=O(\mu_k^{1+\epsilon})} - \mu_k\Big(1 - c(c_{tc}, C_{dp})\Big)\left\|F(x_k^\delta) - y^\delta\right\|^r \\
&\le 0
\end{aligned}
$$

by the choice of μ_k.

Convergence results for Landweber (II)

Theorem (convergence with exact data)
$\delta = 0$, μ_k appropriately chosen (suff. small). Then

$$x_k \to x^\dagger \text{ solution to } F(x) = y \text{ as } k \to \infty$$

Theorem (stability for $\delta > 0$ and convergence as $\delta \to 0$)
μ_k appropriately chosen, c_{tc} suff. small, C_{dp} suff. large.
Y uniformly smooth.
Then for all $k \leq k_*(\delta)$, x_k^δ continuously depends on y^δ and

$$x_{k_*(\delta)}^\delta \to x^\dagger \text{ solution to } F(x) = y \text{ as } \delta \to 0$$

[Schöpfer&Louis&Schuster'06] linear case,
[BK&Schöpfer&Schuster'09] nonlinear case

Remark

- convergence rates can be shown for the *iteratively regularized Landweber iteration*

$$\begin{aligned} J_p(x_{k+1}^\delta - x_0) &= (1-\alpha_k)J_p(x_k^\delta - x_0) - \mu_k F'(x_k^\delta)^* j_r(F'(x_k^\delta) - y^\delta) \\ x_{k+1}^\delta &= x_0 + J^*_{\frac{p}{p-1}}(J_p(x_{k+1}^\delta - x_0)) \end{aligned}$$

$p, r \in (1, \infty)$, and x_0 ... initial guess.

IRGN for inverse problems in Banach space

$$x_{k+1}^\delta \in \mathrm{argmin}_{x\in\mathcal{D}(F)} \left\| F'(x_k^\delta)(x-x_k^\delta)+F(x_k^\delta)-y^\delta \right\|^r + \alpha_k \|x-x_0\|^p \,,$$

$p, r \in (1,\infty)$, and x_0 ... initial guess;

convex minimization problem:
efficient solution see, e.g., [Bonesky, Kazimierski, Maass, Schöpfer, Schuster'07]

for comparison: IRGN in Hilbert space:

$$x_{k+1}^\delta = x_k^\delta - (F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1}\Big(F(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)\Big)$$

Choice of α_k

discrepancy type principle:

$$\underline{\theta}\left\|F(x_k)-y^\delta\right\| \leq \left\|F'(x_k^\delta)(x_{k+1}^\delta(\alpha)-x_k^\delta)+F(x_k^\delta)-y^\delta\right\| \leq \overline{\theta}\left\|F(x_k)-y^\delta\right\|$$

$0<\underline{\theta}<\overline{\theta}<1$

Trade off between stability and approximation:

Choose α_k as large as possible (stability) such that the predicted residual is smaller than the old one (approximation)

see also: inexact Newton (for inverse problems: [Hanke'97, Rieder'99,'01])

Convergence of the IRGN

Theorem [BK&Schöpfer&Schuster'09]
C_{dp}, $\underline{\theta}$, $\overline{\theta}$ sufficiently large, c_{tc} sufficently small.
Additionally, assume that either
(a) $F'(x) : X \to Y$ is weakly closed for all $x \in \mathcal{D}(F)$ and Y reflexive or
(b) $\mathcal{D}(F)$ weakly closed.
Then for all $k \leq k_*(\delta) - 1$ the iterates

$$x_{k+1}^\delta \in \mathrm{argmin} \left\| F'(x_k^\delta)(x - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r + \alpha_k \left\| x - x_0 \right\|^p$$
$$\alpha_k \text{ s.t. } \left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\| \sim \theta \left\| F(x_k) - y^\delta \right\|$$

are well-defined and

$$x_{k_*(\delta)} \to x^\dagger \text{ solution to } F(x) = y \text{ as } \delta \to 0$$

if $x^\dagger$ unique, (and along subsequences otherwise).

Idea of proof

By minimality of x_{k+1}^δ:

$$\left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r + \alpha_k \left\| x_{k+1}^\delta - x_0 \right\|^p$$
$$\leq \underbrace{\left\| F'(x_k^\delta)(x^\dagger - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r}_{\approx \delta^r \leq C_{dp}^{-r} \left\| F(x_k^\delta) - y^\delta \right\|^r} + \alpha_k \left\| x^\dagger - x_0 \right\|^p .$$

Choice of $\alpha_k \Rightarrow \left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\| \geq \underline{\theta} \left\| F(x_k^\delta) - y^\delta \right\|$

Idea of proof

By minimality of x_{k+1}^δ:

$$\left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r + \alpha_k \left\| x_{k+1}^\delta - x_0 \right\|^p$$
$$\leq \underbrace{\left\| F'(x_k^\delta)(x^\dagger - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r}_{\approx \delta^r \leq C_{dp}^{-r} \left\| F(x_k^\delta) - y^\delta \right\|^r} + \alpha_k \left\| x^\dagger - x_0 \right\|^p .$$

Choice of $\alpha_k \;\Rightarrow\; \left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\| \geq \underline{\theta} \left\| F(x_k^\delta) - y^\delta \right\|$

$$\alpha_k \left(\left\| x_{k+1}^\delta - x_0 \right\|^p - \left\| x^\dagger - x_0 \right\|^p \right)$$
$$\leq (c(c_{tc}, C_{dp}) - \underline{\theta}^r) \left\| F(x_k^\delta) - y^\delta \right\|^r$$

Idea of proof

By minimality of x_{k+1}^δ:

$$\left\|F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta\right\|^r + \alpha_k \left\|x_{k+1}^\delta - x_0\right\|^p$$
$$\leq \underbrace{\left\|F'(x_k^\delta)(x^\dagger - x_k^\delta) + F(x_k^\delta) - y^\delta\right\|^r}_{\approx \delta^r \leq C_{dp}^{-r}\left\|F(x_k^\delta) - y^\delta\right\|^r} + \alpha_k \left\|x^\dagger - x_0\right\|^p .$$

Choice of $\alpha_k \;\Rightarrow\; \left\|F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta\right\| \geq \underline{\theta} \left\|F(x_k^\delta) - y^\delta\right\|$

$$\alpha_k \left(\left\|x_{k+1}^\delta - x_0\right\|^p - \left\|x^\dagger - x_0\right\|^p \right)$$
$$\leq \left(c(c_{tc}, C_{dp}) - \underline{\theta}^r\right) \left\|F(x_k^\delta) - y^\delta\right\|^r$$

Choice of $\underline{\theta}^r > c(c_{tc}, C_{dp}) \;\Rightarrow\; \left\|x_{k+1}^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$

Idea of proof

By minimality of x_{k+1}^δ:

$$\left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r + \alpha_k \left\| x_{k+1}^\delta - x_0 \right\|^p$$
$$\leq \underbrace{\left\| F'(x_k^\delta)(x^\dagger - x_k^\delta) + F(x_k^\delta) - y^\delta \right\|^r}_{\approx \delta^r \leq C_{dp}^{-r} \left\| F(x_k^\delta) - y^\delta \right\|^r} + \alpha_k \left\| x^\dagger - x_0 \right\|^p .$$

Choice of $\alpha_k \;\Rightarrow\; \left\| F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta \right\| \geq \underline{\theta} \left\| F(x_k^\delta) - y^\delta \right\|$

$$\alpha_k \left(\left\| x_{k+1}^\delta - x_0 \right\|^p - \left\| x^\dagger - x_0 \right\|^p \right)$$
$$\leq \; (c(c_{tc}, C_{dp}) - \underline{\theta}^r) \left\| F(x_k^\delta) - y^\delta \right\|^r$$

Choice of $\underline{\theta}^r > c(c_{tc}, C_{dp}) \;\Rightarrow\; \left\| x_{k+1}^\delta - x_0 \right\|^p \leq \left\| x^\dagger - x_0 \right\|^p$

Convergence rates for the IRGN

Theorem [BK&Hofmann'10]
Let the assumptions of the previous theorem be satisfied.
Under the *source type condition*

$$J_p(x^\dagger - x_0) \cap \mathcal{R}(F'(x^\dagger)^*) \neq \emptyset \,, \text{ i.e. },$$

$$\exists \hat{\xi} \in J_p(x^\dagger - x_0) \,, \; v \in Y^* \; : \; \hat{\xi} = F'(x^\dagger)^* v$$

we obtain optimal convergence rates

$$D_p(x_{k_*} - x_0, x^\dagger - x_0) = O(\delta) \,,$$

where $D_p^{x_0}(x, \tilde{x}) = D_p(x - x_0, \tilde{x} - x_0)$.

Hilbert space case: $\left\| x_{k_*} - x^\dagger \right\| = O(\sqrt{\delta})$

Idea of proof

recall: $D_p(x, \tilde{x}) = \frac{1}{p} \|x\|^p - \frac{1}{p} \|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x} - x\rangle \ : \ \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x} - x\rangle \; : \; \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

$$D_p(x_k^\delta - x_0, x^\dagger - x_0)$$
$$\leq \underbrace{\frac{1}{p}\left\|x_k^\delta - x_0\right\|^p - \frac{1}{p}\left\|x^\dagger - x_0\right\|^p}_{\leq 0} - \langle \underbrace{\hat{\xi}}_{=F'(x^\dagger)^* v}, x^\dagger - x_k^\delta\rangle$$

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x}-x\rangle \ : \ \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

$$D_p(x_k^\delta - x_0, x^\dagger - x_0)$$
$$\leq \underbrace{\frac{1}{p}\left\|x_k^\delta - x_0\right\|^p - \frac{1}{p}\left\|x^\dagger - x_0\right\|^p}_{\leq 0} - \langle \underbrace{\hat{\xi}}_{=F'(x^\dagger)^* v}, x^\dagger - x_k^\delta\rangle$$
$$\leq \langle v, \underbrace{F'(x^\dagger)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta) - y^\delta}\rangle$$

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x}-x\rangle \ : \ \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

$$D_p(x_k^\delta - x_0, x^\dagger - x_0)$$

$$\leq \underbrace{\frac{1}{p}\left\|x_k^\delta - x_0\right\|^p - \frac{1}{p}\left\|x^\dagger - x_0\right\|^p}_{\leq 0} - \langle \underbrace{\hat{\xi}}_{=F'(x^\dagger)^* v}, x^\dagger - x_k^\delta\rangle$$

$$\leq \langle v, \underbrace{F'(x^\dagger)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta) - y^\delta}\rangle$$

$$\leq \|v\|(1+c_{tc})\|F(x_k^\delta) - y^\delta\|$$

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x}-x\rangle \ : \ \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

$$D_p(x_k^\delta - x_0, x^\dagger - x_0)$$
$$\leq \underbrace{\frac{1}{p}\left\|x_k^\delta - x_0\right\|^p - \frac{1}{p}\left\|x^\dagger - x_0\right\|^p}_{\leq 0} - \langle \underbrace{\hat{\xi}}_{=F'(x^\dagger)^* v}, x^\dagger - x_k^\delta\rangle$$
$$\leq \langle v, \underbrace{F'(x^\dagger)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta)-y^\delta}\rangle$$
$$\leq \|v\|(1+c_{tc})\|F(x_k^\delta) - y^\delta\|$$

Hence, for $k = k_*$: $D_p(x^\dagger - x_0, x_{k_*}^\delta - x_0) \leq \|v\|(1+c_{tc})C_{dp}\,\delta$.

Idea of proof

recall: $D_p(x,\tilde{x}) = \frac{1}{p}\,\|x\|^p - \frac{1}{p}\,\|\tilde{x}\|^p - \inf\{\langle \xi, \tilde{x}-x\rangle \ : \ \xi \in J_p(\tilde{x})\}$

We have, from the previous proof, $\left\|x_k^\delta - x_0\right\|^p \leq \left\|x^\dagger - x_0\right\|^p$ for $k \leq k_*$.

$$
\begin{aligned}
&D_p(x_k^\delta - x_0, x^\dagger - x_0)\\
&\quad \leq \underbrace{\frac{1}{p}\left\|x_k^\delta - x_0\right\|^p - \frac{1}{p}\left\|x^\dagger - x_0\right\|^p}_{\leq 0} - \langle \underbrace{\hat{\xi}}_{=F'(x^\dagger)^* v}, x^\dagger - x_k^\delta\rangle\\
&\leq \langle v, \underbrace{F'(x^\dagger)(x_k^\delta - x^\dagger)}_{\approx F(x_k^\delta) - y^\delta}\rangle\\
&\leq \|v\|(1+c_{tc})\|F(x_k^\delta) - y^\delta\|
\end{aligned}
$$

Hence, for $k = k_*$: $D_p(x^\dagger - x_0, x_{k_*}^\delta - x_0) \leq \|v\|(1+c_{tc})C_{dp}\,\delta$.

Remarks

- rates result can be extended to
  $D_p^{x_0}(x_{k_*}, x^\dagger) = O(\delta^\kappa)$ with $\kappa \in (0,1)$ or
  $D_p^{x_0}(x_{k_*}, x^\dagger) = O(\log(\delta)^{-\kappa})$ with $\kappa > 0$
  under *approximate* source conditions;
- rates can be shown alternatively with *a priori* choice of α_k and k_* instead of the discrepancy principle;
  needs a priori information on smoothness of $x^\dagger$, though

Numerical tests

c-example:

$$-\Delta u + c\, u = 0 \text{ in } \Omega\,.$$

Identify c from measurements of u in Ω.

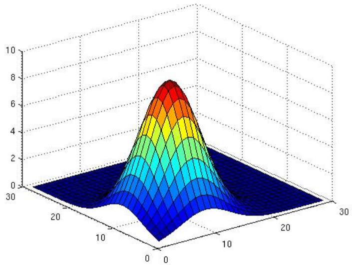
$\Omega = (0,1)^2 \subseteq \mathbb{R}^2$,

"smooth" c: $\quad c(x,y) = 10 \exp\left(-\frac{(x-0.3)^2 + (y-0.3)^2}{0.04}\right)$

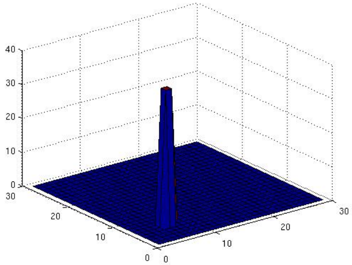
"sparse" c: $\quad c(x,y) = 40\chi_{[0.19,0.24]^2}(x,y)$

Test examples

smooth c:

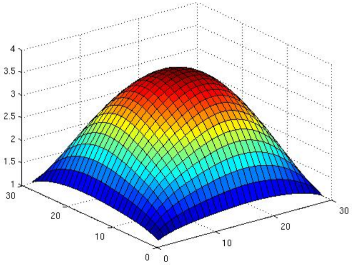


sparse c:

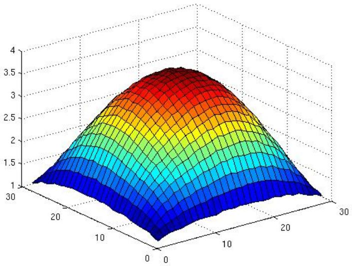


Smooth test example, 1% L^∞-noise

exact data u:

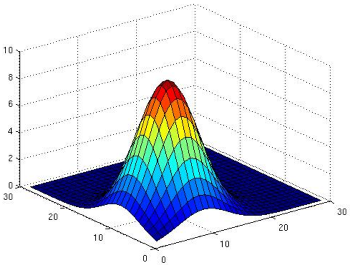


noisy data u^δ, $\|u - u^\delta\|_{L^\infty} = 1\%$:

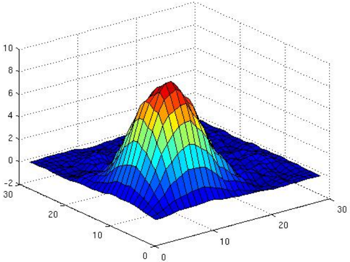


Smooth test example, 1% L^∞-noise

exact potential c:

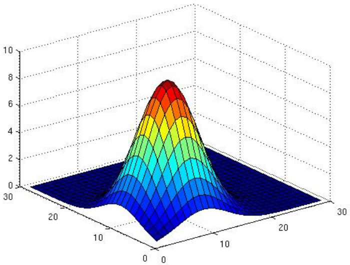


reconstruction $c_{k_*}^\delta$,
$X = L^2$, $Y = L^2$, $p = 2$, $r = 2$:

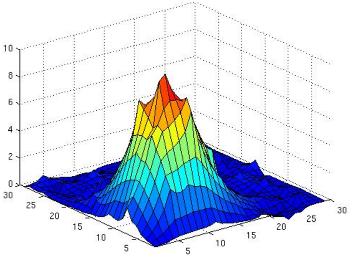


Smooth test example, 1% L^∞-noise

exact potential c:

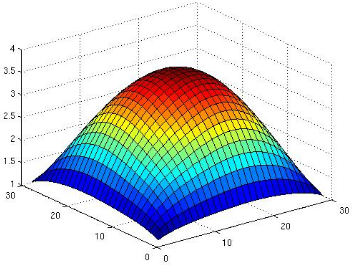


reconstruction $c_{k_*}^\delta$,
$X = L^2$, $Y = L^{22}$, $p = 2$, $r = 2$:

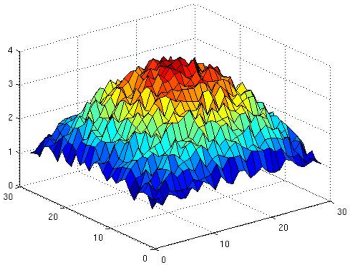


Smooth test example, 10% L^∞-noise

exact data u:

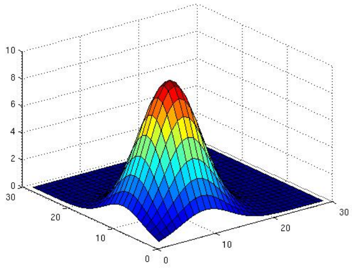


noisy data u^δ, $\|u - u^\delta\|_{L^\infty} = 10\%$:

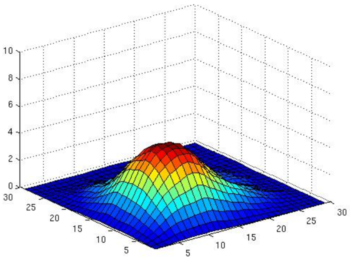


Smooth test example, 10% L^∞-noise

exact potential c:

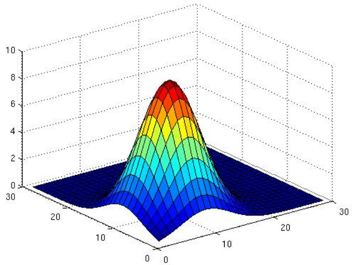


reconstruction $c_{k_*}^{\delta}$,
$X = L^2$, $Y = L^2$, $p = 2$, $r = 2$:

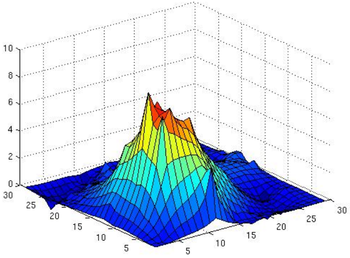


Smooth test example, 10% L^∞-noise

exact potential c:

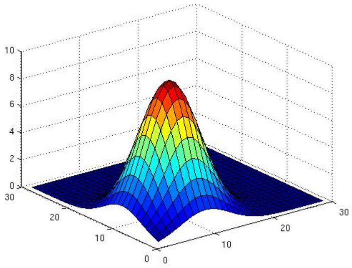


reconstruction $c_{k_*}^\delta$,
$X = L^2$, $Y = L^{22}$, $p = 2$, $r = 2$:

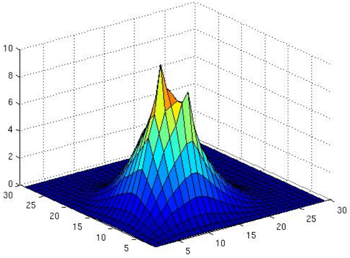


Smooth test example, 10% L^∞-noise

exact potential c:

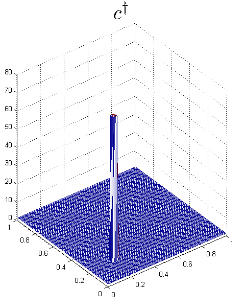


reconstruction $c_{k_*}^\delta$,
$X = L^{1.1}$, $Y = L^{22}$, $p = 1.1$, $r = 2$:

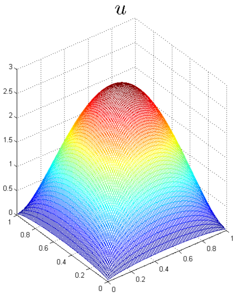


Sparse test example, 3% L^∞-noise

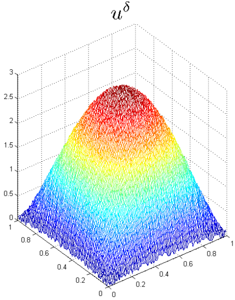
exact potential c:



exact data u:



noisy data u^δ:



computations by Frank Schöpfer

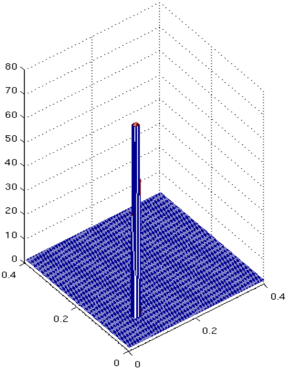
Sparse test example, 3% L^∞-noise

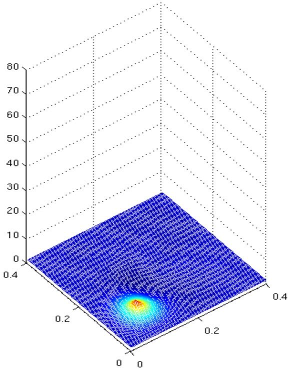
exact potential c:

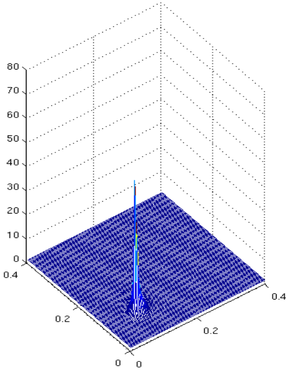
reconstructions $c^\delta_{k_*}$:

$X = L^2,\ Y = L^{11}$ $\quad$ $X = L^{1.1},\ Y = L^{11}$

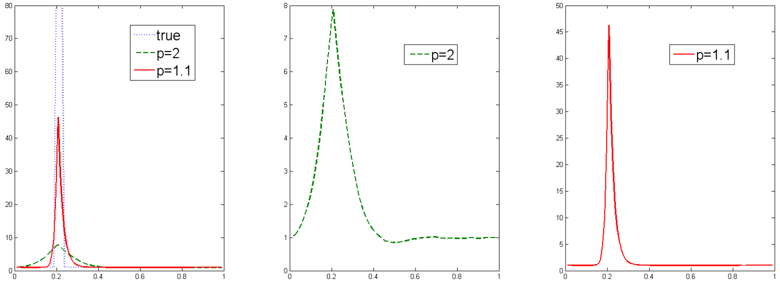
$p = 2,\ r = 2$: $\quad$ $p = 1.1,\ r = 2$:







Sparse test example, 3% L^∞-noise



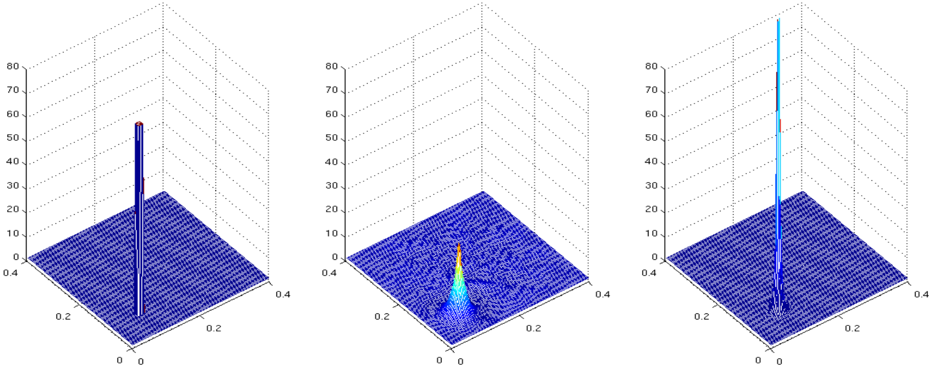
Sparse test example, 1% L^∞-noise

exact potential c:

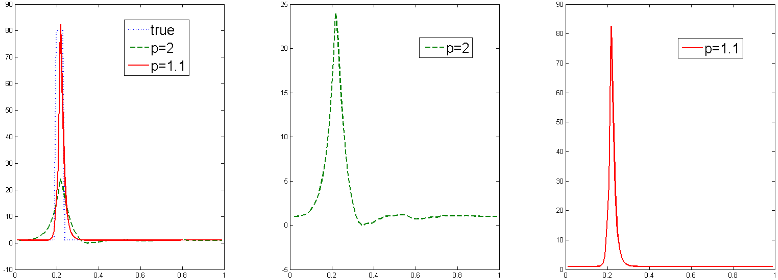
reconstructions $c_{k_*}^\delta$:

$X = L^2$, $Y = L^{11}$ $p = 2$, $r = 2$:

$X = L^{1.1}$, $Y = L^{11}$ $p = 1.1$, $r = 2$:



Sparse test example, 1% L^∞-noise



Summary and Outlook

- motivation for solving inverse problems in Banach spaces: more natural norms, possible reduction of ill-posedness, sparsity
- use of Banach spaces instead of Hilbert spaces may add nonlinearity and nonsmoothness, but keeps convexity
- gradient (Landweber) and Gauss-Newton methods for nonlinear inverse problems
- formulation and convergence analysis in Banach space

$\longrightarrow$ replace Tikhonov for Newton step by an inner iteration

$\longrightarrow$ ...

Thank you for your attention!